ON CYCLIC FIXED POINTS OF SPECTRA

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ABSTRACT. For a finite $p$-group $G$ and a bounded below $G$-spectrum $X$ of finite type mod $p$, the $G$-equivariant Segal conjecture for $X$ asserts that the canonical map $X^G \to X^{hG}$, from $G$-fixed points to $G$-homotopy fixed points, is a $p$-adic equivalence. Let $C_p^n$ be the cyclic group of order $p^n$. We show that if the $C_p^n$-equivariant Segal conjecture holds for a $C_p^n$-spectrum $X$, as well as for each of its geometric fixed point spectra $\Phi^n(X)$ for $0 < e < n$, then the $C_p^n$-equivariant Segal conjecture holds for $X$. Similar results also hold for weaker forms of the Segal conjecture, asking only that the canonical map induces an equivalence in sufficiently high degrees, on homotopy groups with suitable finite coefficients.

1. Introduction

Let $p$ be any prime number. Graeme Segal’s Burnside ring conjecture [1] for a finite $p$-group $G$ asserts that when $X = S_G$ is the genuinely $G$-equivariant sphere spectrum, then the canonical map $X^G \to X^{hG} = F(EG, X)^G$ is a $p$-adic equivalence. For cyclic groups $C = C_p$ of prime order the conjecture was proved by Lin [14] and Gunawardena [10]. Thereafter Ravenel [18], [19] gave an inductive proof of the Segal conjecture for finite cyclic $p$-groups $G = C_p^n$ of order $p^n$, starting from Lin and Gunawardena’s theorems. Ravenel’s result was superseded by Carlsson’s proof [6] of the Segal conjecture for all finite $p$-groups, but as we shall show here, Ravenel’s methods are also of interest in a more general context, where $X$ is a quite general $G$-spectrum. As was elucidated by Miller and Wilkerson [17], Ravenel’s methods give two proofs of the Segal conjecture for cyclic groups—one computational using the modified Adams spectral sequence, and one non-computational, using explicit geometric constructions.

The object of this paper is to generalize the non-computational, geometric proof of the Segal conjecture to deduce when $X^G \to X^{hG}$ is “close to” a $p$-adic equivalence for $G = C_p^n$, assuming that $X^C \to X^{hC}$ and similar maps are “close to” such an equivalence for $C = C_p$.

Our main technical results are Theorem 2.4 and Corollary 2.5. Their statements involve $(W, k)$-coconnected maps and geometric fixed points, which are discussed in Definitions 2.1 and 2.3, respectively.

In the special cases $X = B^{t_p^n}$ or $X = THH(B)$, where $B^{t_p^n}$ is a specific $C_p^n$-equivariant model for the $p^n$-th smash power of a spectrum $B$, and $THH(B)$ is the topological Hochschild homology of a symmetric ring spectrum $B$, the geometric fixed points are well understood, as explained in Theorems 2.7 and 2.8. In the special cases $W = S^{-1}/p^\infty$ or $W = F(V, S)$, where $V$ is a finite $p$-torsion spectrum, the $(W, k)$-coconnected maps are well understood in terms of $p$-completion or homotopy with $V$-coefficients, as explained in Examples 2.9 and 2.10. In the doubly special case when $X = THH(B)$ and $W = S^{-1}/p^\infty$, our results recover the main theorem of Tsalidis [20].

2. Statement of results

We first formalize the notion of being “close to” a $p$-adic equivalence.

Definition 2.1. Let $S^{-1}/p^\infty$ be the Moore spectrum with homology $\mathbb{Z}/p^\infty$ concentrated in degree $-1$, so that the function spectrum $F(S^{-1}/p^\infty, Y) = Y_p^\infty$ is the $p$-adic completion of an arbitrary spectrum $Y$.

Let $W$ be an object in the localizing ideal [12, Def. 1.4.3(d)] of spectra generated by $S^{-1}/p^\infty$, i.e., the smallest thick subcategory of spectra that contains $S^{-1}/p^\infty$ and is closed under arbitrary wedge sums, as well as under smash products with arbitrary spectra. This assumption on $W$ implies that $F(W, Y)$ is contractible whenever $Y_p^\infty$ is contractible.

Let $k$ be an integer, or $-\infty$. We say that a spectrum $Y$ is $(W, k)$-coconnected if $\pi_*(F(W, Y)) = 0$ for all $* \geq k$. We say that a map of spectra $f : Y_1 \to Y_2$ is $(W, k)$-coconnected if $hofib(f)$ is $(W, k)$-coconnected, or equivalently, if $\pi_*(F(W, Y_1)) \to \pi_*(F(W, Y_2))$ is injective for $* = k$ and an isomorphism for all $* > k$.

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Example 2.2. The most obvious choice for $W$ is $W = S^{-1/p\infty}$, in which case $F(W,Y) = Y_p^\infty$ so a map $f: Y_1 \to Y_2$ is $(W,k)$-coconnected if and only if the $p$-completed map $\hat{f}_p: (Y_1)^\infty_p \to (Y_2)^\infty_p$ induces an injection on $\pi_*$ for $* = k$ and an isomorphism for $* > k$. When $k = -\infty$, this is the same as being a $p$-adic equivalence.

For another class of examples we may take $W = F(V,S)$, where $V$ is a finite CW spectrum whose integral homology is $p$-torsion, in which case $F(W,Y) \simeq V \wedge Y$ by Spanier–Whitehead duality. In this case $f: Y_1 \to Y_2$ is $(W,k)$-coconnected if and only if the map $1 \wedge f: V \wedge Y_1 \to V \wedge Y_2$ induces an injection $\pi_*(Y_1) = \pi_*(V \wedge Y_1) \to \pi_*(V \wedge Y_2) = \pi_*(Y_2)$ for $* = k$ and an isomorphism for $* > k$.

Hereafter we assume that a pair $(W,k)$ has been chosen as in the definition above. Next, we recall some comparison maps between fixed points, geometric fixed points and homotopy fixed points.

Definition 2.3. Let $C = C_p \subset C_p = G$ and $\tilde{G} = G/C \cong C_{p^{n-1}}$. Let $\lambda = C(1)$ be the basic faithful $G$-representation of complex rank one, and $S^\lambda$ its one-point compactification. Let $\infty\lambda$ be the direct sum of a countable number of copies of $\lambda$. Its unit sphere $S(\infty\lambda) = EG$ is a free contractible $G$-CW space, and its one-point compactification $S^{\infty\lambda} = \tilde{E}G$ sits in a $G$-homotopy cofiber sequence $EG \to S^0 \to \tilde{E}G$, where the first map collapses $EYG$ to the non-basepoint. Note the $G$-homeomorphism $\tilde{E}G \wedge ( S^{\infty\lambda} \cong \tilde{E}G$ for each $j \geq 0$. Let $X$ be any genuine $G$-spectrum [13], and consider the vertical map

$$
\begin{array}{c}
\begin{CD}
EG_+ \wedge X @>\simeq_g>> X @>\Phi^C(X)^G>> \tilde{E}G \wedge X
\end{CD}
\end{array}
$$

of horizontal $G$-homotopy cofiber sequences. Passing to $G$-fixed point spectra we obtain a vertical map

$$
\begin{array}{c}
\begin{CD}
X_{hG} @>N>> X^G @>\Phi^C(X)^G>> X_{hG}
\end{CD}
\end{array}
$$

of horizontal homotopy cofiber sequences, called the norm–restriction sequences [9, Diag. (C), (D)]. Here

$X_{hG} = EG_+ \wedge GX$ (homotopy orbits)

$X^G = F(EG_+, X)^G$ (homotopy fixed points)

$X_{hG}^G = [\tilde{E}G \wedge F(EG_+, X)]^G$ (Tate construction)

and there is a $\tilde{G}$-equivariant equivalence

$$\Phi^C(X) \simeq [\tilde{E}G \wedge X]^C$$

inducing the upper right hand equivalence $[\tilde{E}G \wedge X]^G \simeq \Phi^C(X)^G$. For more details, see e.g. [11, Prop. 2.1].

The right hand square above is homotopy cartesian, so $\Gamma_n$ is $(W,k)$-coconnected if and only if $\hat{\Gamma}_n$ is $(W,k)$-coconnected. This observation can be combined with the conclusions of all of the theorems below. We write $H_*(X) = H_*(X; \mathbb{F}_p)$ for the mod $p$ homology of any spectrum.

Theorem 2.4. Let $X$ be a $G$-spectrum. Assume that $\pi_*(X)$ is bounded below and $H_*(X)$ is of finite type. Suppose that $\Gamma_1: X^G \to X^{hG}$ and $\Gamma_{n-1}: \Phi^C(X)^G \to \Phi^C(X)^{hG}$ are $(W,k)$-coconnected maps. Then $\Gamma_n: X^G \to X^{hG}$ is $(W,k)$-coconnected.

Informally, the theorem asserts that if $X^G \to X^{hG}$ is close to a $p$-adic equivalence, and we can inductively prove that $Y^G \to Y^{hG}$ is close to a $p$-adic equivalence for $Y = \Phi^C(X)$, then $X^G \to X^{hG}$ is close to a $p$-adic equivalence.

Corollary 2.5. Let $X$ be a $C_{p^n}$-spectrum. Suppose for each of the geometric fixed point spectra

$$Y = X, \Phi^{C_p}(X), \ldots, \Phi^{C_{p^{n-1}}}(X)$$

that $Y$ is bounded below with $H_*(Y)$ of finite type, and that $\Gamma_1: Y^{C_p} \to Y^{hC_p}$ is $(W,k)$-coconnected. Then $\Gamma_n: X^{C_{p^n}} \to X^{hC_{p^n}}$ is $(W,k)$-coconnected.

The proofs of Theorem 2.4 and Corollary 2.5 are given near the end of Section 3.
Definition 2.6. Let \( B \) be any spectrum. When \( B \) is realized as a symmetric spectrum (or an FSP), the \( r \)-fold smash power \( B^{\wedge r} \) can be defined as a genuine \( C_\ast \)-spectrum by the construction

\[
B^{\wedge r} = sd_r THH(B)_0 = THH(B)_{r-1}
\]

from [11, §2.4]. Its \( V \)-th space is defined by a homotopy colimit

\[
(B^{\wedge r})_V = \text{hocolim}_{(i_1, \ldots, i_r) \in I^r} \text{Map}(S^{i_1} \wedge \cdots \wedge S^{i_r}, B_{i_1} \wedge \cdots \wedge B_{i_r} \wedge S^V),
\]

and \( C_i \) cyclically permutes the smash factors, in addition to its natural action on \( S^V \). We are principally interested in the case \( r = p^n \).

In [15, Thm. 5.13], the third and fourth authors prove that \( \Gamma_1 : (B^{\wedge p})^C_p \to (B^{\wedge p})^{hC_p} \) is a \( p \)-adic equivalence whenever \( \pi_\ast(B) \) is bounded below and \( H_\ast(B) \) is of finite type. This provides the inductive beginning for the following application of Corollary 2.5.

Theorem 2.7. Let \( B \) be a spectrum with \( \pi_\ast(B) \) bounded below and \( H_\ast(B) \) of finite type. Then

\[
\Gamma_n : (B^{\wedge p^n})^C_p \to (B^{\wedge p^n})^{hC_p}
\]

is a \( p \)-adic equivalence, for each \( n \geq 1 \).

When \( B \) is a symmetric ring spectrum, its topological Hochschild homology \( THH(B) \) is a genuine \( \mathbb{T} \)-spectrum [11, §2.4], where \( \mathbb{T} \) is the circle group. It is not true in general that \( \Gamma_1 : THH(B)^C_p \to THH(B)^{hC_p} \) is a \( p \)-adic equivalence, but when it is “approximately” true, then the following theorem is useful.

Theorem 2.8. Let \( B \) be a connective symmetric ring spectrum with \( H_\ast(B) \) of finite type, and suppose that

\[
\Gamma_1 : THH(B)^C_p \to THH(B)^{hC_p}
\]

is \((W, k)\)-coconnected. Then

\[
\Gamma_n : THH(B)^{C_p^n} \to THH(B)^{hC_p^n}
\]

is \((W, k)\)-coconnected, for each \( n \geq 2 \).

The proofs of Theorems 2.7 and 2.8 are given at the end of Section 3.

In the case \( B = S \) there is a \( G \)-equivariant equivalence \( THH(S) \simeq S_G \), and \( \Gamma_1 \) is a \( p \)-adic equivalence by the classical Segal conjecture. Also in the cases \( B = MU \) (the complex cobordism spectrum) and \( B = BP \) (the Brown–Peterson spectrum) it turns out that \( \Gamma_1 \) for \( THH(B) \) is a \( p \)-adic equivalence, as the third and fourth authors show in [16, Thm. 1.1]. This provides examples with \( k = -\infty \) for the following special case.

Example 2.9. Taking \( W = S^{-1/p^\infty} \), the assumption in Theorem 2.8 is that the \( p \)-completed map

\[
\Gamma_1 : (THH(B)^C_p)^\wedge_p \to (THH(B)^{hC_p})^\wedge_p
\]

is \( k \)-coconnected, i.e., that it induces an injection on \( \pi_k \) and an isomorphism on \( \pi_\ast \) for \( * > k \), and the conclusion is that the \( p \)-completed map

\[
\Gamma_n : (THH(B)^{C_p^n})^\wedge_p \to (THH(B)^{hC_p^n})^\wedge_p
\]

is also \( k \)-coconnected, for all \( n \geq 2 \). This recovers a theorem of Tsalidis [20, Thm. 2.4].

Example 2.10. Taking \( W = F(V, S) \) and \( V = V(1) = S/(p, v_1) \), the Smith–Toda complex of chromatic type 2, the assumption in Theorem 2.8 is that

\[
\Gamma_1 : V(1)_\ast(THH(B)^C_p) \to V(1)_\ast(THH(B)^{hC_p})
\]

is \( k \)-coconnected, and the conclusion is that

\[
\Gamma_n : V(1)_\ast(THH(B)^{C_p^n}) \to V(1)_\ast(THH(B)^{hC_p^n})
\]

is also \( k \)-coconnected, for all \( n \geq 2 \). This recovers the generalization of Tsalidis’ theorem used (implicitly) by Ausoni and Rognes [3, Thm. 5.7] in the special case when \( B = \ell \), the Adams summand of connective \( p \)-local complex \( K \)-theory, and \( k = 2p - 2 \). The generalized result is used again in [4, Cor. 5.10].
Lemma 3.2. Let $X$ be a genuine $G$-spectrum. There is a natural homotopy cofiber sequence

$$\text{holim}_j (\Sigma^{-j\lambda} X)^G \longrightarrow (X^C)^G \xrightarrow{\Gamma_{n-1}} (X^C)^{hG}$$

where the right hand map is $\Gamma_{n-1}$ for the $G$-spectrum $X^C$.

Proof. By mapping the $G$-homotopy cofiber sequence $EG_+ \to S^0 \to \widetilde{EG}$ into $X^C$, we get the homotopy (co-)fiber sequence

$$F(\widetilde{EG}, X^C)^G \longrightarrow (X^C)^G \xrightarrow{\Gamma_{n-1}} F(EG_+, X^C)^G.$$ 

Here

$$F(\widetilde{EG}, X^C)^G \simeq \text{holim}_j F(S^{j\lambda}, X^C)^G \simeq \text{holim}_j (\Sigma^{-j\lambda} X)^G.$$ 

This gives the asserted homotopy cofiber sequence. □

Proposition 3.3. Let $X$ be a genuine $G$-spectrum. There is a vertical map of homotopy cofiber sequences

$$\text{holim}_j \Phi^C(\Sigma^{-j\lambda} X)^G \longrightarrow \Phi^C(X)^G \xrightarrow{\Gamma_{n-1}} \Phi^C(X)^{hG} \quad \text{holim}_j (\Sigma^{-j\lambda} X)^{hG} \longrightarrow X^{hG} \xrightarrow{\Gamma_{n-1}} (X^C)^{hG}.$$ 

The right hand horizontal maps are $\Gamma_{n-1}$ for the $G$-spectra $\Phi^C(X) \simeq [\widetilde{EG} \wedge X]^C$ and $X^{1C} \simeq [\widetilde{EG} \wedge F(EG_+, X)]^C$, respectively.

Proof. We replace $X$ in the lemma above by the $G$-spectra $\widetilde{EG} \wedge X$ and $\widetilde{EG} \wedge F(EG_+, X)$. This gives the two claimed homotopy cofiber sequences, in view of the $G$-equivalences

$$\Phi^C(\Sigma^{-j\lambda} X) \simeq [\widetilde{EG} \wedge F(S^{j\lambda} p, X)]^C \simeq F(S^{j\lambda}, [\widetilde{EG} \wedge X]^C)$$

and

$$(\Sigma^{-j\lambda} X)^{1C} \simeq [\widetilde{EG} \wedge F(EG_+, F(S^{j\lambda} p, X))]^C \simeq F(S^{j\lambda}, [\widetilde{EG} \wedge F(EG_+, X)]^C),$$

respectively. These all follow from the $G$-dualizability of $S^{j\lambda} p$. □

Lemma 3.4. If $\tilde{\Gamma}_1: \Phi^C(X) \to X^{1C}$ is $(W, k)$-coconnected, then $(\tilde{\Gamma}_1)^{hG}$ is $(W, k)$-coconnected.

Proof. This is a special case of a more general result. The homotopy fixed point spectral sequence

$$E_2^{s,t} = H^{-s}(G; \pi_t(Y)) \implies \pi_{s+t}(Y^{hG})$$

shows that $Y^{hG}$ is $k$-coconnected whenever $Y$ is a $k$-coconnected $G$-spectrum. Commutation of function spectra, homotopy fibers and homotopy fixed points shows that $Y_i^{hG} \to Y_{i+1}^{hG}$ is $(W, k)$-coconnected whenever $Y_i \to Y_{i+1}$ is a $(W, k)$-coconnected $G$-map. The lemma follows by applying this for the $G$-map $\tilde{\Gamma}_1$. □

Definition 3.5. The Greenlees filtration [8, p. 437] of $\widetilde{EG}$ is an integer-indexed $G$-cellular filtration of spectra, whose $2i$-th term is $S^{i\lambda}$ for each integer $i$. The $(2i+1)$-th term is obtained from $S^{i\lambda}$ by attaching a single $G$-free $(2i+1)$-cell, and $S^{(2i+1)\lambda}$ is in turn obtained from it by attaching a single $G$-free $(2i+2)$-cell. The Greenlees filtration induces an increasing filtration of $X^G = [\widetilde{EG} \wedge F(EG_+, X)]^G$, and a tower of homotopy cofibres with $(2i+1)$-th term

$$X^{1G}(i) = [\widetilde{EG} / S^{i\lambda} \wedge F(EG_+, X)]^G.$$
which we call the Tate tower. The associated spectral sequence is the homological $G$-equivariant Tate spectral sequence

$$E^2_{s,t} = \hat{H}^{-s}(G; H_t(X))$$

converging to the continuous homology groups

$$H^s_*(X^{tG}) = \lim_{i} H_*(X^{tG} (i))$$

of $X^{tG}$, when $X$ is a bounded below spectrum with $H_*(X)$ of finite type. See [15, Def. 2.3, Prop. 4.15]. Note that $i$ tends to $-\infty$ in this limit. We shall also refer to the continuous cohomology groups

$$H_+(X^{tG}) = \colim_i H^+(X^{tG} (i)),$$

and note that $H^+_*(X^{tG}) \cong H^+_*(X^{tG})^*$ (the Hom dual) when $H_*(X)$ is of finite type, because then each $H_+(X^{tG} (i))$ is also of finite type.

**Definition 3.6.** Let the $G$-map $\xi: S^\lambda \to S^{\lambda^p}$ be the suspension of the degree $p$ covering map $\pi: S^1 = S(\lambda) \to S(\lambda^p) = S^1/C$ of unit spheres, as in the following vertical map of horizontal homotopy cofiber sequences:

$$
\begin{array}{ccc}
S(\lambda) & \to & S^0 \\
\downarrow & & \downarrow \xi \\
S(\lambda^p) & \to & S^0 \\
\end{array}
$$

Then $\xi$ has degree $p$ on the top cell, so $\xi_*: H_*(S^\lambda) \to H_*(S^{\lambda^p})$ is the zero homomorphism (since we work with reduced homology and mod $p$ coefficients).

**Proposition 3.7.** Let $X$ be a $G$-spectrum with $H_*(X)$ bounded below. Then

$$\lim_j H_+^*((\Sigma^{-j\lambda^p} X)^{tG}) = \lim_{i,j} H_+^*((\Sigma^{-j\lambda^p} X)^{tG} (i)) = 0$$

and

$$\colim_{i,j} H_+^*((\Sigma^{-j\lambda^p} X)^{tG} (i)) = \colim_{i,j} H^+((\Sigma^{-j\lambda^p} X)^{tG} (i)) = 0.$$

**Proof.** In the notation of (3.1) we have a natural equivalence

$$(\Sigma^{-j\lambda^p} X)^{tG} (i) \cong (\Sigma^{(\lambda-\lambda^p)} X)^{tG} (i-j)$$

for each $i$ and $j$, since $S^{i\lambda}$ is $G$-dualizable. Under this identification, the $z$-tower map

$$z: (\Sigma^{-j(1+]} \lambda^p X)^{tG} (i) \to (\Sigma^{-j\lambda^p} X)^{tG} (i)$$

induced by smashing with $z: S^0 \to S^{\lambda^p}$ corresponds to the composite of the Tate tower map

$$(\Sigma^{(j+1)(\lambda-\lambda^p)} X)^{tG} (i-j-1) \to (\Sigma^{(j+1)(\lambda-\lambda^p)} X)^{tG} (i-j)$$

induced by smashing with $S^0 \to S^\lambda$, and the map

$$\xi: (\Sigma^{(j+1)(\lambda-\lambda^p)} X)^{tG} (i-j) \to (\Sigma^{(\lambda-\lambda^p)} X)^{tG} (i-j)$$

induced by smashing with $\xi: S^\lambda \to S^{\lambda^p}$:

$$
\begin{array}{ccc}
[\hat{E}/S^{i\lambda} \wedge F(EG_+, \Sigma^{-j(1+]} \lambda^p X)]^G & \cong & [\hat{E}/S^{(i-j-1)\lambda} \wedge F(EG_+, \Sigma^{(j+1)(\lambda-\lambda^p)} X)]^G \\
\downarrow & & \downarrow \\
[\hat{E}/S^{(i-j)\lambda} \wedge F(EG_+, \Sigma^{(j+1)(\lambda-\lambda^p)} X)]^G & \cong & [\hat{E}/S^{(i-j)\lambda} \wedge F(EG_+, \Sigma^{(\lambda-\lambda^p)} X)]^G \\
\end{array}
$$

Passing to the limit over $i$, the homomorphism

$$z_*: H^+_*((\Sigma^{-j(1+]} \lambda^p X)^{tG}) \to H^+_*((\Sigma^{-j\lambda^p} X)^{tG})$$

is identified with the homomorphism

$$\xi_*: H^+_*((\Sigma^{(j+1)(\lambda-\lambda^p)} X)^{tG}) \to H^+_*((\Sigma^{(\lambda-\lambda^p)} X)^{tG}),$$

(3.2)
so it suffices to show that the limit over \(j\) of the latter homomorphisms is zero. Let

\[
\hat{E}_s^2(j) = \hat{H}^{-s}(G; H_t(\Sigma^{(\lambda^{-j\lambda})} X)) \implies H^c_{s+t}(\Sigma^{(\lambda^{-j\lambda})} X)^G
\]

be the homological Tate spectral sequence for the \(j\)-th term in the \(\xi\)-tower.

The map \(\xi_s\) above is compatible with the spectral sequence map \(\hat{E}_s^2(j+1) \rightarrow \hat{E}_s^2(j)\) that is induced on Tate cohomology by the \(G\)-module homomorphism

\[
\xi_s: H_*(\Sigma^{(\lambda^{-j\lambda})} X) \rightarrow H_*(\Sigma^{(\lambda^{-j\lambda})} X).
\]

This homomorphism is zero, since \(\xi_s\): \(H_*(S^{\lambda}) \rightarrow H_*(S^{\lambda})\) is zero. Hence the map of spectral sequences is also zero. It follows that the homomorphism \(\xi_s\) in (3.2) strictly reduces the Tate filtration (= \(s\)) of each nonzero continuous homology class. Equivalently, \(\xi_s\) strictly increases the vertical degree (= \(t\)) of the spectral sequence representative of each nonzero class.

By assumption, there is an integer \(\ell\) such that \(H_t(X) = 0\) for all \(t < \ell\). Then \(\hat{E}_s^{2}\) is contractible. This follows by induction on \(\ell\).

**Proof.**

Consider the diagram in Proposition 3.3. By assumption, the maps

\[
\Gamma_{n-1}: \Phi^C(X)^G \rightarrow \Phi^C(X)^{hG}\quad \text{and} \quad \Gamma_1: X^C \rightarrow X^{hC}
\]

are \((W, k)\)-coconnected. Hence \(\hat{\Gamma}_1: \Phi^C(X) \rightarrow X^C\) is \((W, k)\)-coconnected, so by Lemma 3.4 also

\[
(\hat{\Gamma}_1)^{hG}: \Phi^C(X)^{hG} \rightarrow (X^{hC})^{hG}
\]

is \((W, k)\)-coconnected. By Proposition 3.8, the map \(\Gamma_{n-1}: X^{hG} \rightarrow X^{hG}\) is a p-adic equivalence, hence \((W, -\infty)\)-coconnected, by our standing assumption that \(W\) is in the localizing ideal of spectra generated by \(S^{-1}/p^\infty\). It follows easily that \(\hat{\Gamma}_1: \Phi^C(X)^G \rightarrow X^{hG}\) is \((W, k)\)-coconnected, which is equivalent to \(\Gamma_{n-1}: X^G \rightarrow X^{hG}\) being \((W, k)\)-coconnected.

**Proof of Corollary 2.5.** This follows by induction on \(n\), using Theorem 2.4 and the observation that

\[
\Phi^{C_{e+1}}(\Phi^{C_{e}}(X)) \cong \Phi^{C_{e+1}}(X)
\]

for all \(0 \leq e < n\).
Proof of Theorem 2.7. This will follow from Corollary 2.5 in the case $X = B^\wedge p^n$, $W = S^{-1/p\infty}$ and $k = -\infty$, once we show that for each $0 \leq e < n$ there is a $C_{p,-e}$-equivalence

$$Y = \Phi^{C_p}(B^\wedge p^n) \simeq B^\wedge p^{n-e},$$

the right hand side is bounded below with mod $p$ homology of finite type, and $\Gamma_1 : Y^{C_p} \to Y^{hC_p}$ is a $p$-adic equivalence.

The first claim follows from the proof in simplicial degree 0 of [11, Prop. 2.5]. Writing $Y \simeq Z^{\wedge p}$, where $Z = B^\wedge p^{n-e-1}$ is bounded below with $H_* (Z)$ of finite type, the other claims also follow, since $\Gamma_1 : (Z^{\wedge p})^{C_p} \to (Z^{\wedge p})^{hC_p}$ is a $p$-adic equivalence by [15, Thm. 5.13], generalizing [5, §II.5].

Proof of Theorem 2.8. There is a $C_{p,-e}$-equivalence

$$r : \Phi^{C_p} THH(B) \xrightarrow{\sim} THH(B)$$

(the cyclotomic structure map of $THH(B)$, see [11, §2.5]), whose $e$-fold iterate is a $C_{p,-e}$-equivalence $\Phi^{C_p}(THH(B)) \simeq THH(B)$. It is clear from the simplicial definition that $THH(B)$ is connective and has mod $p$ homology of finite type, hence the theorem follows from Corollary 2.5.

References