

CHROMATIC REDSHIFT

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1. INTRODUCTION

Consider a commutative ring R , with sum and product operations. The category of representations of R inherits a commutative rig structure, given by direct sum and tensor product. In other words, the category $\text{Mod}(R)$ of R -modules inherits a bipermutative structure. Continuing, one can consider the categorical representations of $\text{Mod}(R)$, and these in turn form a 2-category $\text{Mod}(\text{Mod}(R))$, with a ring-like structure. Iterating, one can consider an n -category of higher representations, for each $n \geq 1$.

All of these constructions can take place within the limiting context of structured ring spectra, or commutative S -algebras. From the category of (finite cell) modules over a commutative S -algebra B we can distill a new commutative S -algebra, the algebraic K -theory spectrum $K(B)$. Continuing, one can form $K(K(B))$, etc. When $B = HR$ is the Eilenberg–Mac Lane spectrum of an ordinary ring, the n -fold algebraic K -theory $K^{(n)}(B)$ is extracted from the n -category of higher representations, as above. In this sense, n -fold iterated algebraic K -theory has something to do with n -categories.

From this point of view it is surprising that n -fold iterated algebraic K -theory also has something to do with formal group laws of height n , i.e., one-dimensional commutative formal group laws F in characteristic p where the series expansion $[p]_F(x)$ for the multiplication-by- p map starts with a unit times x^{p^n} . This is essentially a statement about the formal coproduct on $K^{(n)}(B)^*(\mathbb{C}P^\infty)$ that comes from the product on $\mathbb{C}P^\infty$. Hesselholt–Madsen asked about the chromatic filtration of iterated topological cyclic homology in [HM97, p. 61], but could almost as well have asked about the chromatic filtration of iterated algebraic K -theory.

In a strong form, this connection implies that the algebraic K -theory of a structured ring spectrum related to formal group laws of height n will be related to formal group laws of height $n+1$. In terms of the periodic families of stable homotopy theory, if the homotopy of B is v_n -periodic but not v_{n+1} -periodic, then frequently $K(B)$ is v_{n+1} -periodic but not v_{n+2} -periodic.

Since the (fundamental) period $|v_{n+1}| = 2p^{n+1} - 2$ of v_{n+1} -periodicity is longer than the period $|v_n| = 2p^n - 2$ of v_n -periodicity, we think of this phenomenon as an increase, or lengthening, of wavelengths. This is what we informally call a “redshift”. In a related fashion, the v_{n+1} -periodic phenomena are usually hidden or nested behind the v_n -periodic ones, hence more subtle and difficult to detect. Again this corresponds informally to less energetic light, propagating at lower frequencies.

The height filtration is also related to the sequence of Hopf subalgebras

$$0 \subset \cdots \subset \mathcal{E}(n) = E(Q_0, \dots, Q_n) \subset \cdots$$

of the Steenrod algebra \mathcal{A} , and their annihilating subalgebras

$$\mathcal{A}_* \supset \cdots \supset (\mathcal{A} // \mathcal{E}(n))^* = P(\bar{\xi}_k \mid k \geq 1) \otimes E(\bar{\tau}_k \mid k \geq n+1) \supset \cdots$$

The latter nested sequence of \mathcal{A}_* -comodule subalgebras are invariant under the Dyer–Lashof operations that arise from thinking of the dual Steenrod algebra \mathcal{A}_* as $H_*(H)$, where $H = H\mathbb{F}_p$ is a commutative structured ring spectrum.

2. REDSHIFT IN THE K -THEORY OF RINGS

We start with examples of chromatic redshift in the algebraic K -theory of discrete rings.

Let k be a finite field of characteristic p , with algebraic closure \bar{k} . Quillen proved [Qui72, §11] that $H_i(BGL(\bar{k}); \mathbb{F}_p) = 0$ for $i > 0$, so that $K(\bar{k})_p \simeq H\mathbb{Z}_p$. Furthermore, he deduced that $\pi_* K(k)_p \cong \pi_* K(\bar{k})_p^{hG_k}$ for $* \geq 0$, where the absolute Galois group G_k acts continuously on $K(\bar{k})$, so $K(k)_p \simeq H\mathbb{Z}_p$.

Multiplication by p acts injectively on $\pi_*K(\bar{k})_p$, hence also on $\pi_*K(k)_p$. Think of p as a lift of $p = v_0 \in \pi_*BP$, where BP is the Brown–Peterson ring spectrum with $\pi_*BP = \mathbb{Z}_{(p)}[v_n \mid n \geq 1]$.

For a separably closed field \bar{F} of characteristic $\neq p$ (including 0), Lichtenbaum conjectured that $\pi_tK(\bar{F})_p$ is \mathbb{Z}_p for $t \geq 0$ even and 0 for t odd. This was proved by Suslin [Sus84, Cor. 3.13], and implies that $K(\bar{F})_p \simeq ku_p$ and $\hat{L}_1K(\bar{F}) \simeq KU_p$. Here ku is the connective cover of the complex topological K -theory ring spectrum KU , and $\hat{L}_n = L_{K(n)}$ denotes Bousfield localization [Bou79] with respect to the Morava K -theory ring spectrum $K(n)$. Multiplication by the Bott element $u \in \pi_2ku_p$ acts bijectively on $\pi_*K(\bar{F})_p$, for $* \geq 0$.

Let F be a number field, with a ring of S -integers A .

$$\begin{array}{ccc} A & \longrightarrow & F \\ \uparrow & & \uparrow \\ \mathbb{Z} & \longrightarrow & \mathbb{Q} \end{array}$$

Quillen conjectured [Qui75, §9] that there is a spectral sequence

$$E_{s,t}^2 = H_{\text{ét}}^{-s}(\text{Spec } A; \mathbb{Z}_p(t/2)) \implies \pi_{s+t}K(A)_p$$

converging in total degrees ≥ 1 . Here $H_{\text{ét}}^*(-)$ denotes étale cohomology, which is only well-behaved if $1/p \in A$, and $\mathbb{Z}_p(t/2) \cong \pi_tK(\bar{F})_p$. For $A = F$ this means that $\pi_*K(F)_p \cong \pi_*K(\bar{F})_p^{hG_F}$ for $* \geq 1$, where G_F is the absolute Galois group. The general case requires the more elaborate framework of étale homotopy types. Passing to mod p homotopy, a lift $\beta \in \pi_{2p-2}(S/p)$ of $u^{p-1} \in \pi_{2p-2}(ku; \mathbb{Z}/p)$ would act bijectively on $\pi_*(K(A); \mathbb{Z}/p)$, for $* \geq 1$. Think of $\beta = v_1$ as a lift of $v_1 \in \pi_*(BP; \mathbb{Z}/p)$.

Thomason [Tho85, Thm. 4.1] proved Quillen’s conjecture, up to the localization given by inverting β . In particular, $\pi_*(K(F); \mathbb{Z}/p)[1/\beta] \cong \pi_*(K(\bar{F})^{hG_F}; \mathbb{Z}/p)$ for $* \geq 2$. It remained to show that $\pi_*(K(A); \mathbb{Z}/p) \rightarrow \pi_*(K(A); \mathbb{Z}/p)[1/\beta]$ is an isomorphism for $* \geq 2$. Waldhausen [Wal84, p. 193] noted that this amounts to asking that $K(A) \rightarrow L_1K(A)$ is a p -adic equivalence, in sufficiently high degrees. Here $L_n = L_{E(n)}$ denotes Bousfield localization with respect to the Johnson–Wilson ring spectrum $E(n)$, or equivalently with respect to $BP[1/v_n]$.

Using topological cyclic homology, Hesselholt–Madsen [HM03, Thm. A] confirmed Quillen’s conjecture for valuation rings in local number fields, after special cases were treated by Bökstedt–Madsen [BM94], [BM95] and Rognes [Rog99a], [Rog99b].

Finally, Voevodsky’s proof [Voe03], [Voe11] of the Milnor and Bloch–Kato conjectures confirmed Quillen’s conjecture for rings of integers in global number fields.

3. REDSHIFT IN THE K -THEORY OF RING SPECTRA

We continue with examples of chromatic redshift in the context of algebraic K -theory of structured ring spectra.

Let $L = E(1)$ be the Adams summand of $KU_{(p)}$, and $\ell = BP\langle 1 \rangle$ its connective cover. Using topological cyclic homology, Ausoni–Rognes [AR02, Thm. 0.4] computed $V(1)_*K(\ell_p)$, and Ausoni [Aus10, Thm. 1.1] computed $V(1)_*K(ku_p)$, where $p \geq 5$ and $V(1) = S/\langle p, v_1 \rangle$ is the Smith–Toda spectrum of chromatic type 2. Using a localization sequence of Blumberg–Mandell [BM08, p. 157], this also calculates $V(1)_*K(L_p)$ and $V(1)_*K(KU_p)$. In each case, a lift $v_2 \in \pi_{2p^2-2}V(1)$ of $v_2 \in V(1)_*BP$ acts bijectively on the answer $V(1)_*K(B)$, for $* \geq 2p - 2$.

The results are compatible with the existence of a spectral sequence

$$E_{s,t}^2 = H_{\text{mot}}^{-s}(\text{Spec } B; \mathbb{F}_{p^2}(t/2)) \implies V(1)_{s+t}K(B)$$

for suitable “ ℓ_p -algebras of S -integers” B , converging in sufficiently high total degrees. Here $H_{\text{mot}}^*(-)$ refers to a hypothetical form of motivic cohomology for commutative structured ring spectra, and $\mathbb{F}_{p^2}(t/2) \cong V(1)_tE_2$ where E_2 is the $K(2)$ -local Lubin–Tate ring spectrum with $\pi_*E_2 = \mathbb{W}\mathbb{F}_{p^2}[[u_1]][u]$.

The appearance of the field \mathbb{F}_{p^2} is needed to account for the sign in Ausoni’s relation $b^{p-1} = -v_2$ in $V(1)_*K(ku_p)$, since if b is represented by αu^{p+1} and v_2 by u^{p^2-1} then $\alpha^{p-1} = -1$, which cannot be satisfied for $\alpha \in \mathbb{F}_p$.

4. AN ANALOGUE OF THE LICHTENBAUM–QUILLEN CONJECTURES

Consider a Galois extension $L_p[1/p] \rightarrow M$, like in [Rog08, §4]. By an ℓ_p -algebra of integers in M we mean a connected (only trivial idempotents) commutative ℓ_p -algebra B , with a structure map to M ,

such that B is semi-finite (retract of a finite cell module), or perhaps dualizable, as an ℓ_p -module:

$$\begin{array}{ccccc}
 & & \Omega_1 & & \\
 & & \uparrow & & \\
 & & B & \xrightarrow{\quad} & M \\
 & & \uparrow & & \uparrow G \\
 \ell_p & \longrightarrow & L_p & \longrightarrow & L_p[1/p] \\
 & & \uparrow & & \\
 & & J_p & &
 \end{array}$$

For S -integers we may allow localizations that invert p or v_1 . Let Ω_1 be the p -completed homotopy colimit of all such B , i.e., the ℓ_p -algebraic integers. ((Does this exist?))

By analogy with Quillen's conjecture/Voevodsky's theorem we predict that v_2 acts bijectively on $V(1)_*K(B)$, for $* \gg 0$. By analogy with Lichtenbaum's conjecture/Suslin's theorem, we predict that $V(1)_*K(\Omega_1) \cong V(1)_*E_2$, in all sufficiently high degrees, and that $\hat{L}_2K(\Omega_1) \simeq E_2$.

In the case when $B \rightarrow \Omega_1$ is an unramified G -Galois extension, the hypothetical motivic cohomology would reduce to group cohomology, and $V(1)_*K(B) \cong V(1)_*K(\Omega_1)^{hG}$ for $* \gg 0$. The general case would require a more elaborate construction than the familiar homotopy fixed points. Even establishing the existence of a ring spectrum map $K(ku) \rightarrow E_2$ seems to be an open problem.

Similarly, for $n \geq 1$ let E_n be the $K(n)$ -local Lubin–Tate ring spectrum, and let e_n be its connective cover, so that $E_n = e_n[1/u]$. Consider Galois extensions $E_n[1/p] \rightarrow M$ and connected commutative e_n -algebras B , with a structure map to M , such that B is semi-finite as an e_n -module:

$$\begin{array}{ccccc}
 & & \Omega_n & & \\
 & & \uparrow & & \\
 & & B & \xrightarrow{\quad} & M \\
 & & \uparrow & & \uparrow \\
 e_n & \longrightarrow & E_n & \longrightarrow & E_n[1/p] \\
 & & \uparrow & & \\
 & & \hat{L}_n S & &
 \end{array}$$

Let Ω_n be the p -completed homotopy colimit of all such B , i.e., the e_n -algebraic integers.

Let F be a finite p -local spectrum admitting a v_{n+1} self map $v: \Sigma^d F \rightarrow F$, cf. Hopkins–Smith [HS98, Def. 8]. The finite localization functor L_{n+1}^f , which annihilates all finite $E(n+1)$ -acyclic spectra [Mil92, Thm. 4], is a smashing localization such that $F_*L_{n+1}^f X \cong F_*X[1/v]$ for all spectra X .

I stated something like the following at Schloß Ringberg in January 1999 and in Oberwolfach in September 2000:

Conjecture 4.1. *Let $B \rightarrow \Omega_n$ and (F, v) be as above.*

(a) *Multiplication by v acts bijectively on $F_*K(B)$ for $* \gg 0$, and $K(B) \rightarrow L_{n+1}^f K(B)$ is a p -adic equivalence in sufficiently high degrees.*

(b) *There are isomorphisms $F_*K(\Omega_n) \cong F_*E_{n+1}$ for $* \gg 0$, and $\hat{L}_{n+1}K(\Omega_n) \simeq E_{n+1}$.*

The cases $n = -1$ and $n = 0$ correspond to Quillen's results and the proven Lichtenbaum–Quillen conjectures, respectively.

5. THE CYCLOTOMIC TRACE MAP

We can detect chromatic redshift in algebraic K -theory using the cyclotomic trace map to topological cyclic homology, or one of its variants.

The topological Hochschild homology $\mathrm{THH}(B)$ of a commutative S -algebra B is an S^1 -equivariant spectrum whose underlying spectrum with S^1 -action can be constructed as $B \otimes S^1$, where \otimes refers to

the tensored structure of commutative S -algebras over spaces. Let

$$\mathrm{THH}(B)^{hS^1} = F(ES_+^1, \mathrm{THH}(B))^{S^1}$$

be the S^1 -homotopy fixed points of $\mathrm{THH}(B)$, and let

$$\mathrm{THH}(B)^{tS^1} = [\widetilde{ES^1} \wedge F(ES_+^1, \mathrm{THH}(B))]^{S^1}$$

be the associated S^1 -Tate construction, previously denoted $t_{S^1} \mathrm{THH}(B)^{S^1}$ or $\hat{\mathbb{H}}(S^1, \mathrm{THH}(B))$. Here ES^1 is a free contractible S^1 -space, and $\widetilde{ES^1}$ is the mapping cone of the collapse map $ES_+^1 \rightarrow S^0$. Homotopy fixed point spectra model group cohomology, and the Tate construction models Tate cohomology.

Think of B as a ring spectrum of functions on a brave new scheme X . Then $B \wedge \cdots \wedge B$ is the ring of functions on $X \times \cdots \times X$, so $\mathrm{THH}(B)$ plays the role of the ring of functions on the free loop space $\mathrm{Map}(S^1, X) = \Lambda X$, and $\mathrm{THH}(B)^{hS^1}$ is like the ring of functions on the Borel construction $ES_+^1 \wedge_{S^1} \Lambda X$. The Tate construction is a periodicized version of the Borel construction.

There is a natural trace map

$$K(B) \longrightarrow \mathrm{THH}(B)$$

that factors through the fixed point spectra $\mathrm{THH}(B)^{C_r}$ for all finite subgroups $C_r \subset S^1$. In particular, there is a limiting map

$$K(B) \longrightarrow TF(B; p) = \mathrm{holim}_n \mathrm{THH}(B)^{C_{p^n}}.$$

Continuing with the canonical map from fixed points to homotopy fixed points, the target of

$$\mathrm{holim}_n \mathrm{THH}(B)^{C_{p^n}} \longrightarrow \mathrm{holim}_n \mathrm{THH}(B)^{hC_{p^n}}$$

is p -adically equivalent to $\mathrm{THH}(B)^{hS^1}$. The cyclotomic structure of $\mathrm{THH}(B)$ gives a similar map

$$\mathrm{holim}_n \mathrm{THH}(B)^{C_{p^n}} \longrightarrow \mathrm{holim}_n \mathrm{THH}(B)^{tC_{p^{n+1}}}$$

whose target is p -adically equivalent to $\mathrm{THH}(B)^{tS^1}$.

The topological Hochschild construction itself does not introduce a redshift, since $\mathrm{THH}(B)$ is a commutative B -algebra. However, in all the computations made so far, any v_{n+1} -periodicity that is seen in the algebraic K -theory $K(B)$ has already been visible in the S^1 -Tate construction $\mathrm{THH}(B)^{tS^1}$.

Furthermore, it is possible to see in homological terms where the redshift arises, in terms of these S^1 -equivariant constructions.

6. CIRCLE-EQUIVARIANT REDSHIFT

The algebra $H_*(e_n)$ appears to be unwieldy for $n \geq 2$, but there is a map $BP\langle n \rangle \rightarrow e_n$ of (not necessarily commutative) S -algebras, covering the usual map $E(n) \rightarrow E_n$, and the augmentation $BP\langle n \rangle \rightarrow H$ induces an identification

$$H_*(BP\langle n \rangle) \cong P(\bar{\xi}_k \mid k \geq 1) \otimes E(\bar{\tau}_k \mid k \geq n+1)$$

of subalgebras of the dual Steenrod algebra

$$\mathcal{A}_* = P(\bar{\xi}_k \mid k \geq 1) \otimes E(\bar{\tau}_k \mid k \geq 0).$$

Forgetting some structure, we can therefore think of the homology $H_*(B)$ of a commutative e_n -algebra B as a commutative $H_*(BP\langle n \rangle)$ -algebra. This makes the Adams spectral sequence

$$E_2^{s,t}(B) = \mathrm{Ext}_{\mathcal{A}_*}^{s,t}(\mathbb{F}_p, H_*(B)) \Longrightarrow \pi_{t-s}(B_p^\wedge)$$

an algebra over the Adams spectral sequence

$$E_2^{s,t} = \mathrm{Ext}_{\mathcal{A}_*}^{s,t}(\mathbb{F}_p, H_*(BP\langle n \rangle)) \Longrightarrow \pi_{t-s}(BP\langle n \rangle_p^\wedge)$$

which collapses at the E_2 -term

$$E_2^{*,*} = P(v_0, \dots, v_n)$$

and converges to the homotopy

$$\pi_* BP\langle n \rangle_p^\wedge \cong \mathbb{Z}_p[v_1, \dots, v_n].$$

The Bökstedt spectral sequence

$$E_{s,t}^2(B) = HH_s(H_*(B))_t \Longrightarrow H_{s+t}(\mathrm{THH}(B))$$

is then an algebra spectral sequence over

$$E_{*,*}^2 = HH_*(H_*(BP\langle n \rangle)) \cong H_*(BP\langle n \rangle) \otimes E(\sigma \bar{\xi}_k \mid k \geq 1) \otimes \Gamma(\sigma \bar{\tau}_k \mid k \geq n+1)$$

converging to $H_*(\mathrm{THH}(BP\langle n \rangle))$. Here σ denotes the suspension operator, coming from the S^1 -action on THH , and $\Gamma(x) = \mathbb{F}_p\{\gamma_j x \mid j \geq 0\}$ denotes the divided power algebra on x .

The Dyer–Lashof operations $Q^{p^k}(\bar{\tau}_k) = \bar{\tau}_{k+1}$ in \mathcal{A}_* (coming from the commutative S -algebra structure on H), imply multiplicative extensions $(\sigma\bar{\tau}_k)^p = \sigma\bar{\tau}_{k+1}$, for $k \geq n+1$, which in turn imply that the Bockstein images $\beta(\sigma\bar{\tau}_{k+1}) = \sigma\bar{\xi}_{k+1}$ vanish in the abutment. This argument, see Ausoni [Aus05, Lem. 5.3], implies differentials

$$d^{p-1}(\gamma_j \sigma\bar{\tau}_k) \doteq \sigma\bar{\xi}_{k+1} \cdot \gamma_{j-p} \sigma\bar{\tau}_k$$

for all $j \geq p$, which leave

$$E_{*,*}^p = E_{*,*}^\infty \cong H_*(BP\langle n \rangle) \otimes E(\sigma\bar{\xi}_1, \dots, \sigma\bar{\xi}_{n+1}) \otimes P_p(\sigma\bar{\tau}_k \mid k \geq n+1)$$

converging to

$$H_*(\mathrm{THH}(BP\langle n \rangle)) \cong H_*(BP\langle n \rangle) \otimes E(\sigma\bar{\xi}_1, \dots, \sigma\bar{\xi}_{n+1}) \otimes P(\sigma\bar{\tau}_{n+1}).$$

This will still have trivial v_{n+1} -periodic homotopy, but note how building in a circle action gives rise to the class $\sigma\bar{\tau}_{n+1}$.

The homological Tate spectral sequence

$$E_{s,t}^2(B) = \hat{H}^{-s}(S^1; H_t(\mathrm{THH}(B))) \implies H_{s+t}^c(\mathrm{THH}(B)^{tS^1})$$

converges to a limit that we call the continuous homology of $\mathrm{THH}(B)^{tS^1}$. It is an algebra spectral sequence over

$$E_{*,*}^2 = \hat{H}^{-*}(S^1; H_*(\mathrm{THH}(BP\langle n \rangle))) \cong P(t^{\pm 1}) \otimes H_*(\mathrm{THH}(BP\langle n \rangle))$$

converging to $H_*^c(\mathrm{THH}(BP\langle n \rangle)^{tS^1})$. Here

$$d^2(t^i \cdot x) = t^{i+1} \cdot \sigma x$$

for all x , which leaves

$$E_{*,*}^3 = P(t^{\pm 1}) \otimes P(\bar{\xi}_1^p, \dots, \bar{\xi}_{n+1}^p, \bar{\xi}_k \mid k \geq n+2) \otimes E(\tau'_k \mid k \geq n+2) \otimes E(\bar{\xi}_1^{p-1} \sigma\bar{\xi}_1, \dots, \bar{\xi}_{n+1}^{p-1} \sigma\bar{\xi}_{n+1})$$

where $\tau'_k = \bar{\tau}_k - \bar{\tau}_{k-1}(\sigma\bar{\tau}_{k-1})^{p-1}$ for each $k \geq n+2$. Note that $\bar{\tau}_{n+1}$ supports a nontrivial d^2 -differential to $t \cdot \sigma\bar{\tau}_{n+1}$, and does not survive to the E^∞ -term, while the τ'_k for $k \geq n+2$ are d^2 -cycles, due to the known multiplicative extension.

This spectral sequence often collapses at this stage [BR05, Prop. 6.1], and there can be \mathcal{A}_* -comodule extensions that combine p^{n+1} shifted copies of $P(\bar{\xi}_1^p, \dots, \bar{\xi}_{n+1}^p, \bar{\xi}_k \mid k \geq n+2) \otimes E(\tau'_k \mid k \geq n+2)$ to a copy of $P(\bar{\xi}_k \mid k \geq 1) \otimes E(\tau'_k \mid k \geq n+2) \cong H_*(BP\langle n+1 \rangle)$. The PhD theses of Sverre Lunøe–Nielsen [LNR12], [LNR11] and Knut Berg (to appear) address these questions. Note the transition from $H_*(BP\langle n \rangle)$ to $H_*(BP\langle n+1 \rangle)$, with non-trivial v_{n+1} -periodic homotopy groups.

The typical result is that $H_*^c(\mathrm{THH}(B)^{tS^1})$ is an algebra over $H_*^c(\mathrm{THH}(BP\langle n \rangle)^{tS^1})$, which has an associated graded of the form

$$P(t^{\pm p^{n+1}}) \otimes H_*(BP\langle n+1 \rangle) \otimes E(\nu_1, \dots, \nu_{n+1})$$

where ν_k is a t -power multiple of $\bar{\xi}_k^{p-1} \sigma\bar{\xi}_k$, but that there is room for further \mathcal{A}_* -comodule extensions.

This implies that the inverse limit Adams spectral sequence

$$E_2^{s,t}(B) = \mathrm{Ext}_{\mathcal{A}_*}^{s,t}(\mathbb{F}_p, H_*^c(\mathrm{THH}(B)^{tS^1})) \implies \pi_{t-s} \mathrm{THH}(B)_p^{tS^1}$$

is an algebra over the Adams spectral sequence

$$E_2^{s,t} = \mathrm{Ext}_{\mathcal{A}_*}^{s,t}(\mathbb{F}_p, H_*^c(\mathrm{THH}(BP\langle n \rangle)^{tS^1})) \implies \pi_{t-s} \mathrm{THH}(BP\langle n \rangle)_p^{tS^1}$$

which contains factors like

$$\mathrm{Ext}_{\mathcal{A}_*}^{*,*}(\mathbb{F}_p, H_*(BP\langle n+1 \rangle)) \cong P(v_0, \dots, v_n, v_{n+1}).$$

Due to the exterior factors $E(\nu_1, \dots, \nu_{n+1})$ there is room for differentials that might truncate the periodic v_{n+1} -action visible above, but empirically this does not happen. A theory that explains the general picture is, however, currently lacking.

7. BEYOND ELLIPTIC COHOMOLOGY

Do $K(tmf)$ and $\mathrm{THH}(tmf)^{tS^1}$ detect v_3 -periodic families? Work in progress for $p = 2$ with Bruner (2008).

8. WALDHAUSEN'S LOCALIZATION TOWER

The chromatic localization functors (L_n and) \hat{L}_n and the finite localizations functors L_n^f fit in a diagram of commutative structured ring spectra

$$\begin{array}{ccccccc}
 & & E_n & & KU_p & & \\
 & & \uparrow \mathbb{G}_n & & \uparrow \mathbb{Z}_p^\times & & \\
 & & \hat{L}_n S & & J_p & & HQ \\
 & & \uparrow & & \uparrow & & \uparrow \simeq \\
 S_{(p)} & \longrightarrow \cdots \longrightarrow & L_n^f S & \longrightarrow & L_{n-1}^f S & \longrightarrow \cdots \longrightarrow & L_1^f S \longrightarrow L_0^f S
 \end{array}$$

where $L_n^f S \rightarrow L_n S$ is an equivalence for $n \leq 1$, but probably not for $n \geq 2$, according to the wisdom concerning Ravenel's telescope conjecture [MRS01]. Applying algebraic K -theory to the lower row one gets a telescopic localization tower

$$K(S_{(p)}) \longrightarrow \cdots \longrightarrow K(L_n^f S) \longrightarrow K(L_{n-1}^f S) \longrightarrow \cdots \longrightarrow K(L_1 S) \longrightarrow K(\mathbb{Q})$$

similar to that of [Wal84, p. 174], interpolating between the geometrically significant algebraic K -theory of spaces on the left hand side, and the arithmetically significant algebraic K -theory of number fields on the right hand side. Waldhausen worked with L_n , and explicitly assumed that it is a finite localization functor, but we can work with L_n^f instead. This ensures that each finite cell $L_n^f S$ -module is L_n^f -equivalent to a finite cell S -module, as can be proved by induction on the number of $L_n^f S$ -cells.

Let \mathcal{C}_0 be the category of finite p -local spectra, and let $w_n \mathcal{C}_0$ be the subcategory of $E(n)_*$ -equivalences, or equivalently of L_n^f -equivalences, for $n \geq 0$. Let $\mathcal{C}_n = \mathcal{C}_0^{w_{n-1}}$ denote the full subcategory of $E(n-1)_*$ -acyclic spectra, i.e., the finite spectra of type $\geq n$, for $n \geq 1$. Then $K(\mathcal{C}_0, w_n) \simeq K(L_n^f S)$, and Waldhausen's localization theorem [Wal84, §3] recognizes the homotopy fiber of $K(L_n^f S) \rightarrow K(L_{n-1}^f S)$ as $K(\mathcal{C}_n, w_n)$, i.e., the algebraic K -theory of finite spectra of type $\geq n$, with respect to the $E(n)_*$ -equivalences. We get a homotopy fiber sequence

$$K(\mathcal{C}_n, w_n) \longrightarrow K(L_n^f S) \longrightarrow K(L_{n-1}^f S).$$

Let \mathcal{K}_n^{sm} be the category of small $K(n)$ -local spectra, and let \mathcal{K}'_n be the full subcategory of $K(n)$ -localizations of finite spectra of type $\geq n$. Hovey–Strickland [HS99, Thm. 8.5] show that the inclusion $\mathcal{K}'_n \subset \mathcal{K}_n^{sm}$ is an idempotent completion, so the induced map $K(\mathcal{K}'_n) \rightarrow K(\mathcal{K}_n^{sm})$ induces an isomorphism on π_i for each $i \geq 1$. The localization functors L_n and \hat{L}_n agree on \mathcal{C}_n , hence induce an equivalence $K(\mathcal{C}_n, w_n) \simeq K(\mathcal{K}'_n)$. Thus we have a map

$$K(\mathcal{C}_n, w_n) \longrightarrow K(\mathcal{K}_n^{sm}),$$

which induces a π_i -isomorphism for each $i \geq 1$. We view \mathcal{K}_n^{sm} as a category of suitably small $\hat{L}_n S$ -modules. ((What happens at the level of π_0 ?)

Let \mathcal{E}_n^{df} be the category of E_n -module spectra that have degreewise finite homotopy groups. Base change along the $K(n)$ -local pro- \mathbb{G}_n -Galois extension $\hat{L}_n S \rightarrow E_n$ takes \mathcal{K}_n^{sm} to \mathcal{E}_n^{df} , and conversely [HS99, Cor. 12.16], so it is plausible that a Galois descent comparison map

$$K(\mathcal{K}_n^{sm}) \longrightarrow K(\mathcal{E}_n^{df})^{h\mathbb{G}_n}$$

is close to an equivalence. Finally, $K(\mathcal{E}_n^{df})$ is related to the algebraic K -theory of E_n and its various localizations. For $n = 1$ we have $E_1 = KU_p$, and $K(\mathcal{E}_1^{df})$ is the algebraic K -theory of p -nilpotent finite cell KU_p -modules, which sits [Bar13, Prop. 11.15] in a homotopy fiber sequence

$$K(\mathcal{E}_1^{df}) \longrightarrow K(KU_p) \longrightarrow K(KU_p[1/p]).$$

In general, this fiber sequence is replaced by an n -dimensional cubical diagram. Note that the transfer map $K(KU/p) \rightarrow K(\mathcal{E}_1^{df})$ associated to $KU_p \rightarrow KU/p$ is far from an equivalence, by the calculations of [AR12, Cor. 1.3], so there does not appear to be any easy way to describe the algebraic K -theory of degreewise finite E_n -modules in terms of dévissage, cf. [Wal84, p. 188].

$$\begin{array}{ccccc}
& & K(\mathcal{E}_n^{df}) & \longrightarrow & K(E_n) \\
& & \uparrow \mathbb{G}_n & & \\
K(\mathcal{C}_n, w_n) & \longrightarrow & K(\mathcal{K}_n^{sm}) & & \\
& \searrow & & & \\
\cdots & \longrightarrow & K(L_n^f S) & \longrightarrow & K(L_{n-1}^f S) \longrightarrow \cdots
\end{array}$$

Conjecture 4.1 about the structure of the algebraic K -theory of E_n (and various localizations) is therefore also a statement about $K(\mathcal{E}_n^{df})$, and conjecturally about $K(\mathcal{K}_n^{sm})$, which rather precisely measures the difference between $K(L_n^f S)$ and $K(L_{n-1}^f S)$.

9. THE SPHERICAL CASE

Calculations of $TC(S; p)$, $K(\mathbb{Z})$ and $TC(\mathbb{Z}; p)$ were assembled to a calculation of $K(S)$ at $p = 2$ in [Rog02] and at odd regular primes in [Rog03]. These results describe the cohomology of $K(S)$ as an \mathcal{A} -module in all degrees (up to an extension in the odd case), and lead to Adams spectral sequence calculations in a finite range of degrees.

The algebraic K -groups of S are at least as complicated as those of its stable homotopy groups. The complex cobordism spectrum MU has turned out to be a convenient halfway house

$$S \longrightarrow MU \longrightarrow H$$

between homology and homotopy. The Thom equivalence $MU \wedge MU \simeq MU \wedge BU_+$ makes $S \rightarrow MU$ a Hopf–Galois extension [Rog08, §12], and the cosimplicial Amitsur resolution

$$[q] \longmapsto MU \wedge MU^{\wedge q}$$

of S is equivalent to the cobar resolution $[q] \longmapsto MU \wedge BU_+^q$ for the $S[BU] = \Sigma^\infty(BU_+)$ -comodule algebra MU . Applying algebraic K -theory, an analogue of Quillen’s conjecture would predict that $K(S)$ is well approximated by the totalization of the cosimplicial spectrum

$$[q] \longmapsto K(MU \wedge MU^{\wedge q})$$

rewriteable as $[q] \longmapsto K(MU \wedge BU_+^q)$. If, by analogy with the Galois case, there are compatible maps $K(MU \wedge BU_+^q) \rightarrow K(MU) \wedge BU_+^q$, then this might in turn be approximated by the totalization of the cobar resolution $[q] \longmapsto K(MU) \wedge BU_+^q$ for an $S[BU]$ -comodule algebra structure on $K(MU)$.

Conceivably, this leads to a more conceptual understanding of $\pi_* K(S)$ in terms of $\pi_* K(MU)$ and Hopf–Galois descent, by analogy with the Adams–Novikov spectral sequence description of $\pi_* S$ in terms of $\pi_* MU$ and its $H_*(BU)$ -coaction. This has been a motivating factor for the study of $K(MU)$, advertised in [BR05] and [Rog09], and pursued in [LNR11].

10. HIGHER REDSHIFT

For a Lie group G of rank k , consider $(B \otimes G)^{hG}$ or something like $(B \otimes G)^{tG}$. If B is v_n -periodic but not v_{n+1} -periodic, then apparently $(B \otimes G)^{tG}$ is v_{n+k} -periodic. Tested for $B = H$ and $G = T^k$ for small k , as well as for $G = SO(3)$ and $G = S^3$. Work in progress (Rognes, 2008–2011) and in Torleif Veen’s PhD thesis (2013).

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