ALGEBRAIC K-THEORY AND THE CHROMATIC FILTRATION

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March 20th 2006

1. Brave new schemes and stacks.

By a brave new ring, we mean a (usually commutative) structured ring spectrum. This can variously interpreted as an S-algebra, a symmetric ring spectrum, an orthogonal ring spectrum, or perhaps, an A_∞/E_∞ ring spectrum. We will generically talk about $\mathbb{S}$-algebras, where $\mathbb{S}$ denotes the sphere spectrum.

A commutative $\mathbb{S}$-algebra A can be viewed as an affine brave new (or derived, homotopical or topological) scheme $(\operatorname{Spec} A, \mathcal{O})$, with underlying space the prime ideal spectrum $\operatorname{Spec} \pi_0(A)$, but locally brave new ringed by the commutative $\mathbb{S}$-algebra $A[1/f]$ with $\pi_*(A[1/f]) = \pi_*(A)[1/f]$ over the open subset $U_f = \operatorname{Spec} \pi_0(A)[1/f]$, for non-zero divisors $f \in \pi_0(A)$.

We say that A is connective if $\pi_i(A) = 0$ for $i < 0$. For connective A the Hurewicz map $A \to H\pi_0(A)$ maps $\operatorname{Spec} \pi_0(A)$ into $\operatorname{Spec} A$, and we can view the brave new scheme as a thickening, or deformation, of the underlying algebraic scheme. More generally, the Postnikov tower

$$A \to \dots \to P^n A \to P^{n-1} A \to \dots \to P^0 A = H\pi_0(A)$$

define an increasing sequence of infinitesimal deformations $\operatorname{Spec} P^n A$ of $\operatorname{Spec} \pi_0(A)$, and to the extent that we can identify $A = \operatorname{holim}_n P^n A$ with this tower, $\operatorname{Spec} A$ is the formal deformation of $\operatorname{Spec} \pi_0(A)$ given by this increasing union. These brave new schemes are locally brave new ringed by the Postnikov sections $P^n A[1/f]$ over U_f. For example, this applies to the connective commutative $\mathbb{S}$-algebras $\mathbb{S}$, ku (connective K-theory) and $H\mathbb{Z}$ (the Eilenberg–Mac Lane spectrum of the integers).

We say that A is even periodic if there exists a unit $u \in \pi_2(A)$ and $\pi_1(A) = 0$, so $\pi_*(A) \cong \pi_0(A)[u^{\pm 1}]$. Then $\operatorname{Spec} \pi_0(A)$ is locally ringed by the even periodic spectra $A[1/f]$ with $\pi_*(A[1/f]) = \pi_0(A)[1/f][u^{\pm 1}]$. The connective cover $P_0 A \to A$ defines a canonical morphism $\operatorname{Spec} A \to \operatorname{Spec} P_0 A$ of affine brave new schemes. ((Such a structure on the formal deformation $\operatorname{Spec} P_0 A$ of $\operatorname{Spec} \pi_0(A)$ might deserve a name.)) For example, this applies to the even periodic commutative $\mathbb{S}$-algebras KU (complex K-theory) and E_n (the Lubin–Tate spectra), where $\pi_0 E_n = \mathbb{W}\mathbb{F}_{p^n}[[u_1, \dots, u_n]]$ is the complete local ring that supports the universal deformation of the height n Honda formal group Γ_n over $\mathbb{F}_{p^n}$. In the case $n = 1$, $E_1 = KU_p$ is p-adic K-theory.

Globalizing, a brave new scheme X with a Zariski topology is glued together from affine brave new schemes $\operatorname{Spec} A$ along open subobjects like $\operatorname{Spec} A[1/f]$. A presheaf over X is glued together from presheaves over the affine pieces. There is a

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(closed) model structure on these, with cofibrations given locally over open affine pieces, and weak equivalences given stalkwise. The sheaves over X are the fibrant presheaves.

The prime ideal spectrum of $\pi_0(A)$ is not always the most interesting brave new geometry underlying A. We say that a map $A \to B$ of commutative $\mathbb{S}$-algebras is a G-Galois extension if G is a finite group acting on B through commutative A-algebra maps, such that the canonical maps $i\colon A \to B^{hG}$ and $h\colon B \wedge_A B \to \prod_G B$ are homotopy equivalences. In symbols, $h(b_1 \wedge b_2)\colon g \mapsto b_1 g(b_2)$. Then we wish to view $\operatorname{Spec} B \to \operatorname{Spec} A$ as a regular covering space, with group G of deck transformations. However, it is not necessarily the case that $\pi_0(A) \to \pi_0(B)$ is a G-Galois extension of rings, i.e., that the orbit scheme for the G-action on $\operatorname{Spec} B$ is realized by $\operatorname{Spec} A$ as defined above. In this case, we will instead interpret $\operatorname{Spec} A = [\operatorname{Spec} B/G]$ as the orbit stack for this G-action, i.e., as the translation category, so that $\operatorname{Spec} A$ is a brave new stack, rather than a brave new scheme. For example, this applies to the $\mathbb{Z}/2$-Galois extension $KO \to KU$ (of the real K-theory spectrum) given by complexification map and the action by complex conjugation, where $\pi_0(KO) = \pi_0(KU) = \mathbb{Z}$, so the geometry underlying $\operatorname{Spec} KO$ is the orbit category for the $\mathbb{Z}/2$-action on $\operatorname{Spec} KU$.

Given a homology theory E_*, represented by a spectrum E, there is also the notion of an E-local G-Galois extension $A \to B$, where the two maps i and h above are only assumed to be E_*-isomorphisms, or equivalently, to become homotopy equivalences after applying the E-localization functor L_E. We are mostly interested in this when $E = K(n)$ is the n-th Morava K-theory spectrum, with $\pi_* K(n) = \mathbb{F}_p[v_n^{\pm 1}]$. There is also a notion of a pro-G-Galois extension $A \to B$, given as a filtered direct limit of G_α-Galois extensions $A \to B_\alpha$, with $G = \lim_\alpha G_\alpha$. For example, the n-th extended Morava stabilizer group $\mathbb{G}_n = \operatorname{Aut}(\Gamma_n, \mathbb{F}_{p^n}) \rtimes \operatorname{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p)$ acts on the even periodic spectrum E_n, and there is a $K(n)$-local pro-$\mathbb{G}_n$-Galois extension $L_{K(n)}\mathbb{S} \to E_n$. We might then think of $\operatorname{Spec} L_{K(n)}\mathbb{S}$ as the orbit stack $[\operatorname{Spec} E_n/\mathbb{G}_n]$, keeping in mind that $\mathbb{G}_n$ is a profinite group. In the case $n = 1$, $L_{K(1)}\mathbb{S} = J_p$ is the p-adic image of J spectrum, $\mathbb{G}_1 = \mathbb{Z}_p^\times$ and $k \in \mathbb{Z}_p^\times$ acts on $E_1 = KU_p$ by the p-adic Adams operation ψ^k.

Piecing together affine brave new schemes along Galois covers of open subschemes yields a brave new scheme with an étale topology. We are particularly interested in the chromatic tower from stable homotopy theory. As usual, we fix a rational prime p, implicit in the notation, and write $L_n X = L_{E(n)} X$ and $\hat{L}_n X = L_{K(n)} X$, where $E(n)$ is the n-th Johnson–Wilson spectrum with $\pi_* E(n) = \mathbb{Z}_{(p)}[v_1, \dots, v_n^{\pm 1}]$. Then there is a tower

$$\mathbb{S}_{(p)} \to \cdots \to L_n\mathbb{S} \to L_{n-1}\mathbb{S} \to \dots L_0\mathbb{S} = H\mathbb{Q}$$

that can be viewed as an infinite unfolding, in brave new geometric terms, of the short algebraic filtration

$$\mathbb{Z}_{(p)} \to \mathbb{Q}\,.$$

By chromatic convergence, $\mathbb{S}_{(p)} \simeq \operatorname{holim}_n L_n\mathbb{S}$, so to the extent that we can identify this tower with its homotopy limit, then $\operatorname{Spec} \mathbb{S}_{(p)}$ is the increasing union of the open subobjects $\operatorname{Spec} L_n\mathbb{S}$. However, for $n \geq 1$ the spectra $L_n\mathbb{S}$ are neither connective nor even periodic, and the geometric structure of this tower appears to be very much richer than that of $\operatorname{Spec} \mathbb{S}_{(p)}$ for $\mathbb{S}_{(p)}$ viewed as a connective brave new ring, i.e., $\operatorname{Spec} \mathbb{Z}_{(p)} = \{(p), (0)\}$ with the local brave new rings $\mathbb{S}_{(p)}$ and $\mathbb{S}_{(p)}[1/p] \simeq H\mathbb{Q}$.

For there is a homotopy cartesian square of commutative $\mathbb{S}$-algebras

$$\begin{tikzcd}
L_n\mathbb{S} \arrow[r] \arrow[d] & L_{n-1}\mathbb{S} \arrow[d] \\
\hat{L}_n\mathbb{S} \arrow[r] & L_{n-1}\hat{L}_n\mathbb{S}
\end{tikzcd}$$

for each chromatic height $1 \le n < \infty$, that is analogous to the arithmetic square

$$\begin{tikzcd}
\mathbb{Z}_{(p)} \arrow[r] \arrow[d] & \mathbb{Q} \arrow[d] \\
\mathbb{Z}_p \arrow[r] & \mathbb{Q}_p
\end{tikzcd}$$

of ordinary rings. Furthermore, $L_{n-1}\mathbb{S} \wedge_{L_n\mathbb{S}} \hat{L}_n\mathbb{S} \simeq L_{n-1}\hat{L}_n\mathbb{S}$, like $\mathbb{Q} \otimes_{\mathbb{Z}_{(p)}} \mathbb{Z}_p \cong \mathbb{Q}_p$.

We view $L_n\mathbb{S} \to \hat{L}_n\mathbb{S}$ as a completion map, so that $\operatorname{Spec} \hat{L}_n\mathbb{S}$ is a formal neighborhood in $\operatorname{Spec} L_n\mathbb{S}$, and the brave new arithmetic square expresses $\operatorname{Spec} L_n\mathbb{S}$ as being glued together from the open/formal neighborhoods $\operatorname{Spec} L_{n-1}\mathbb{S}$ and $\operatorname{Spec} \hat{L}_n\mathbb{S}$ along their meet $\operatorname{Spec} L_{n-1}\hat{L}_n\mathbb{S}$. And as remarked above, $\operatorname{Spec} \hat{L}_n\mathbb{S}$ is geometrically the orbit stack $[\operatorname{Spec} E_n/\mathbb{G}_n]$. Presumably, $\operatorname{Spec} L_{n-1}\hat{L}_n\mathbb{S}$ is a reasonable open subobject of this orit stack. So from this point of view, $\operatorname{Spec} \mathbb{S}_{(p)}$ is to be interpreted as the ind-object given by an increasing union of brave new stacks $\operatorname{Spec} L_n\mathbb{S}$, where the n-th object is obtained from the $(n-1)$-st by attaching an orbit stack along an open subobject.

2. Algebraic K-theory.

To understand these geometric objects, we might study the brave new algebraic analogues of the classical homotopy invariants from algebraic topology, such as the fundamental group, cohomology and topological K-theory. The algebraic analogue of the fundamental group is given by the Galois groups, and a Galois theory for commutative $\mathbb{S}$-algebras, based on the definitions above, has been developed. It suffices for the study of unramified extensions, but does not directly handle ramified extensions. For this, an appropriate machinery for localization away from the ramification "locus" will be needed.

The theory of étale cohomology or motivic cohomology for commutative $\mathbb{S}$-algebras has not yet been properly developed. In a sufficiently local situation, étale cohomology might specialize to Galois cohomology, i.e., the continuous cohomology of the absolute Galois group, which is then handled by the Galois theory just mentioned, but in general the correct passage to a local situation needs to be found.

However, the algebraic K-theory of brave new rings (commutative or not) was already defined quite long ago. To each $\mathbb{S}$-algebra A there is a category $\mathcal{M}_A$ of A-modules, and a full subcategory $\mathcal{C}_A$ of semi-finite cell A-modules. The latter are those A-modules that can be obtained from A in finitely many steps by the formation of mapping cones, retracts and desuspensions, and passage to homotopy equivalent A-modules. In other words, $\mathcal{C}_A$ is the thick subcategory of $\mathcal{M}_A$ generated by A. For example, $\mathcal{M}_{\mathbb{S}}$ is the category of spectra, and $\mathcal{C}_{\mathbb{S}}$ is the subcategory of finite spectra. ((For connective A we can work with cell spectra or CW spectra interchangably, but for non-connective A this is not so clear.))

Let $h\mathcal{C}_A \subset \mathcal{C}_A$ be the subcategory of homotopy equivalences. Then the algebraic K-theory of A, a space $K(A)$, is defined as a suitable group completion of the nerve of this subcategory, with a map $|h\mathcal{C}_A| \to K(A)$. For a discrete ring R, with Eilenberg–Mac Lane ring spectrum HR, this recovers Quillen's algebraic K-theory as $K(R) \simeq K(HR)$. The ability to talk about such module categories for more general $\mathbb{S}$-algebras A, not just the associated homotopy categories, was one of the original motivations for developing the theory of structured ring spectra.

As in algebraic geometry, one might expect an analogue of the Atiyah–Hirzebruch spectral sequence from singular cohomology to topological K-theory, in the form of a spectral sequence from a motivic cohomology of a commutative $\mathbb{S}$-algebra to its algebraic K-theory. With suitably finite coefficients, one might also expect an étale descent spectral sequence from an étale cohomology to étale K-theory. In a sufficiently local situation, this might reduce to a Galois descent spectral sequence for étale K-theory, which might agree with algebraic K-theory in sufficiently high degrees, or after the introduction of suitable periodicity by localization.

The definition of $K(A)$ in terms of semi-finite cell A-modules seems appropriate for connective or even periodic $\mathbb{S}$-algebras A. For more general A, like $A = \hat{L}_n\mathbb{S}$, it may also be appropriate to study the full subcategory $\mathcal{D}_A \subset \mathcal{M}_A$ of dualizable A-modules, i.e., those A-modules X with the property that $\nu\colon F_A(X, A) \wedge_A X \to F_A(X, X)$ is a homotopy equivalence, perhaps after some E-localization. For connective A, $A = L_n\mathbb{S}$ or $A = E_n$, these are the same as the semi-finite A-modules. But for $A = \hat{L}_n\mathbb{S}$, the inclusion $\mathcal{C}_A \subset \mathcal{D}_A$ is proper. In fact, for the map $K(\hat{L}_n\mathbb{S}) \to K(E_n)^{h\mathbb{G}_n}$ to be close to a homotopy equivalence, the source should probably be interpreted as a group completion of $|h\mathcal{D}_{\hat{L}_n\mathbb{S}}|$.

One class of important examples is given by the $\mathbb{S}$-algebra $A = \mathbb{S}[\Omega M] = \Sigma^\infty \Omega M_+$ given as a spherical group ring for the loop group ΩM on a manifold M. By Waldhausen's stable parametrized h-cobordism theorem, the algebraic K-theory $K(\mathbb{S}[\Omega M])$ of these $\mathbb{S}$-algebras is tightly connected to the automorphism groups of M and the moduli space of manifolds in the isomorphism class of M. ((Elaborate?)) In the smooth category this is interesting already for $M = D^m$, an m-dimensional disc, when $\mathbb{S}[\Omega D^m] \simeq \mathbb{S}$, so the algebraic K-theory in question is $K(\mathbb{S})$, that of the sphere spectrum.

As a first approximation, the linearization map $\mathbb{S} \to H\mathbb{Z}$ induces a rational equivalence

$$K(\mathbb{S}) \to K(\mathbb{Z})$$

so Borel's rational calculation of $K(\mathbb{Z})$ yields a rational calculation of $K(\mathbb{S})$ and thus of the diffeomorphism groups of discs, $\mathrm{Diff}(D^m)$, in a stable range. It is known that $K(\mathbb{S})$ has finite type, like $K(\mathbb{Z})$, so to address the finer question of intergral homotopy types, it suffices to treat one rational prime p at a time, either by p-completion or p-localization.

On one hand, there is a homotopy cartesian square

$$\begin{tikzcd}
K(\mathbb{S})_p \arrow[r] \arrow[d] & K(\mathbb{Z})_p \arrow[d] \\
TC(\mathbb{S})_p \arrow[r] & TC(\mathbb{Z})_p
\end{tikzcd}$$

due to Dundas, where $TC(A)_p$ is the p-complete topological cyclic homology of an $\mathbb{S}$-algebra A. Using this, a calculation of $K(\mathbb{S})_p$ can be given for $p = 2$ or p an odd regular prime. However, the answer is about as complicated as the description of the stable homotopy groups of $\mathbb{C}P^\infty$. It might be desirable to find a more conceptual description that separates these stable homotopy groups into periodic families, by way of the chromatic filtration.

For this, one may work p-locally. There is a homotopy equivalence

$$\mathrm{hofib}(K(\mathbb{S}) \to K(\mathbb{Z}))_{(p)} \simeq \mathrm{hofib}(K(\mathbb{S}_{(p)}) \to K(\mathbb{Z}_{(p)}))$$

due to Waldhausen. This is almost the same as the homotopy fiber of the composite map $K(\mathbb{S}_{(p)}) \to K(\mathbb{Q})$ from the chromatic tower, due to the localization sequence

$$K(\mathbb{F}_p) \to K(\mathbb{Z}_{(p)}) \to K(\mathbb{Q})$$

where $K(\mathbb{F}_p)$ is well understood, and $K(\mathbb{F}_p)_{(p)} \simeq H\mathbb{Z}_{(p)}$. In more detail, the homotopy fiber of $K(\mathbb{Z}_{(p)}) \to K(\mathbb{Q})$ can be decribed as the algebraic K-theory $K^{tors}(\mathbb{Z}_{(p)})$ of the category of finite torsion $\mathbb{Z}_{(p)}$-modules, and by the dévissage theorem, this is equivalent to the algebraic K-theory of $\mathbb{F}_p$, when each $\mathbb{F}_p$-module is viewed as a torsion $\mathbb{Z}_{(p)}$-module.

We are therefore interested in obtaining an understanding of the algebraic K-theory chromatic tower

$$K(\mathbb{S}_{(p)}) \to \cdots \to K(L_n\mathbb{S}) \to K(L_{n-1}\mathbb{S}) \to \cdots \to K(L_0\mathbb{S}) \simeq K(\mathbb{Q})\,.$$

To be fair, we should point out that we do not yet know if this tower converges, i.e., if $K(\mathbb{S}_{(p)}) \simeq \mathrm{holim}_n\, K(L_n\mathbb{S})$, but this seems plausible in view of the chromatic convergence theorem.

3. Monochromatic layers.

We therefore study the diagram in algebraic K-theory

$$\begin{tikzcd}
 & K(L_n\mathbb{S}) \arrow[r] \arrow[d] & K(L_{n-1}\mathbb{S}) \arrow[d] \\
K^{sm}(\hat{L}_n\mathbb{S}) \arrow[r] \arrow[d, "\mathbb{G}_n"] \arrow[ur, bend left] & K(\hat{L}_n\mathbb{S}) \arrow[r] \arrow[d, "\mathbb{G}_n"] & K(L_{n-1}\hat{L}_n\mathbb{S}) \\
K^{fin}(E_n) \arrow[r] & K(E_n) &
\end{tikzcd}$$

where the upper right hand square is derived from the following square of categories

$$\begin{tikzcd}
h\mathcal{C}_{L_n\mathbb{S}} \arrow[r] \arrow[d] & h\mathcal{C}_{L_{n-1}\mathbb{S}} \arrow[d] \\
h\mathcal{C}_{\hat{L}_n\mathbb{S}} \arrow[r] & h\mathcal{C}_{L_{n-1}\hat{L}_n\mathbb{S}}\,.
\end{tikzcd}$$

and receives a map from the upper right-hand square in the following diagram

$$\begin{tikzcd}
h\mathcal{C}^{e_{n-1}}_{L_n\mathbb{S}} \arrow[r] \arrow[d, "\simeq"] & h\mathcal{C}_{L_n\mathbb{S}} \arrow[r] \arrow[d] & e_{n-1}\mathcal{C}_{L_n\mathbb{S}} \arrow[d] \\
h\mathcal{C}^{e_{n-1}}_{\hat{L}_n\mathbb{S}} \arrow[r] \arrow[d] & h\mathcal{C}_{\hat{L}_n\mathbb{S}} \arrow[r] \arrow[d] & e_{n-1}\mathcal{C}_{\hat{L}_n\mathbb{S}} \\
h\mathcal{C}^{fin}_{E_n} \arrow[r] & h\mathcal{C}_{E_n} &
\end{tikzcd}$$

The prefix e_{n-1} indicates the subcategory of $E(n-1)$-equivalences, and the superscript e_{n-1} indicates the full subcategory of $E(n-1)$-acyclic objects. So $h\mathcal{C}^{e_{n-1}}_{L_n\mathbb{S}}$ is the category of semi-finite cell $E(n)$-local spectra of type $\geq n$, which by Hovey–Strickland is equivalent to the category of (categorically) small $K(n)$-local spectra, $h\mathcal{C}^{e_{n-1}}_{\hat{L}_n\mathbb{S}}$.

The map $e_{n-1}\mathcal{C}_{L_n\mathbb{S}} \to h\mathcal{C}_{L_{n-1}\mathbb{S}}$ is an idempotent completion, i.e., it only adjoins retracts of objects, so the induced map in algebraic K-theory is 0-coconnected, i.e., injective on π_0 and an isomorphism in higher degrees. By Waldhausen's generic fibration theorem, there is therefore a homotopy fiber sequence

$$K^{sm}(\hat{L}_n\mathbb{S}) \to K(L_n\mathbb{S}) \to K(L_{n-1}\mathbb{S})$$

where the right hand map is perhaps not surjective on π_0, and $K^{sm}(\hat{L}_n\mathbb{S})$ denotes the algebraic K-theory of the category of small $K(n)$-local spectra.

The lower vertical maps are induced by base change in the $K(n)$-local pro-$\mathbb{G}_n$-Galois extension $\hat{L}_n\mathbb{S} \to E_n$. The small $K(n)$-local spectra X are characterized by their base changes, $E_n \wedge_{\hat{L}_n\mathbb{S}} X \simeq \hat{L}_n(E_n \wedge X)$, whose homotopy is denoted $E^\vee_{n*}(X)$. Namely, X is small if and only if $E^\vee_{n*}(X)$ is finite in each degree. Hence there is a natural map

$$K^{sm}(\hat{L}_n\mathbb{S}) \to K^{fin}(E_n)^{h\mathbb{G}_n}$$

where $K^{fin}(E_n)$ is the algebraic K-theory of the semi-finite cell E_n-modules that have degreewise finite homotopy groups, $h\mathcal{C}^{fin}_{E_n}$. By the brave new analogue of the Lichtenbum–Quillen conjecture, it is then to be expected that this map and the Galois descent map

$$K(\hat{L}_n\mathbb{S}) \to K(E_n)^{h\mathbb{G}_n}$$

(where the source might really be $K^d(\hat{L}_n\mathbb{S})$ formed from $h\mathcal{D}_{\hat{L}_n\mathbb{S}}$, but this is not critical for this argument) are both close to equivalences. More precisely, these maps should be homotopy equivalences in v_{n+1}-periodic homotopy, obtained by smashing with a finite type $n+1$ spectrum $F(n+1)$ and forming the mapping telescope for the v_{n+1}-self map, or equivalently, smashing with $T(n+1) = v^{-1}_{n+1}F(n+1)$.

Let $K_n = E_n/I$ be the even periodic Morava K-theory spectrum, with $\pi_*(K_n) = \mathbb{F}_{p^n}[u^{\pm 1}]$. The $\mathbb{S}$-algebra homomorphism $\pi\colon E_n \to K_n$ makes K_n a finite cell E_n-module spectrum, and more generally, composition with π makes each semi-finite cell K_n-module a semi-finite cell E_n-module with degreewise finite homotopy groups. Optimistically, this inclusion $h\mathcal{C}_{K_n} \to h\mathcal{C}^{fin}_{E_n}$ induces a dévissage equivalence

$$K(K_n) \to K^{fin}(E_n)$$

in algebraic K-theory.

These arguments support the idea that the difference between $K(L_n\mathbb{S})$ and $K(L_{n-1}\mathbb{S})$ in the algebraic K-theory chromatic tower consists of a π_0-contribution, arising from the presence of properly semi-finite (not finite) cell spectra, and a relative term $K^{sm}(\hat{L}_n\mathbb{S})$ given by the algebraic K-theory of the small $K(n)$-local spectra. The belief in Galois descent for algebraic K-theory, along the $K(n)$-local pro-Galois extension $\hat{L}_n\mathbb{S} \to E_n$, translates this into a contribution from $K^{fin}(E_n)$ given by the algebraic K-theory of the semi-finite cell E_n-module spectra that have degreewise finite homotopy. This may, by dévissage, amount to the algebraic K-theory of the (non-commutative) $\mathbb{S}$-algebra K_n, the even periodic variant of Morava's $K(n)$.

For $n = 1$, the diagram can be rewritten as

$$\begin{tikzcd}
 & K(\tilde{J}_{(p)}) \arrow[r] \arrow[d] & K(\mathbb{Q}) \arrow[d] \\
K^{sm}(J_p) \arrow[ur, bend left] \arrow[r] \arrow[d, "\mathbb{Z}_p^\times"] & K(J_p) \arrow[r] \arrow[d, "\mathbb{Z}_p^\times"] & K(J\mathbb{Q}_p) \\
K^{fin}(KU_p) \arrow[r] & K(KU_p) &
\end{tikzcd}$$

where $\tilde{J}_{(p)} = L_1\mathbb{S}$, $J_p = \hat{L}_1\mathbb{S}$, $J\mathbb{Q}_p = J_p[1/p] \simeq H\mathbb{Q}_p \vee \Sigma^{-1}H\mathbb{Q}_p$, and $KU/p = K_1$ maps $K(KU/p) \to K^{fin}(KU_p)$.

4. Algebraic K-theory of topoplogical K-theory.

We are thus led to study $K(KU_p)$ and $K(KU/p)$. There is a fiber sequence

$$K(\mathbb{Z}) \to K(ku) \to K(KU)$$

due to Blumberg–Mandell, and the same proof should yield fiber sequences

$$K(\mathbb{Z}_p) \to K(ku_p) \to K(KU_p)$$

and

$$K(\mathbb{Z}/p) \to K(ku/p) \to K(KU/p)\,.$$

These are analogous to the dévissage fiber sequence for $K(\mathbb{Z}_p) \to K(\mathbb{Q}_p)$, except that p-torsion is replaced by u-torsion. The advantage of this, is that the terms on the left and in the middle are connective $\mathbb{S}$-algebras A with $\pi_0 = \mathbb{Z}/p$ or $\mathbb{Z}_p$, for which the cyclotomic trace map identifies $K(A)_p$ with the connective cover of $TC(A)_p$.