CONNECTIVE STRUCTURES ON GERBES AND TWO-VECTOR BUNDLES

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GERBES AND 2-VECTOR BUNDLES

Let $\mathcal{V}$ be a bipermutative groupoid of finite-dimensional complex vector spaces, under direct sum and tensor product. For example, we may take $\mathcal{V}$ to have objects $\mathbb{C}^n$ for $n \geq 0$, so that $\mathbb{C}^n$ has automorphism group $GL_n(\mathbb{C})$

Recall from Brylinski, Section 5.2:

Definition. Let X be a smooth manifold, perhaps infinite-dimensional. A **smooth $\mathbb{C}^*$-gerbe** over X is a sheaf of groupoids $\mathcal{G}$ that is locally equivalent to the sheaf of smooth functions to the groupoid $GL_1(\mathcal{V})$, with objects the 1-dimensional complex vector spaces and isomorphisms the complex linear isomorphisms between these.

In more detail:

(P1) for each local homeomorphism $f\colon Y \to X$ there is a groupoid $\mathcal{G}(Y) = \mathcal{G}(Y \xrightarrow{f} X)$,

(P2) for every further local homeomorphism $g\colon Z \to Y$ there is a functor

$$g^{-1}\colon \mathcal{G}(Y) \to \mathcal{G}(Z)\,,$$

(P3) and for every further local homeomorphism $h\colon W \to Z$ there is a natural isomorphism $\theta_{g,h}\colon h^{-1}g^{-1} \xrightarrow{\cong} (gh)^{-1}$ of functors $\mathcal{G}(Y) \to \mathcal{G}(W)$, such that

(P4) for each further local homeomorphism $k\colon T \to W$ the identity $\theta_{gh,k}\theta_{h,k} = \theta_{g,hk}\theta_{g,h}$ holds. This makes $\mathcal{G}$ a **presheaf of groupoids**.

For $\mathcal{G}$ to be a **sheaf of groupoids**, two descent conditions must hold:

(D1) For each local homeomorphism $f\colon Y \to X$ and each pair of objects $P, Q \in \mathcal{G}(Y)$, the assignment taking a further local homeomorphism $g\colon Z \to Y$ to the morphism set $\mathcal{G}(Z)(g^{-1}P, g^{-1}Q)$ is a sheaf of sets on Y, denoted $\underline{\mathcal{G}(Y)}(P, Q)$.

(D2) For each open subset $V \subseteq X$, each surjective local homeomorphism $f\colon Y \to V$, and each object $P \in \mathcal{G}(Y)$ equipped with a descent datum, that is, an isomorphism $\phi\colon p_2^{-1}P \xrightarrow{\cong} p_1^{-1}P$ in $\mathcal{G}(Y \times_X Y)$ such that $p_{12}^{-1}\phi \circ p_{23}^{-1}\phi \equiv p_{13}^{-1}\phi$ (modulo θ's), there exists an object $Q \in \mathcal{G}(V)$ and an isomorphism $\psi\colon f^{-1}(Q) \xrightarrow{\cong} P$ in $\mathcal{G}(Y)$, such that $\phi \equiv (p_1^{-1}\psi)(p_2^{-1}\psi)^{-1}$ (modulo θ's).

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For $\mathcal{G}$ to be a **smooth $\mathbb{C}^*$-gerbe**, three gerbe conditions must hold:

(G3) There is a surjective local homeomorphism $f\colon Y \to X$ such that $\mathcal{G}(Y)$ is non-empty.

(G2) For any local homeomorphism $f\colon Y \to X$ and any two objects P and Q in $\mathcal{G}(Y)$, there exists a surjective local homeomorphism $g\colon Z \to Y$ such that $g^{-1}P$ and $g^{-1}Q$ are isomorphic in $\mathcal{G}(Z)$.

(G1) Given any local homeomorphism $f\colon Y \to X$ and any object P of $\mathcal{G}(Y)$, there is a preferred isomorphism from the sheaf $\underline{\mathcal{G}(Y)}(P,P)$ of automorphisms of P to the sheaf $\underline{\mathbb{C}}^*_Y$ of smooth $\mathbb{C}^*$-valued functions.

((Omitted: When are two gerbes equivalent?)

Here is a definition from Baas–Dundas–Rognes.

Definition. By an **ordered open cover** $(\mathcal{U}, \mathcal{I})$ of a topological space X, we mean a partially ordered indexing set $\mathcal{I}$, and an open subset $U_\alpha \subseteq X$ for each $\alpha \in \mathcal{I}$, so that for each $x \in X$ the partial ordering on $\mathcal{I}$ restricts to a total (= linear) ordering on the subset $\mathcal{I}_x = \{\alpha \in \mathcal{I} \mid x \in U_\alpha\}$. It follows that for any subset $\mathcal{J} \subseteq \mathcal{I}$ of indices with $\bigcap_{\alpha\in\mathcal{J}} U_\alpha \neq \emptyset$, $\mathcal{J}$ is totally ordered.

The following defines a charted 2-vector bundle of rank 1. We use the abbreviations $U_{\alpha\beta} = U_\alpha \cap U_\beta$, $U_{\alpha\beta\gamma} = U_\alpha \cap U_\beta \cap U_\gamma$, etc. See also Murray's paper on bundle gerbes.

Definition. Let X be a topological space, with an ordered open cover $(\mathcal{U}, \mathcal{I})$. A **charted 2-line bundle** $\mathcal{L}$ over X consists of

(1) a complex line bundle $L^{\alpha\beta}$ over $U_{\alpha\beta}$ for each pair $\alpha < \beta$ in $\mathcal{I}$, and

(2) an isomorphism
$$\phi^{\alpha\beta\gamma}\colon L^{\alpha\beta} \otimes L^{\beta\gamma} \xrightarrow{\cong} L^{\alpha\gamma}$$
of complex line bundles over $U_{\alpha\beta\gamma}$ for each triple $\alpha < \beta < \gamma$ in $\mathcal{I}$, such that

(3) the diagram
$$\begin{array}{ccccc} L^{\alpha\beta}\otimes(L^{\beta\gamma}\otimes L^{\gamma\delta}) & & \xrightarrow{\underline{\alpha}} & & (L^{\alpha\beta}\otimes L^{\beta\gamma})\otimes L^{\gamma\delta} \\ \downarrow{\scriptstyle id\otimes\phi^{\beta\gamma\delta}} & & & & \downarrow{\scriptstyle \phi^{\alpha\beta\gamma}\otimes id} \\ L^{\alpha\beta}\otimes L^{\beta\delta} & \xrightarrow[\phi^{\alpha\beta\delta}]{} & L^{\alpha\delta} & \xleftarrow[\phi^{\alpha\gamma\delta}]{} & L^{\alpha\gamma}\otimes L^{\gamma\delta} \end{array}$$
of line bundle isomorphisms over $U_{\alpha\beta\gamma\delta}$ commutes for each chain $\alpha < \beta < \gamma < \delta$ in $\mathcal{I}$, where $\underline{\alpha}$ is the coherent natural associativity isomorphism.

The $L^{\alpha\beta}$ are the gluing line bundles and the $\phi^{\alpha\beta\gamma}$ are the coherence isomorphisms of the charted 2-line bundle.

If X is a smooth manifold, each $L^{\alpha\beta}$ is smooth line bundle, and each $\phi^{\alpha\beta\gamma}$ is a smooth isomorphism, we get a smooth charted 2-line bundle. This notion is closely related to that of a bundle gerbe.

((Omitted: When are two 2-line bundles equivalent?)

Here is the general definition, for $n \geq 1$.

Definition. Let X be a topological space, with an ordered open cover $(\mathcal{U}, \mathcal{I})$. A **charted 2-vector bundle** $\mathcal{E}$ of rank n over X consists of

(1) an $n \times n$ matrix
$$E^{\alpha\beta} = (E_{ij}^{\alpha\beta})_{i,j=1}^{n}$$
of complex vector bundles over $U_{\alpha\beta}$, with dimension matrix $\dim(E^{\alpha\beta})$ of determinant ± 1 everywhere, for each pair $\alpha < \beta$ in $\mathcal{I}$, and

(2) an $n \times n$ matrix
$$\phi^{\alpha\beta\gamma} = (\phi_{ik}^{\alpha\beta\gamma})_{i,k=1}^{n} \colon E^{\alpha\beta} \cdot E^{\beta\gamma} \xrightarrow{\cong} E^{\alpha\gamma}$$
of vector bundle isomorphisms over $U_{\alpha\beta\gamma}$, where
$$(E^{\alpha\beta} \cdot E^{\beta\gamma})_{ik} = \bigoplus_{j=1}^{n} E_{ij}^{\alpha\beta} \otimes E_{jk}^{\beta\gamma}$$
for $i, k = 1, \dots, n$, for each triple $\alpha < \beta < \gamma$ in $\mathcal{I}$, such that

(3) the diagram
$$\begin{array}{ccccc}
E^{\alpha\beta} \cdot (E^{\beta\gamma} \cdot E^{\gamma\delta}) & & \xrightarrow{\quad \underline{\alpha} \quad} & & (E^{\alpha\beta} \cdot E^{\beta\gamma}) \cdot E^{\gamma\delta} \\
\downarrow {\scriptstyle id \cdot \phi^{\beta\gamma\delta}} & & & & \downarrow {\scriptstyle \phi^{\alpha\beta\gamma} \cdot id} \\
E^{\alpha\beta} \cdot E^{\beta\delta} & \xrightarrow[\phi^{\alpha\beta\delta}]{} & E^{\alpha\delta} & \xleftarrow[\phi^{\alpha\gamma\delta}]{} & E^{\alpha\gamma} \cdot E^{\gamma\delta}
\end{array}$$
of $n \times n$ matrices of vector bundle isomorphisms over $U_{\alpha\beta\gamma\delta}$ commutes for each chain $\alpha < \beta < \gamma < \delta$ in $\mathcal{I}$, where $\underline{\alpha}$ is the coherent natural associativity isomorphism.

The $E^{\alpha\beta}$ are the **gluing bundle matrices** and the $\phi^{\alpha\beta\gamma}$ are the **coherence isomorphism matrices** of the charted 2-vector bundle.

If X is a smooth manifold, each $E_{ij}^{\alpha\beta}$ is smooth vector bundle, and each $\phi_{ik}^{\alpha\beta\gamma}$ is a smooth isomorphism, we get a smooth charted 2-vector bundle.

The following construction is adapted from Brylinski's Definition and Proposition 7.2.1, and the proof of surjectivity in his Theorem 5.2.8.

Proposition. *To each smooth charted* 2*-line bundle* $\mathcal{L}$ *over* X *there is a naturally associated smooth* $\mathbb{C}^*$*-gerbe* $\mathcal{G}$ *over* X*.*

Proof. We work in the smooth category. Let $f \colon Y \to X$ be a local homeomorphism, and let $Y_\alpha = f^{-1}(U_\alpha)$, $Y_{\alpha\beta} = f^{-1}(U_{\alpha\beta})$, etc. be open subsets of Y. Let $\mathcal{G}(Y) = \mathcal{G}(Y \xrightarrow{f} X)$ be the groupoid with objects consisting of

(1) a complex line bundle P^α over Y_α for each $\alpha \in \mathcal{I}$,

(2) an isomorphism
$$\xi^{\alpha\beta} \colon L^{\alpha\beta} \otimes P^\beta \xrightarrow{\cong} P^\alpha$$
of complex line bundles over $Y_{\alpha\beta}$, for each pair $\alpha < \beta$ in $\mathcal{I}$, such that

(3) the diagram

$$\begin{array}{ccccc}
L^{\alpha\beta} \otimes (L^{\beta\gamma} \otimes P^{\gamma}) & & \xrightarrow{\alpha} & & (L^{\alpha\beta} \otimes L^{\beta\gamma}) \otimes P^{\gamma} \\
\downarrow{\scriptstyle id\otimes\xi^{\beta\gamma}} & & & & \downarrow{\scriptstyle \phi^{\alpha\beta\gamma}\otimes id} \\
L^{\alpha\beta} \otimes P^{\beta} & \xrightarrow[\xi^{\alpha\beta}]{} & P^{\alpha} & \xleftarrow[\xi^{\alpha\gamma}]{} & L^{\alpha\gamma} \otimes P^{\gamma}
\end{array}$$

of isomorphisms over $U_{\alpha\beta\gamma}$ commutes for each triple $\alpha < \beta < \gamma$ in $\mathcal{I}$.

The isomorphisms in $\mathcal{G}(Y)$ from $(P^{\alpha}, \xi^{\alpha\beta})$ to $(Q^{\alpha}, \eta^{\alpha\beta})$ consist of

(1) isomorphisms $\psi^{\alpha} \colon P^{\alpha} \to Q^{\alpha}$ over Y_{α} for each $\alpha \in \mathcal{I}$, such that
(2) the diagram

$$\begin{array}{ccc}
L^{\alpha\beta} \otimes P^{\beta} & \xrightarrow{\xi^{\alpha\beta}} & P^{\alpha} \\
\downarrow{\scriptstyle id\otimes\psi^{\beta}} & & \downarrow{\scriptstyle \psi^{\alpha}} \\
L^{\alpha\beta} \otimes Q^{\beta} & \xrightarrow[\eta^{\alpha\beta}]{} & Q^{\alpha}
\end{array}$$

of isomorphisms over $Y_{\alpha\beta}$ commutes for each pair $\alpha < \beta$ in $\mathcal{I}$.

For each local homeomorphism $g \colon Z \to Y$ the functor $g^{-1} \colon \mathcal{G}(Y) \to \mathcal{G}(Z)$ is given on objects by pullback of the line bundles P^{α} along $Z_{\alpha} \to Y_{\alpha}$ and of the isomorphisms $\xi^{\alpha\beta}$ along $Z_{\alpha\beta} \to Y_{\alpha\beta}$, while on isomorphisms it is given by pullback of the isomorphisms ψ^{α} along $Z_{\alpha} \to Y_{\alpha}$.

This accounts for (P1) and (P2). We omit to discuss (P3), (P4), (D1), (D2), (G3) and (G2).

The automorphisms of $P = (P^{\alpha}, \xi^{\alpha\beta})$ consist of isomorphisms $\psi^{\alpha} \colon P^{\alpha} \to P^{\alpha}$ over Y_{α}, for each $\alpha \in \mathcal{I}$, which amounts to multiplication by a (smooth) function $a^{\alpha} \colon Y_{\alpha} \to \mathbb{C}^*$, and these satisfy $a^{\alpha} = a^{\beta}$ over $Y_{\alpha\beta}$, hence these functions glue together to a (smooth) $\mathbb{C}^*$-valued function a on Y. More generally this works for the pullback over any local homeomorphism $g \colon Z \to Y$, so we get the isomorphism of sheaves between $\underline{\mathcal{G}(Y)}(P, P)$ and $\underline{\mathbb{C}}^*_Y$, required in (G1). $\square$

Definition. Let $GL_n(\mathcal{V})$ be the (smooth) groupoid with objects the $n \times n$ matrices $V = (V_{ij})_{i,j=1}^n$ of complex vector spaces with dimension matrix $\dim(V) = \dim(V_{ij})_{i,j=1}^n$ of determinant ± 1, and morphisms $n \times n$ matrices $\phi \colon V \to W$ of complex linear isomorphisms between these. Matrix multiplication $V \cdot W$ with

$$(V \cdot W)_{ik} = \bigoplus_{j=1}^{n} V_{ij} \otimes W_{jk}$$

defines a monoidal structure on $GL_n(\mathcal{V})$.

There is one isomorphism class of objects in $GL_n(\mathcal{V})$, for each $n \times n$ matrix $D = (d_{ij})_{i,j=1}^n$ with non-negative integer entries and determinant ± 1. These matrices form the monoid $GL_n(\mathbb{N}_0) = \mathrm{Mat}_n(\mathbb{N}_0) \cap GL_n(\mathbb{Z})$, whose invertible elements are exactly the permutation matrices $\Sigma_n \subseteq GL_n(\mathbb{N}_0)$.

The automorphisms in $GL_n(\mathcal{V})$ of the object $\mathbb{C}^D = (\mathbb{C}^{d_{ij}})_{i,j=1}^n$ form the product group

$$GL_D(\mathbb{C}) = \prod_{i,j=1}^{n} GL_{d_{ij}}(\mathbb{C}) \,.$$

For example, if D is a permutation matrix, $GL_D(\mathbb{C}) \cong GL_1(\mathbb{C})^n = (\mathbb{C}^*)^n$.

((Let $\|D\| = \sum_{i,j=1}^n d_{ij}$. Then $\|D\| \geq n$, with equality if and only if D is a permutation matrix. If $A \cdot B = C$ then $\|B\| \leq \|C\|$, with equality if and only if A is a permutation matrix. Hence a given matrix D can be factored in at most finitely many ways, up to permutation matrices (= units).))

Proposition. *To each smooth charted 2-vector bundle $\mathcal{E}$ of rank n over X there is a naturally associated sheaf of groupoids $\mathcal{P}$ over X with a right action by the monoidal sheaf of smooth functions to $GL_n(\mathcal{V})$.*

((Can say more about local structure, under further hypotheses on the ordered open cover.))

Proof. We work in the smooth category. Let $f\colon Y \to X$ be a local homeomorphism, and let $Y_\alpha = f^{-1}(U_\alpha)$, $Y_{\alpha\beta} = f^{-1}(U_{\alpha\beta})$, etc. be open subsets of Y. Let $\mathcal{P}(Y) = \mathcal{P}(Y \xrightarrow{f} X)$ be the groupoid with objects consisting of

(1) an $n \times n$ matrix $P^\alpha = (P^\alpha_{ij})_{i,j=1}^n$ of complex vector bundles over Y_α, with dimension matrix $\dim(P^\alpha)$ of determinant ± 1 everywhere, for each $\alpha \in \mathcal{I}$, and
(2) an $n \times n$ matrix

$$\xi^{\alpha\beta} = (\xi^{\alpha\beta}_{ik})_{i,k=1}^n \colon E^{\alpha\beta} \cdot P^\beta \xrightarrow{\cong} P^\alpha$$

of vector bundle isomorphisms over $Y_{\alpha\beta}$, for each pair $\alpha < \beta$ in $\mathcal{I}$, such that
(3) the diagram

$$\begin{array}{ccccc} E^{\alpha\beta} \cdot (E^{\beta\gamma} \cdot P^\gamma) & & \xrightarrow{\quad\alpha\quad} & & (E^{\alpha\beta} \cdot E^{\beta\gamma}) \cdot P^\gamma \\ \Big\downarrow{\scriptstyle id\cdot\xi^{\beta\gamma}} & & & & \Big\downarrow{\scriptstyle \phi^{\alpha\beta\gamma}\cdot id} \\ E^{\alpha\beta} \cdot P^\beta & \xrightarrow[\xi^{\alpha\beta}]{} & P^\alpha & \xleftarrow[\xi^{\alpha\gamma}]{} & E^{\alpha\gamma} \cdot P^\gamma \end{array}$$

of $n \times n$ matrices of vector bundle isomorphisms over $U_{\alpha\beta\gamma}$ commutes for each triple $\alpha < \beta < \gamma$ in $\mathcal{I}$.

The isomorphisms in $\mathcal{P}(Y)$ from $(P^\alpha, \xi^{\alpha\beta})$ to $(Q^\alpha, \eta^{\alpha\beta})$ consist of

(1) an $n \times n$ matrix

$$\psi^\alpha = (\psi^\alpha_{ij})_{i,j=1}^n \colon P^\alpha \to Q^\alpha$$

of vector bundle isomorphisms over Y_α, for each $\alpha \in \mathcal{I}$, such that
(2) the diagram

$$\begin{array}{ccc} E^{\alpha\beta} \cdot P^\beta & \xrightarrow{\xi^{\alpha\beta}} & P^\alpha \\ \Big\downarrow{\scriptstyle id\cdot\psi^\beta} & & \Big\downarrow{\scriptstyle \psi^\alpha} \\ E^{\alpha\beta} \cdot Q^\beta & \xrightarrow[\eta^{\alpha\beta}]{} & Q^\alpha \end{array}$$

of $n \times n$ matrices of vector bundle isomorphisms over $Y_{\alpha\beta}$ commutes, for each pair $\alpha < \beta$ in $\mathcal{I}$.

For each local homeomorphism $g\colon Z \to Y$ the functor $g^{-1}\colon \mathcal{P}(Y) \to \mathcal{P}(Z)$ is given on objects by pullback of the vector bundle matrix P^α along $Z_\alpha \to Y_\alpha$ and of the isomorphism matrix $\xi^{\alpha\beta}$ along $Z_{\alpha\beta} \to Y_{\alpha\beta}$, while on isomorphisms it is given by pullback of the isomorphism matrix ψ^α along $Z_\alpha \to Y_\alpha$.

This accounts for (P1) and (P2). The remaining presheaf conditions (P3) and (P4), and the sheaf conditions (D1) and (D2) should be clear.

The value of the sheaf of smooth functions to $GL_n(\mathcal{V})$ at Y can be viewed as the groupoid of $n \times n$ matrices $R = (R_{ij})_{i,j=1}^n$ of complex vector bundles over Y, with $\dim(R)$ of determinant ± 1, and $n \times n$ matrices $\zeta\colon R \to S$ of vector bundle isomorphisms between these. The right action of R on an object $(P^\alpha, \xi^{\alpha\beta})$ gives the object consisting of

(1) the $n \times n$ matrix $P^\alpha \cdot R$ of vector bundles over Y_α, for each $\alpha \in \mathcal{I}$, and
(2) the $n \times n$ matrix of composite vector bundle isomorphisms

$$E^{\alpha\beta} \cdot (P^\beta \cdot R) \xrightarrow{\alpha} (E^{\alpha\beta} \cdot P^\beta) \cdot R \xrightarrow{\xi^{\alpha\beta} \cdot id} P^\alpha \cdot R$$

over $Y_{\alpha\beta}$, for each pair $\alpha < \beta$ in $\mathcal{I}$.

The right action of $\zeta\colon R \to S$ on an isomorphism (ψ^α) from $(P^\alpha, \xi^{\alpha\beta})$ to $(Q^\alpha, \eta^{\alpha\beta})$ is the collection of isomorphisms

$$\psi^\alpha \cdot \zeta\colon P^\alpha \cdot R \xrightarrow{\cong} Q^\alpha \cdot S$$

for $\alpha \in \mathcal{I}$. □

When does $\mathcal{P}$ have objects locally?

Proposition. *Suppose that $x \in X$ has an open neighborhood V such that the subset $\mathcal{I}_V = \{\alpha \in \mathcal{I} \mid V \cap U_\alpha \neq \emptyset\}$ of $\mathcal{I}$ is finite and totally ordered. Suppose also that the $V_\alpha = V \cap U_\alpha$ for $\alpha \in \mathcal{I}_V$, as well as all of their intersections, are contractible. ((Need a condition about extension to closures, to ensure that the union of two trivial bundles along a contractible open subspace is again trivial.)) Then $\mathcal{P}(V)$ is not the empty category, hence has objects.*

((The missing condition might be that X is a normal space (for the Tietze extension theorem), the gluing bundle matrices $E^{\alpha\beta}$ are defined over the closed intersections $\bar{U}_\alpha \cap \bar{U}_\beta$, and that the coherence isomorphism matrices $\phi^{\alpha\beta\gamma}$ are defined over the closed intersections $\bar{U}_\alpha \cap \bar{U}_\beta \cap \bar{U}_\gamma$, with the usual compatibility. Furthermore, the $\bar{V} \cap \bar{U}_\alpha$ and all of their intersections, should be contractible.))

Proof. Write $\mathcal{I}_V = \{\alpha < \cdots < \omega\}$. For all $\beta < \gamma$ in $\mathcal{I}_V$ we (should) know that $E^{\beta\gamma}$ is trivial over the contractible open set $V_{\beta\gamma}$. Choose an $n \times n$ matrix of trivial vector bundles P^ω over V_ω. As the last step of a descending induction over $\mathcal{I}_V$, we shall assume that we have chosen

(1) matrices of trivial vector bundles P^β over V_β for all β with $\alpha < \beta \le \omega$, and
(2) matrices of isomorphisms $\xi^{\beta\gamma}\colon E^{\beta\gamma} \cdot P^\gamma \to P^\beta$ over $V_{\beta\gamma}$ for all β and γ with $\alpha < \beta < \gamma \le \omega$,

subject to the usual compatibility condition over $V_{\beta\gamma\delta}$ for all $\alpha < \beta < \gamma < \delta \le \omega$. Then we can

(1) form the matrix of trivial vector bundles $E^{\alpha\beta} \cdot P^\beta$ over $V_{\alpha\beta}$, for each β with $\alpha < \beta \le \omega$,

(2) these are related by a chain of matrices of isomorphisms

$$\begin{array}{ccc} E^{\alpha\beta} \cdot (E^{\beta\gamma} \cdot P^{\gamma}) & \xrightarrow{\ \alpha\ } & (E^{\alpha\beta} \cdot E^{\beta\gamma}) \cdot P^{\gamma} \\ {\scriptstyle id\cdot\xi^{\beta\gamma}}\downarrow & & \downarrow{\scriptstyle \phi^{\alpha\beta\gamma}\cdot id} \\ E^{\alpha\beta} \cdot P^{\beta} & & E^{\alpha\gamma} \cdot P^{\gamma} \end{array}$$

over $V_{\alpha\beta\gamma}$, for all β and γ with $\alpha < \beta < \gamma \le \omega$, and

(3) these chains of isomorphisms are strictly compatible over $V_{\alpha\beta\gamma\delta}$, for all $\alpha < \beta < \gamma < \delta \le \omega$.

Hence we can find a matrix of vector bundles $\tilde{P}^{\alpha}$ over

$$\tilde{V}_{\alpha} = \bigcup_{\alpha<\beta\le\omega} V_{\alpha\beta} \subseteq V_{\alpha}$$

with isomorphisms $\xi^{\alpha\beta} \colon E^{\alpha\beta} \cdot P^{\beta} \to \tilde{P}^{\alpha}$ over $V_{\alpha\beta}$ for all $\alpha < \beta \le \omega$, such that the completed diagram

$$\begin{array}{ccccc} E^{\alpha\beta} \cdot (E^{\beta\gamma} \cdot P^{\gamma}) & & \xrightarrow{\ \alpha\ } & & (E^{\alpha\beta} \cdot E^{\beta\gamma}) \cdot P^{\gamma} \\ {\scriptstyle id\cdot\xi^{\beta\gamma}}\downarrow & & & & \downarrow{\scriptstyle \phi^{\alpha\beta\gamma}\cdot id} \\ E^{\alpha\beta} \cdot P^{\beta} & \xrightarrow[\xi^{\alpha\beta}]{} & \tilde{P}^{\alpha} & \xleftarrow[\xi^{\alpha\gamma}]{} & E^{\alpha\gamma} \cdot P^{\gamma} \end{array}$$

commutes over $V_{\alpha\beta\gamma}$.

((Now we argue that $\tilde{P}^{\alpha}$ over $\tilde{V}_{\alpha}$ is a matrix of trivial bundles, hence can be extended to a matrix P^{α} over V_{α} of trivial bundles.)) □

When are objects of $\mathcal{P}$ locally isomorphic?

((Objects $(P^{\alpha}, \xi^{\alpha\beta})$ and $(Q^{\alpha}, \eta^{\alpha\beta})$ in $\mathcal{P}(V)$ can only be isomorphic if they have the same dimension matrices $\dim(P^{\alpha}) = \dim(Q^{\alpha})$ over $V_{\alpha} = V \cap U_{\alpha}$ for all α. Assuming this, we can build a local isomorphism for small V, assuming that isomorphisms can be extended over inclusions $\tilde{V}_{\alpha} \subseteq V_{\alpha}$, as in the previous result.))

Connective structures on gerbes

Let $q \ge 0$ and let Y be a smooth manifold. The assignment taking a local homeomorphism $g \colon Z \to Y$ to the complex vector space $A^q(Z;\mathbb{C}) = \Omega^q(Z) \otimes \mathbb{C}$ of complex alternating q-forms on Z, defines a complex vector space sheaf $\underline{A}^q_{Y,\mathbb{C}}$ over Y. For each complex line bundle P over Y, the set of connections ∇ on P forms an affine space (a single free orbit) under $A^1(Y;\mathbb{C})$. For a more general complex vector bundle E, the connections form an affine space under the 1-forms in the endomorphism bundle of E, but when $E = P$ is a line bundle, the endomorphism bundle is a trivial $\mathbb{C}^1$-bundle. Passing to restrictions along the local homeomorphisms $g \colon Z \to Y$, the collection of connections over the various Z, denoted $Co(P)$, forms a torsor under $\underline{A}^1_{Y,\mathbb{C}}$. If $\psi \colon P \xrightarrow{\cong} Q$ is an isomorphism of complex line bundles over Y, there is an induced isomorphism $\psi_* \colon Co(P) \to Co(Q)$ that takes a connection ∇ on P to the connection $\psi_*(\nabla)$ on Q, determined on smooth sections s in $Q \to Y$ by

$$\psi_*(\nabla)_w(s) = \psi(\nabla_v(\psi^{-1}s))$$

when $T\psi\colon TP \to TQ$ takes the tangent vector v to w. When ψ is the automorphism given by multiplication by a smooth function $a\colon Y \to \mathbb{C}^*$, we can compute that

$$\begin{aligned} \psi_*(\nabla)_w(s) &= a \cdot \nabla_v(a^{-1} \cdot s) \\ &= a \cdot a^{-1} \cdot \nabla_w(s) + a \cdot d(a^{-1})(w) \cdot s \\ &= \nabla_w(s) - a^{-1}\, da(w) \cdot s \end{aligned}$$

so that $\psi_*(\nabla) = \nabla - a^{-1}\, da = \nabla - d\log a$. In a $\mathbb{C}^*$-gerbe, the local groupoids are non-canonically equivalent to the groupoid of complex line bundles, and a connective structure specifies corresponding torsors under the complex vector space sheaf $\underline{A}^1_{Y,\mathbb{C}}$.

Definition. Let $\mathcal{G}$ be a smooth $\mathbb{C}^*$-gerbe over X. A **connective structure** Co on $\mathcal{G}$ consists of

(1) a rule that for each local homeomorphism $f\colon Y \to X$ and each object $P \in \mathcal{G}(Y)$ associates an $\underline{A}^1_{Y,\mathbb{C}}$-torsor $Co(P)$,
(2) an isomorphism

$$co^g\colon g^{-1}Co(P) \xrightarrow{\cong} Co(g^{-1}P)$$

of $\underline{A}^1_{Z,\mathbb{C}}$-torsors for each further local homeomorphism $g\colon Z \to Y$, and
(3) an isomorphism

$$\psi_*\colon Co(P) \to Co(Q)$$

of $\underline{A}^1_{Y,\mathbb{C}}$-torsors for each isomorphism $\psi\colon P \to Q$ in $\mathcal{G}(Y)$. These are required to be functorial and compatible, in the sense that
(4) $co^{gh} \equiv co^h \circ co^g$ (modulo canonical isomorphisms and θ's) for further local homeomorphisms $h\colon W \to Z$,
(5) $(\psi' \circ \psi)_* = \psi'_* \circ \psi_*$ for further isomorphisms $\psi'\colon Q \to R$ in $\mathcal{G}(Y)$, and
(6) the diagram

$$\begin{array}{ccc} g^{-1}Co(P) & \xrightarrow{g^{-1}(\psi_*)} & g^{-1}Co(Q) \\ \downarrow{\scriptstyle co^g} & & \downarrow{\scriptstyle co^g} \\ Co(g^{-1}P) & \xrightarrow[(g^{-1}\psi)_*]{} & Co(g^{-1}Q) \end{array}$$

commutes. Finally, the isomorphisms ψ_* behave as expected for gauge automorphisms, meaning that
(7) when $\psi\colon P \to P$ is the automorphism induced by multiplication by a smooth function $a\colon Y \to \mathbb{C}^*$, we have

$$\psi_*(\nabla) = \nabla - a^{-1}\, da$$

for all ∇ in the $\underline{A}^1_{Y,\mathbb{C}}$-torsor $Co(P)$.

See Brylinski's Definition 5.3.1. He writes α_g for our co^g, and labels (6) as (R2) and (7) as (R1). The following combines Brylinski's Proposition 5.3.2 and 5.3.6.

Proposition. *Let $\mathcal{G}$ be a smooth $\mathbb{C}^*$-gerbe over a smooth manifold X. Then $\mathcal{G}$ admits connective structures, and the isomorphism classes of torsors under $\underline{A}^1_{X,\mathbb{C}}$ act freely and transitively on the isomorphism classes of connective structures on $\mathcal{G}$.*

Definition. Let X be a smooth manifold, with an ordered open cover $(\mathcal{U}, \mathcal{I})$. Let $\mathcal{L} \to X$ be a smooth charted 2-line bundle, with gluing line bundles $L^{\alpha\beta}$ and coherence isomorphisms $\phi^{\alpha\beta\gamma}$. A **connective structure** ∇ on $\mathcal{L}$ is

(1) a connection $\nabla^{\alpha\beta}$ on $L^{\alpha\beta}$ over $U_{\alpha\beta}$, for each $\alpha < \beta$ in $\mathcal{I}$, such that
(2) the isomorphism $\phi^{\alpha\beta\gamma} \colon L^{\alpha\beta} \otimes L^{\beta\gamma} \xrightarrow{\cong} L^{\alpha\gamma}$ over $U_{\alpha\beta\gamma}$ takes the tensor product connection $\nabla^{\alpha\beta} + \nabla^{\beta\gamma}$ on $L^{\alpha\beta} \otimes L^{\beta\gamma}$ to the connection $\nabla^{\alpha\gamma}$ on $L^{\alpha\gamma}$, so that
$$(\phi^{\alpha\beta\gamma})_*(\nabla^{\alpha\beta} + \nabla^{\beta\gamma}) = \nabla^{\alpha\gamma}$$
over $U_{\alpha\beta\gamma}$.

For sections s in $L^{\alpha\beta}$ and t in $L^{\beta\gamma}$, we have

$$(\nabla^{\alpha\beta} + \nabla^{\beta\gamma})(s \otimes t) = \nabla^{\alpha\beta}(s) \otimes t + s \otimes \nabla^{\beta\gamma}(t)$$

as 1-forms of sections in $L^{\alpha\beta} \otimes L^{\beta\gamma}$ over $U_{\alpha\beta\gamma}$.

The following is close to the first part of the proof of Brylinski's Proposition 5.3.2. For an ordered open cover $(\mathcal{U}, \mathcal{I})$ of X and an abelian sheaf $\underline{A}$ over X, the ordered Čech complex consists of the abelian groups

$$\check{C}^p(\mathcal{U}; \underline{A}) = \prod_{\alpha_0 < \cdots < \alpha_p} A(U_{\alpha_0 \dots \alpha_p}) \,.$$

An ordered Čech p-cochain η associates to each sequence $\alpha_0 < \cdots < \alpha_p$ an element $\eta(\alpha_0, \dots, \alpha_p) \in A(U_{\alpha_0 \dots \alpha_p})$. Its coboundary $d\eta$ is the ordered Čech $(p+1)$-cochain that associates to each sequence $\alpha_0 < \cdots < \alpha_{p+1}$ the alternating sum $\sum_{i=0}^{p+1} (-1)^i \eta(\alpha_0, \dots, \widehat{\alpha_i}, \dots, \alpha_{p+1})$, interpreted via various restriction maps to lie in $A(U_{\alpha_0 \dots \alpha_{p+1}})$. Its cohomology is denoted $\check{H}^*(\mathcal{U}; \underline{A})$. The colimit over all ordered open covers $(\mathcal{U}, \mathcal{I})$ defines the Čech cohomology $\check{H}^*(X; \underline{A})$.

Proposition. *Let $\mathcal{L}$ be a smooth charted 2-line bundle over a smooth manifold X. There exists a connective structure ∇ on $\mathcal{L}$.*

Proof I. First choose arbitrary connections $\tilde{\nabla}^{\alpha\beta}$ on $L^{\alpha\beta}$ for each $\alpha < \beta$ in $\mathcal{I}$. Then

$$(\phi^{\alpha\beta\gamma})_*(\tilde{\nabla}^{\alpha\beta} + \tilde{\nabla}^{\beta\gamma}) = \tilde{\nabla}^{\alpha\gamma} + \omega^{\alpha\beta\gamma}$$

for a unique complex 1-form $\omega^{\alpha\beta\gamma}$ on $U_{\alpha\beta\gamma}$. Letting $\alpha < \beta < \gamma$ vary, we get a Čech 2-cochain ω in $\check{C}^2(\mathcal{U}; \underline{A}^1_{X,\mathbb{C}})$. By condition (3) in the definition of a charted 2-line bundle, this Čech 2-cochain is a Čech 2-cocycle.

As discussed in Brylinski's Section 1.4, the sheaves $\underline{A}^q_{X,\mathbb{C}}$ are **soft**, so that the restriction maps $A^q(X; \mathbb{C}) \to A^q(Z; \mathbb{C})$ are surjective for all closed $Z \subseteq X$, hence **acyclic**. So for reasonable X, $\check{H}^p(\mathcal{U}; \underline{A}^q_{X,\mathbb{C}}) \cong H^p(\mathcal{U}; \underline{A}^q_{X,\mathbb{C}}) = 0$ for $p > 0$. Murray gives a more direct proof if this fact. In particular, $\check{H}^2(\mathcal{U}; \underline{A}^1_{X,\mathbb{C}}) = 0$, so the Čech 2-cocycle ω is a Čech 2-coboundary. ((Does it matter that we consider "ordered" Čech cochains?))

Hence there exists a Čech 1-cochain η with $d\eta = \omega$. This η consists of complex 1-forms $\eta^{\alpha\beta}$ over $U_{\alpha\beta}$ for each $\alpha < \beta$, such that $\eta^{\beta\gamma} - \eta^{\alpha\gamma} + \eta^{\alpha\beta} = \omega^{\alpha\beta\gamma}$ over $U_{\alpha\beta\gamma}$. Let $\nabla^{\alpha\beta} = \tilde{\nabla}^{\alpha\beta} - \eta^{\alpha\beta}$ be a new connection on $L^{\alpha\beta}$, for all $\alpha < \beta$ in $\mathcal{I}$. Then

$$(\phi^{\alpha\beta\gamma})_*(\nabla^{\alpha\beta} + \nabla^{\beta\gamma}) = \nabla^{\alpha\gamma} ,$$

so ∇ is a connective structure on $\mathcal{L}$. □

Proof II. First choose 1-forms $\eta^{\alpha\beta}$ on $U_\alpha\beta$ for all $\alpha < \beta$ with $\alpha + 1 = \beta$, so that β is the successor of α. For $\alpha + 1 = \beta$ and $\beta + 1 = \gamma$ the 1-form $\eta^{\alpha\gamma}$ is determined on $U_{\alpha\beta\gamma}$ from the relation

$$\eta^{\beta\gamma} - \eta^{\alpha\gamma} + \eta^{\alpha\beta} = \omega^{\alpha\beta\gamma} .$$

We (should then be able to) choose an extension of $\eta^{\alpha\gamma}$ over $U_{\alpha\beta\gamma} \subseteq U_{\alpha\gamma}$. For $\alpha + 1 = \beta$, $\beta + 1 = \gamma$ and $\gamma + 1 = \delta$, the 1-form $\eta^{\alpha\delta}$ is determined on $U_{\alpha\beta\delta}$ by

$$\eta^{\beta\delta} - \eta^{\alpha\delta} + \eta^{\alpha\beta} = \omega^{\alpha\beta\delta}$$

and on $U_{\alpha\gamma\delta}$ by

$$\eta^{\gamma\delta} - \eta^{\alpha\delta} + \eta^{\alpha\gamma} = \omega^{\alpha\gamma\delta} .$$

These 1-forms agree on $U_{\alpha\beta\gamma\delta}$, as a consequence of the relation

$$\omega^{\beta\gamma\delta} - \omega^{\alpha\gamma\delta} + \omega^{\alpha\beta\delta} - \omega^{\alpha\beta\gamma} = 0\, m$$

so they define a 1-form on $\tilde{U}_{\alpha\delta} = U_{\alpha\beta\delta} \cup U_{\alpha\gamma\delta}$. We (should then be able to) choose an extension of $\eta^{\alpha\delta}$ over $\tilde{U}_{\alpha\delta} \subseteq U_{\alpha\delta}$.

Continuing like this, we determine $\eta^{\alpha\beta}$ for all $\alpha < \beta$ where $\alpha + k = \beta$ for finite k. □

(Classify the choices of connective structure.)

Proposition. *Let ∇ and $\bar{\nabla}$ be two connective structures on $\mathcal{L}$ over X. Then ((ETC))*

Proof. There are unique complex 1-forms $\eta^{\alpha\beta}$ on $U_{\alpha\beta}$ so that $\bar{\nabla}^{\alpha\beta} = \nabla^{\alpha\beta} + \eta^{\alpha\beta}$. This defines a Čech 1-cochain η. Since both ∇ and $\bar{\nabla}$ are connective structures, $\eta^{\alpha\beta} + \eta^{\beta\gamma} = \eta^{\alpha\gamma}$, so η is a Čech 1-cocycle. Since $\check{H}^1(\mathcal{U}; \underline{A}^1_{X,\mathbb{C}}) = 0$, this Čech 1-cocycle is a Čech 1-coboundary. ((ETC))

Proposition. *To each connective structure ∇ on a smooth charted 2-line bundle $\mathcal{L}$ over X there is a naturally associated connective structure Co on the associated smooth $\mathbb{C}^*$-gerbe $\mathcal{G}$ over X.*

Proof. Consider a local homeomorphism $f\colon Y \to X$ and an object $P \in \mathcal{G}(Y)$, consisting of complex line bundles P^α over $Y_\alpha = f^{-1}(U_\alpha)$ for each $\alpha \in \mathcal{I}$, and isomorphisms $\xi^{\alpha\beta}\colon L^{\alpha\beta} \otimes P^\beta \xrightarrow{\cong} P^\alpha$ over $Y_{\alpha\beta} = f^{-1}(U_{\alpha\beta})$ for each $\alpha < \beta$. We need to define the $\underline{A}^1_{Y,\mathbb{C}}$-torsor $Co(P)$ of connections on P, and its restrictions over further local homeomorphisms $g\colon Z \to Y$.

Over Y the elements $D = (D^\alpha)_{\alpha\in\mathcal{I}}$ of $Co(P)$ consist of

(1) connections D^α on P^α over Y_α for all $\alpha \in \mathcal{I}$, such that
(2) the isomorphism $\xi^{\alpha\beta}\colon L^{\alpha\beta} \otimes P^\beta \xrightarrow{\cong} P^\alpha$ takes the tensor product connection $\nabla^{\alpha\beta} + D^\beta$ to D^α, for all $\alpha < \beta$, so that

$$(\xi^{\alpha\beta})_*(\nabla^{\alpha\beta} + D^\beta) = D^\alpha$$

over $Y_{\alpha\beta}$.

Given another such element, $\bar{D} = (\bar{D}^\alpha)_{\alpha\in\mathcal{I}}$, we can write $\bar{D}^\alpha = D^\alpha + \epsilon^\alpha$ for a unique $\epsilon^\alpha \in A^1(Y_\alpha;\mathbb{C})$, for all $\alpha \in \mathcal{I}$. By condition (2) for D and $\bar{D}$, it follows that $\epsilon^\alpha = \epsilon^\beta$ over $Y_{\alpha\beta}$, so that the ϵ^α are all restrictions of a global complex 1-form $\epsilon \in A^1(Y;\mathbb{C})$. This makes it clear that $A^1(Y;\mathbb{C})$ acts freely and transitively on the elements of $Co(P)$ over Y. The extension to a sheaf is clear, as is the conclusion that $Co(P)$ is a torsor under $\underline{A}^1_{Y,\mathbb{C}}$.

((Explain rest of requirements?)) □

Proposition. *To each smooth charted* 2*-line bundle* $\mathcal{L}$ *over* X*, there is an associated Čech cohomology class* $[\mathcal{L}]$ *in* $\check{H}^2(X;\underline{\mathbb{C}}^*_X) \cong H^2(X;\underline{\mathbb{C}}^*_X) \cong H^3(X;\mathbb{Z})$*.*

Proof. After possibly refining the ordered cover $(\mathcal{U},\mathcal{I})$, we may assume that the gluing line bundles $L^{\alpha\beta}$ admit smooth sections $s^{\alpha\beta}$, giving trivializations $U_{\alpha\beta}\times\mathbb{C} \xrightarrow{\cong} L^{\alpha\beta}$. Comparing these trivializations via $\phi^{\alpha\beta\gamma}$, there is a unique smooth function $h^{\alpha\beta\gamma} \colon U_{\alpha\beta\gamma} \to \mathbb{C}^*$ such that

$$\phi^{\alpha\beta\gamma}(s^{\alpha\beta} \otimes s^{\beta\gamma}) = h^{\alpha\beta\gamma} \cdot s^{\alpha\gamma} \,.$$

Letting $\alpha < \beta < \gamma$ vary, the $h^{\alpha\beta\gamma}$ combine to a Čech 2-cochain $h \in \check{C}^2(\mathcal{U};\underline{\mathbb{C}}^*_X)$. By property (3) in the definition of a charted 2-line bundle, h is a Čech 2-cocycle, and $[\mathcal{L}]$ is its cohomology class in $\check{H}^2(X;\underline{\mathbb{C}}^*_X)$.

((Should check independence of choice of sections $s^{\alpha\beta}$.)) □

(For a suitable notion of equivalence of smooth charted 2-line bundles, this rule induces an isomorphism between the group of equivalence classes of smooth charted 2-line bundles over X and the cohomology group $H^3(X;\mathbb{Z})$.)

Associated to the complex

$$0 \to \underline{\mathbb{C}}^*_X \xrightarrow{d\log} \underline{A}^1_{X,\mathbb{C}} \to 0$$

of abelian sheaves over X, and an ordered open cover $(\mathcal{U},\mathcal{I})$ of X, we can associate the Čech bicomplex

$$0 \to \check{C}^*(\mathcal{U};\underline{\mathbb{C}}^*_X) \xrightarrow{d\log} \check{C}^*(\mathcal{U};\underline{A}^1_{X,\mathbb{C}}) \to 0$$

with $\check{C}^p(\mathcal{U};\underline{\mathbb{C}}^*_X)$ in bidegree $(p,0)$ and $\check{C}^p(\mathcal{U};\underline{A}^1_{X,\mathbb{C}})$ in bidegree $(p,1)$. The cohomology of the corresponding total complex is the hypercohomology

$$\check{H}^*(\mathcal{U};\underline{\mathbb{C}}^*_X \xrightarrow{d\log} \underline{A}^1_{X,\mathbb{C}}) \,.$$

Its colimit over all ordered open covers is the Čech hypercohomology

$$\check{H}^*(X;\underline{\mathbb{C}}^*_X \xrightarrow{d\log} \underline{A}^1_{X,\mathbb{C}}) \,,$$

which for reasonable X agrees with sheaf hypercohomology H^*. There is a long exact sequence

$$\dots \to H^p(X;\underline{\mathbb{C}}^*_X \xrightarrow{d\log} \underline{A}^1_{X,\mathbb{C}}) \to H^p(X;\underline{\mathbb{C}}^*_X) \xrightarrow{d\log} H^p(X;\underline{A}^1_{X,\mathbb{C}}) \to \dots \,.$$

Here $H^p(X;\underline{A}^1_{X,\mathbb{C}}) = 0$ for $p \neq 0$, so there are isomorphisms

$$H^p(X;\underline{\mathbb{C}}^*_X \xrightarrow{d\log} \underline{A}^1_{X,\mathbb{C}}) \xrightarrow{\cong} H^p(X;\underline{\mathbb{C}}^*_X) \cong H^{p+1}(X,\mathbb{Z})$$

for all $p \geq 2$.

Proposition. *To each smooth charted 2-line bundle $\mathcal{L}$ over X with connective structure ∇, there is an associated Čech cohomology class $[\mathcal{L}, \nabla]$ in*

$$\check{H}^2(X; \underline{\mathbb{C}}^*_X \xrightarrow{d\log} \underline{A}^1_{X,\mathbb{C}}) .$$

(In view of the isomorphisms above, this says that the choice of connective structure is unique up to a suitable notion of equivalence.)

Proof. After possibly refining the ordered cover $(\mathcal{U}, \mathcal{I})$, we may choose sections $s^{\alpha\beta}$ in the complex line bundles $L^{\alpha\beta}$ over $U_{\alpha\beta}$. The connections $\nabla^{\alpha\beta}$ from the connective structure on $\mathcal{L}$ act on these sections, so that

$$\nabla^{\alpha\beta}(s^{\alpha\beta}) = \eta^{\alpha\beta} \cdot s^{\alpha\beta}$$

for unique complex 1-forms $\eta^{\alpha\beta}$ on $U_{\alpha\beta}$. These combine to a Čech 1-cochain $\eta \in \check{C}^1(\mathcal{U}; \underline{A}^1_{X,\mathbb{C}})$. It is not a Čech 1-cocycle, but its coboundary has a well-understood form, as we shall now see.

By assumption, the connections $\nabla^{\alpha\beta} + \nabla^{\beta\gamma}$ on $L^{\alpha\beta} \otimes L^{\beta\gamma}$ and $\nabla^{\alpha\gamma}$ on $L^{\alpha\gamma}$ are compatible under $\phi^{\alpha\beta\gamma}$ over $U_{\alpha\beta\gamma}$. Since

$$(\nabla^{\alpha\beta} + \nabla^{\beta\gamma})(s^{\alpha\beta} \otimes s^{\beta\gamma}) = (\eta^{\alpha\beta} + \eta^{\beta\gamma})(s^{\alpha\beta} \otimes s^{\beta\gamma})$$

we get

$$\nabla^{\alpha\gamma}(h^{\alpha\beta\gamma} \cdot s^{\alpha\gamma}) = (\eta^{\alpha\beta} + \eta^{\beta\gamma})(h^{\alpha\beta\gamma} \cdot s^{\alpha\gamma}) .$$

At the same time

$$\begin{aligned} \nabla^{\alpha\gamma}(h^{\alpha\beta\gamma} \cdot s^{\alpha\gamma}) &= dh^{\alpha\beta\gamma} \cdot s^{\alpha\gamma} + h^{\alpha\beta\gamma} \cdot \nabla^{\alpha\gamma}(s^{\alpha\gamma}) \\ &= dh^{\alpha\beta\gamma} \cdot s^{\alpha\gamma} + h^{\alpha\beta\gamma}\eta^{\alpha\gamma} \cdot s^{\alpha\gamma} \end{aligned}$$

so that

$$(\eta^{\alpha\beta} + \eta^{\beta\gamma})h^{\alpha\beta\gamma} = dh^{\alpha\beta\gamma} + h^{\alpha\beta\gamma}\eta^{\alpha\gamma}$$

which we can rewrite as

$$\eta^{\beta\gamma} - \eta^{\alpha\gamma} + \eta^{\alpha\beta} = (h^{\alpha\beta\gamma})^{-1}\, dh^{\alpha\beta\gamma} = d\log h^{\alpha\beta\gamma} .$$

Hence the pair $(h, -\eta)$ forms a Čech 2-cocycle in the total complex

$$\check{C}^*(\mathcal{U}; \underline{\mathbb{C}}^*_X \xrightarrow{d\log} \underline{A}^1_{X,\mathbb{C}}) .$$

We let $[\mathcal{L}, \nabla]$ be its associated Čech cohomology class.

((Should check independence of choice of sections $s^{\alpha\beta}$.)) □

Connective structures on 2-vector bundles

Definition. Let X be a smooth manifold, with an ordered open cover $(\mathcal{U}, \mathcal{I})$. Let $\mathcal{E} \to X$ be a smooth charted 2-vector bundle, with gluing bundle matrices $E^{\alpha\beta}$ and coherence isomorphism matrices $\phi^{\alpha\beta\gamma}$. A **connective structure** ∇ on $\mathcal{E}$ is

(1) a matrix of connections $\nabla^{\alpha\beta} = (\nabla^{\alpha\beta}_{ij})^n_{i,j=1}$ on $E^{\alpha\beta}$ over $U_{\alpha\beta}$, for each $\alpha < \beta$ in $\mathcal{I}$, such that

(2) the matrix of isomorphisms $\phi^{\alpha\beta\gamma} \colon E^{\alpha\beta} \cdot E^{\beta\gamma} \xrightarrow{\cong} E^{\alpha\gamma}$ over $U_{\alpha\beta\gamma}$ takes the matrix product of connections $\nabla^{\alpha\beta} \cdot id + id \cdot \nabla^{\beta\gamma}$ on $E^{\alpha\beta} \cdot E^{\beta\gamma}$ to the matrix of connections $\nabla^{\alpha\gamma}$ on $E^{\alpha\gamma}$, so that

$$(\phi^{\alpha\beta\gamma})_*(\nabla^{\alpha\beta} \cdot id + id \cdot \nabla^{\beta\gamma}) = \nabla^{\alpha\gamma}$$

over $U_{\alpha\beta\gamma}$.

For matrices of sections $s = (s_{ij})^n_{i,j=1}$ in $E^{\alpha\beta}$ and $t = (t_{jk})^n_{j,k=1}$ in $E^{\beta\gamma}$, with $(s \cdot t)_{ik} = \bigoplus_{j=1}^n s_{ij} \otimes t_{jk}$ in $\bigoplus_{j=1}^n E^{\alpha\beta}_{ij} \otimes E^{\beta\gamma}_{jk}$, we have

$$(\nabla^{\alpha\beta} \cdot id + id \cdot \nabla^{\beta\gamma})(s \cdot t) = \nabla^{\alpha\beta}(s) \cdot t + s \cdot \nabla^{\beta\gamma}(t)$$

as 1-forms of matrices of sections in $E^{\alpha\beta} \cdot E^{\beta\gamma}$ over $U_{\alpha\beta\gamma}$. In the (i,k)-th entry, this asserts that

$$(\nabla^{\alpha\beta} \cdot id + id \cdot \nabla^{\beta\gamma})_{ik}(s \cdot t)_{ik} = \bigoplus_{j=1}^n (\nabla^{\alpha\beta}_{ij}(s_{ij}) \otimes t_{jk} + s_{ij} \otimes \nabla^{\beta\gamma}_{jk}(t_{jk}))\,.$$

Do 2-vector bundles admit connective structures?

Proposition. *Assume ((ETC)). Then $\mathcal{E}$ admits a connective structure ∇.*

Proof. Give each $E^{\alpha\beta}$ a matrix $\tilde\nabla^{\alpha\beta}$ of connections. For each $\alpha < \beta < \gamma$ we get a unique $\mathrm{End}(E^{\alpha\gamma})$-valued 1-form $\omega^{\alpha\beta\gamma}$ on $U_{\alpha\beta\gamma}$, such that

$$(\phi^{\alpha\beta\gamma})_*(\tilde\nabla^{\alpha\beta} \cdot id + id \cdot \tilde\nabla^{\beta\gamma}) = \tilde\nabla^{\alpha\gamma} + \omega^{\alpha\beta\gamma}\,.$$

Here $\omega^{\alpha\beta\gamma}_{ik}$ is a 1-form with values in the endomorphism bundle of $E^{\alpha\gamma}_{ik}$, for each $i, k = 1, \dots, n$.

We can change the connections on $E^{\alpha\beta}$ by $\mathrm{End}(E^{\alpha\beta})$-valued 1-forms $\eta^{\alpha\beta}$ on $U_{\alpha\beta}$, to

$$\nabla^{\alpha\beta} = \tilde\nabla^{\alpha\beta} - \eta^{\alpha\beta}\,.$$

Here $\eta^{\alpha\beta}_{ij}$ is a 1-form with values in the endomorphism bundle of $E^{\alpha\beta}_{ij}$, for each $i, j = 1, \dots, n$. Then

$$(\nabla^{\alpha\beta} \cdot id + id \cdot \nabla^{\beta\gamma})_{ik} = \\ (\tilde\nabla^{\alpha\beta} \cdot id + id \cdot \tilde\nabla^{\beta\gamma})_{ik} - \bigoplus_{j=1}^n (\eta^{\alpha\beta}_{ij} \otimes id_{E^{\beta\gamma}_{jk}} + id_{E^{\alpha\beta}_{ij}} \otimes \eta^{\beta\gamma}_{jk})$$

and

$$\nabla^{\alpha\gamma}_{ik} = \tilde\nabla^{\alpha\gamma}_{ik} - \eta^{\alpha\gamma}_{ik}\,.$$

For the $\nabla^{\alpha\beta}$ to specify a connective structure on $\mathcal{E}$, we need to have

$$-\eta_{ik}^{\alpha\gamma} + (\phi_{ik}^{\alpha\beta\gamma})_* \bigoplus_{j=1}^{n} (\eta_{ij}^{\alpha\beta} \otimes id_{E_{jk}^{\beta\gamma}} + id_{E_{ij}^{\alpha\beta}} \otimes \eta_{jk}^{\beta\gamma}) = \omega_{ik}^{\alpha\beta\gamma} ,$$

which we can rewrite as

$$(\phi^{\alpha\beta\gamma})_*(id_{E^{\alpha\beta}} \cdot \eta^{\beta\gamma}) - \eta^{\alpha\gamma} + (\phi^{\alpha\beta\gamma})_*(\eta^{\alpha\beta} \cdot id_{E^{\beta\gamma}}) = \omega^{\alpha\beta\gamma} .$$

The left hand side gives a generalized Čech-style coboundary operator

$$\prod_{\alpha<\beta} \Omega^1(U_{\alpha\beta}; \mathrm{End}(E^{\alpha\beta})) \xrightarrow{d} \prod_{\alpha<\beta<\gamma} \Omega^1(U_{\alpha\beta\gamma}; \mathrm{End}(E^{\alpha\gamma}))$$

taking the family of 1-forms $\eta = (\eta^{\alpha\beta})$ to $d\eta$. There is a similar operator

$$\prod_{\alpha<\beta<\gamma} \Omega^1(U_{\alpha\beta\gamma}; \mathrm{End}(E^{\alpha\gamma})) \xrightarrow{d} \prod_{\alpha<\beta<\gamma<\delta} \Omega^1(U_{\alpha\beta\gamma\delta}; \mathrm{End}(E^{\alpha\delta}))$$

taking a general family of 1-forms $\omega = (\omega^{\alpha\beta\gamma})$ to the family with value

$$(\phi^{\alpha\beta\delta})_*(id_{E^{\alpha\beta}} \cdot \omega^{\beta\gamma\delta}) - \omega^{\alpha\gamma\delta} + \omega^{\alpha\beta\delta} - (\phi^{\alpha\gamma\delta})_*(\omega^{\alpha\beta\gamma} \cdot id_{E^{\gamma\delta}})$$

at $\alpha < \beta < \gamma < \delta$.

Then $dd = 0$ and $d\omega = 0$, when ω is defined as at the beginning of the proof. ((Spell out the proofs.)) We would like to have $\omega = d\eta$. ((Is the complex exact?))

((ETC)) □

((Easier to give a Proof II like in the 2-line bundle case?))

((Can we specify a "connective structure" Co on the stack $\mathcal{P}$ with right $GL_n(\mathcal{V})$-action, associated to a 2-vector bundle $\mathcal{E}$?))