MAHOWALD'S INFINITE FAMILY

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1. Adams filtrations 1 and 2

Let $p = 2$ and write $H_*(X)$ for $H_*(X; \mathbb{Z}/2)$, H for $H\mathbb{Z}/2$, etc.

The E_2-term of the Adams spectral sequence

$$E_2^{s,t} = \mathrm{Ext}_A^{s,t}(\mathbb{Z}/2, \mathbb{Z}/2) \Longrightarrow \pi_{t-2}(S)_2^\wedge$$

can be computed as the homology of the normalized cobar complex

$$\mathbb{Z}/2 \xrightarrow{d^0} \bar{A}_* \xrightarrow{d^1} \bar{A}_* \otimes \bar{A}_* \xrightarrow{d^2} \bar{A}_* \otimes \bar{A}_* \otimes \bar{A}_* \to \dots$$

where

$$A_* = H_*(H) = \mathbb{Z}/2[\xi_k \mid k \geq 1]$$

is the dual Steenrod algebra with $|\xi_k| = 2^k - 1$, $\bar{A}_* = \mathrm{coker}(\eta\colon \mathbb{Z}/2 \to A_*)$. We write $[a_1|\dots|a_s]$ for $a_1 \otimes \dots \otimes a_s \in \bar{A}_*^{\otimes s}$.

Here $d^0 = 0$, so the Adams 0-line is just

$$E_2^{0,*} = \mathrm{Ext}_A^{0,*}(\mathbb{Z}/2, \mathbb{Z}/2) = \mathbb{Z}/2\{1\}$$

concentrated in bidegree $(s, t) = (0, 0)$. Here 1 detects the unit generator $\iota \in \pi_0(S)$.

Next, d^1 is induced by the coproduct $\psi\colon A_* \to A_* \otimes A_*$, dual to the composition in A,

$$\psi(\xi_k) = \sum_{i+j} \xi_i^{2^j} \otimes \xi_j \,.$$

Here $\xi_0 = 1$. So $\ker(d^1)$ consists of the primitives in A_*, with $\psi(x) = x \otimes 1 + 1 \otimes x$, which are spanned by the powers $\xi_1^{2^i}$ for $i \geq 0$. These are dual to the indecomposables Sq^{2^i} in A. Let $h_i = [\xi_1^{2^i}] \in \mathrm{Ext}_A^{1,2^i}(\mathbb{Z}/2, \mathbb{Z}/2)$. The Adams 1-line

$$E_2^{1,*} = \mathrm{Ext}_A^{1,*}(\mathbb{Z}/2, \mathbb{Z}/2) = \mathbb{Z}/2\{h_i \mid i \geq 0\}$$

is concentrated in bidegrees $(s, t) = (1, 2^i)$. Here h_0, h_1, h_2 and h_3 detect 2ι, η, ν and σ in $\pi_*(S)$.

The map d^2 is induced by the difference (= sum) of $\psi \otimes id$ and $id \otimes \psi\colon A_* \otimes A_* \to A_* \otimes A_* \otimes A_*$. Adams shows that the products

$$h_i h_j = [\xi_1^{2^i} | \xi_1^{2^j}]$$

Typeset by $\mathcal{A}_\mathcal{M}\mathcal{S}$-TEX

generate the kernel of d^2. Furthermore, the coproduct

$$\psi(\xi_2^{2^i}) = \xi_2^{2^i} \otimes 1 + \xi_1^{2^{i+1}} \otimes \xi_1^{2^i} + 1 \otimes \xi_2^{2^i}$$

implies the coboundary $d^1([\xi_2^{2^i}]) = [\xi_1^{2^{i+1}}|\xi_1^{2^i}] = h_{i+1}h_i$ for $i \geq 0$. Thus

$$h_i h_{i+1} = 0$$

in Ext_A. This is the only relation, so

$$E_2^{2,*} = \mathrm{Ext}_A^{2,*}(\mathbb{Z}/2, \mathbb{Z}/2) = \mathbb{Z}/2\{h_i h_j \mid 0 \leq i \leq j, j \neq i+1\}$$

is the Adams 2-line.

The products $h_i h_j h_k$ do not generate all of the Adams 3-line $\mathrm{Ext}_A^{3,*}(\mathbb{Z}/2, \mathbb{Z}/2)$. For example there is an indecomposable class $c_0 = \langle h_1, h_0, h_2^2 \rangle \in E_2^{3,11}$, which detects $\epsilon = \langle \eta, 2\iota, \nu^2 \rangle \in \pi_8(S)$. But Adams showed that the only relations between the product classes are

$$h_i^2 h_{i+2} = h_{i+1}^3$$

and

$$h_i h_{i+2}^2 = 0$$

for $i \geq 0$. Compare with the relations $4\nu = \eta^3$ and $2\nu^2 = 0$ in $\pi_*(S)$, for $i = 0$. The comparison is not as perfect in higher degrees, since $\eta^2\sigma = \nu^3 + \eta\epsilon$, but still $\eta\sigma^2 = 0$. Wang showed that the classes $c_i = \langle h_{i+1}, h_i, h_{i+2}^2 \rangle$ generate the rest of the Adams 3-line.

Adams proved that there is a spectral sequence differential

$$d_2(h_i) = h_0 h_{i-1}^2$$

which is nonzero in $\mathrm{Ext}_A^{3,*}(\mathbb{Z}/2, \mathbb{Z}/2)$ for all $i \geq 4$, so there are no further infinite cycles on the 1-line.

Mahowald and Tangora went on to prove that the differential

$$d_2(h_i h_j) = h_0 h_{i-1}^2 h_j + h_0 h_i h_{j-1}^2$$

is nonzero in $\mathrm{Ext}_A^{4,*}(\mathbb{Z}/2, \mathbb{Z}/2)$ for most cases of $0 \leq i < j$, which together with a few more known differentials (like $d_4(h_3 h_5) = h_0 x$, and later corrections, like $d_3(h_2 h_5) = h_0 p$) leaves only the two infinite families

$$h_i^2 \in E_3^{2,2^{i+1}}$$

for $i \geq 0$ and

$$h_1 h_j \in E_3^{2,2^j+2}$$

for $j \neq 0, 2$, and the finitely many cases

$$h_0 h_2, h_0 h_3, h_2 h_4$$

in the Adams 2-line at the E_3-term. The latter three are infinite cycles, represented in $\pi_*(S)$ by 2ν, 2σ and ν_*, respectively, in Toda's notation.

It is known that h_i^2 survives to E_∞ for $0 \leq i \leq 5$, represented by 4ι, η^2, ν^2, σ^2, θ_4 and θ_5, while for $i \geq 6$ this is the subject of the open Kervaire invariant problem. ((References for $\theta_4 = \langle \sigma, 2\sigma, \sigma, 2\sigma \rangle$, θ_5?))

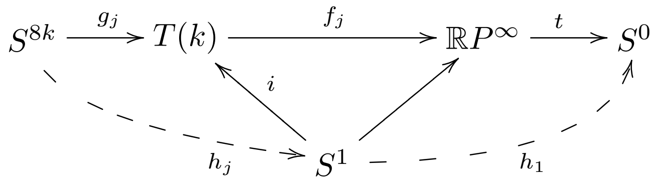
References: Adams (Comm. Math. Helv., 1958), Mahowald–Tangora (Topology, 1967), Mahowald–Tangora (Bol. Soc. Math. Mex., 1967), Barratt–Mahowald–Tangora (Topology, 1970), Ravenel's green book (now mauve).

2. Mahowald's η_j-family

Theorem (Mahowald). *For each $j \geq 3$ there are classes $\eta_j \in \pi_{2^j}(S)$ represented by $h_1 h_j \neq 0$ in the Adams spectral sequence.*

Let $k = 2^{j-3}$, so that $8k = 2^j$.

Sketch of proof. Mahowald constructs a finite CW spectrum $T(k)$ and stable maps



satisfying the following three **claims**:

(1) $H_1(i)$ and $H_1(f_j)$ are isomorphisms, so $f_j \circ i$ represents the bottom 1-cell in $\mathbb{R}P^\infty$,
(2) Sq^{2^j} acts nontrivially on H^1 of the mapping cone $C_{g_j} = T(k) \cup e^{8k+1}$ of g_j, so g_j is represented by $i_*(h_j)$ in $\mathrm{Ext}_A^{1,1}(H^*T(k), \mathbb{Z}/2)$, and
(3) t is the $\mathbb{Z}/2$-transfer map, so Sq^2 acts nontrivially on H^0 of the mapping cone $C_t = S^0 \cup C\mathbb{R}P^\infty$.

Let $\eta_j = t \circ f_j \circ g_j \colon S^{2^j} \to S^0$. Then η_j is represented by the Yoneda product

$$[t \circ f_j] \cdot [g_j] = [t \circ f_j] \cdot i_*(h_j) = [t \circ f_j \circ i] \cdot h_j$$

and since t maps the 1-cell in $\mathbb{R}P^\infty$ to S^0 by η, represented by h_1, this equals $h_1 h_j$. $\square$

Here S^{8k} has homology concentrated in degree $8k$, $T(k)$ has homology concentrated in degrees $1 \leq * \leq k$, $\mathbb{R}P^\infty$ has homology in degrees $* \geq 1$ and S^0 has homology concentrated in degree 0.

We shall discuss claim (1) in section 7, and claim (2) in section 6. Claim (3) is standard, so this will complete the proof.

References: Mahowald (Topology, 1977), Hunter–Kuhn (Math. Proc. Camb. Phil. Soc., 1999).

3. Brown–Gitler spectra

For $n \geq 0$, $T(n)$ will be Spanier–Whitehead n-dual to the Brown–Gitler spectrum $B(n)$, $T(n) = \Sigma^n DB(n)$. In the original paper, $B(n)$ is denoted $B([n/2])$. The following two **properties** characterize the (connective, 2-complete) spectrum $B(n)$ up to homotopy equivalence.

(1) $H^*(B(n)) = M(n) := A/A\{\chi Sq^i \mid 2i > n\}$, and
(2) the natural map $B(n)_n(Z) \to H_n(Z)$ is onto for each space Z.

Considering the admissible monomial basis for A, and conjugating, easily gives:

Lemma. *$H^*(B(n)) = M(n)$ is concentrated in degrees $0 \leq * \leq n - \alpha(n)$, where $\alpha(n)$ is the number of 1's in the binary expansion of n. Dually, $H^*(T(n)) = J(n)$ is concentrated in degrees $\alpha(n) \leq * \leq n$.*

For $n = k = 2^{j-3}$, $\alpha(n) = 1$ and $i\colon S^1 \to T(k)$ is the bottom cell. The top class of $M(k)$, in degree $k-1$, is generated by $\chi(Sq^{k/2} \dots Sq^2 Sq^1)$ on the bottom class 1, in degree 0, so k-dually $Sq^{k/2} \dots Sq^2 Sq^1$ acts isomorphically on $J(k)$ from the bottom class in degree 1, to the top class in degree k.

The Steenrod operation $Sq^{k/2} \cdots Sq^2 Sq^1$ acts isomorphically from H^1 to H^k for both $T(k)$ and $\mathbb{R}P^\infty$, since $H^*(\mathbb{R}P^\infty) = \mathbb{Z}/2[x]$ with $Sq^a(x^b) = \binom{b}{a} x^{a+b}$, so f_j induces an isomorphism on H_1 if and only if it induces an isomorphism on H_k.

References: Brown–Gitler (Topology, 1973), Brown–Peterson (Trans. A.M.S., 1978).

4. The May–Milgram model

Let $C_2(n)$ be the space of n little 2-cubes = squares. It is the n-th space in the little squares operad, with associated monad functor also denoted C_2. For any based space (Y, y_0) (a CW complex, say) there is a natural map

$$C_2(Y) := \coprod_{n \geq 0} C_2(n) \times_{\Sigma_n} Y^n/(\sim) \to \Omega^2 \Sigma^2 Y$$

which is a homotopy equivalence for path-connected Y.

Here, to a diagram c of n little disjoint squares in I^2, and an n-tuple $(y_1, \dots, y_n)$ of points in Y, a partial map $I^2 \to I^2 \times Y$ is defined to take the i-th little square, for $1 \leq i \leq n$ to $I^2 \times \{y_i\}$. Continuing by a map $I^2 \times Y \to \Sigma^2 Y$ that collapses $\partial I^2 \times Y \cup I^2 \times \{y_0\}$ to $*$, the partial map can be extended to all of I^2 by taking what lies outside of the little squares to $*$. Then ∂I^2 goes to $*$, so we have a map $S^2 \to \Sigma^2 Y$, or a point in $\Omega^2 \Sigma^2 Y$.

The left hand side, $C_2(Y)$, is naturally filtered by only allowing n to range up to some fixed number, and the filtration has subquotients

$$F_{n-1} C_2(Y) \subset F_n C_2(Y) \to C_2(n)_+ \wedge_{\Sigma_n} Y^{\wedge n} .$$

Each of these cofiber sequences is stably split, by Snaith, so

$$\Sigma^\infty \Omega^2 \Sigma^2 Y \simeq \bigvee_{n \geq 0} \Sigma^\infty C_2(n)_+ \wedge_{\Sigma_n} Y^{\wedge n} .$$

Mahowald and Brown–Peterson prove:

Proposition. *For Y an odd sphere, $C_2(n)_+ \wedge_{\Sigma^n} Y^{\wedge n}$ is a suspension of a Brown–Gitler spectrum $B(n)$.*

Taking $Y = S^7$, then

$$C_2(n)_+ \wedge_{\Sigma^n} S^{7n} \simeq \Sigma^{7n} B(n)$$

is stably a direct summand of $\Omega^2 S^9$. So to produce stable maps from $\Sigma^{7n} B(n)$, it suffices to produce a map from $\Omega^2 S^9$.

Remark (Arnold, Brown). There is an equivariant homotopy equivalence $C_2(n) \simeq F(\mathbb{R}^2, n)$, the configuration space of n distinct points $(p_1, p_2, \dots, p_n)$ in $\mathbb{R}^2 = \mathbb{C}$. The symmetric group Σ_n acts freely on $F(\mathbb{R}^2, n)$ by permuting the points, as well as on $\mathbb{R}^{2n} = \mathbb{C}^n$ by permuting the complex coordinates $(z_1, z_2, \dots, z_n)$. The associated

$\mathbb{R}^{2n}$-bundle with total space $F(\mathbb{R}^2, n) \times_{\Sigma_n} \mathbb{R}^{2n}$ does in fact admit a trivialization, given by

$$(p_1, p_2, \ldots, p_n; z_1, z_2, \ldots, z_n) \longmapsto (p_1, p_2, \ldots, p_n; \sum_i z_i, \sum_i p_i z_i, \ldots, \sum_i p_i^{n-1} z_i)$$

(using complex multiplications; all sums are over $1 \le i \le n$). Over the orbit of $(p_1, p_2, \ldots, p_n)$ the (Vandermonde) determinant of this linear map is $\pm \prod_{a<b}(p_a - p_b) \neq 0$. So the Thom complex $C_2(n)_+ \wedge_{\Sigma_n} (S^{2\ell+1})^{\wedge n}$ is $\Sigma^{2\ell n}(C_2(n)_+ \wedge_{\Sigma_n} S^n)$.

References: Cohen–Mahowald–Milgram (Proc. Symp. Pure Math., 32(2), 1978), May (The geometry of iterated loop spaces, 1972), Milgram (Annals of Math., 1966), Snaith (J. London Math. Soc., 1974),

5. Thom complexes of bundles over suspensions

Let (X, x_0) be a based space, $g\colon X \to O$ the clutching map of a stable vector bundle $\gamma\colon \Sigma X \to BO$. Let

$$Jg\colon \Sigma^\infty X \to S$$

be the stable map $\Sigma^\infty X \to \Sigma^\infty X_+$ (taking x to $x - x_0$), followed by the left adjoint of

$$X_+ \xrightarrow{g_+} O_+ \xrightarrow{J} \Omega^\infty(S) = Q(S^0)\,.$$

Lemma. *The Thom spectrum $Th(\gamma \downarrow \Sigma X) = (\Sigma X)^\gamma$ is homotopy equivalent to the mapping cone C_{Jg} of the map $Jg\colon \Sigma^\infty X \to S$.*

Proof. Write g as a colimit of maps $g(n)\colon X(n) \to O(n)$. The Thom complex of γ restricted to $\Sigma X(n)$ is the n-th space in the Thom spectrum of γ. It equals the pushout in

$$\begin{array}{ccc} \Sigma^n_+(X) & \xrightarrow{\overline{Jg}_+} & S^n \\ \downarrow & & \downarrow \\ \Sigma^n_+(CX) & \longrightarrow & Th(\gamma)_n \end{array}$$

where $\overline{Jg}_+$ is left adjoint to $Jg_+\colon X(n)_+ \to O(n)_+ \to \Omega^n S^n$. This follows by collapsing one of the two cones in ΣX to a point. Stably, the S^n's on the left hand side corresponding to the base point x_0 can be split off, leaving $\Sigma^n X$ and $C\Sigma^n X$, respectively. $\square$

Corollary. *Under the Thom isomorphism $H^*(\Sigma X) \cong H^*(C_{Jg})$, the i-th Stiefel–Whitney class $w_i(\gamma) = \gamma^*(w_i)$ corresponds to $Sq^i(1)$.*

References: Mahowald (Topology, 1977).

6. Construction of g_j

Let $Y = S^7$, $X = \Omega^2\Sigma^2 Y = \Omega^2 S^9$.

Let $g\colon \Omega^2 S^9 \to O$ be Ω^2 of a map generating $\pi_9(BBO) = \pi_8(BO) \cong \mathbb{Z}$. The adjoint clutching map γ fits in a commutative diagram

$$\begin{array}{ccccc} S^8 & \xrightarrow{\eta} & \Omega S^9 & \xrightarrow{Bg} & BO \\ & & \uparrow{\scriptstyle \epsilon} & \nearrow{\scriptstyle \gamma} & \\ & & \Sigma X = \Sigma\Omega^2 S^9 & & \end{array}$$

where η and ϵ are the unit and counit for the loop–suspension adjunction.

Lemma. *$Bg \circ \eta$ generates $\pi_8(BO)$ and $(Bg \circ \eta)^*(w_8) \neq 0$.*

Proof. The first claim is by definition, and thus $Bg \circ \eta$ factors through $BString = BO\langle 8\rangle \to BO$, and $S^8 \to BString$ induces an isomorphism on π_8 and H^8, by the Hurewicz theorem. The claim is then that w_8 in $H^8(BO)$ restricts to the generator of $H^8(BString)$. To check this, start with $H^*(BSO) = \mathbb{Z}/2[w_i \mid i \geq 2]$ and use the Serre spectral sequence for $K(\mathbb{Z}/2, 1) \to BSpin \to BSO$ (pulled back from w_2) to compute

$$H^*(BSpin) = \mathbb{Z}/2[w_i \mid i \geq 4]/(w_5, Sq^4 w_5, \dots)\,,$$

and then use the Serre spectral sequence for $K(\mathbb{Z}, 3) \to BString \to BSpin$ to compute $H^*(BString) = \{1, w_8, \dots\}$. □

Lemma. *$(Bg)^*(w_{8\ell}) \neq 0$ for all $\ell \geq 1$.*

Proof. Map $S^8 \times \cdots \times S^8 \to \Omega S^9$ (ℓ copies of S^8) by the loop sum of ℓ copies of η. The H-map Bg takes loop sum in ΩS^9 to Whitey sum in BO, so by the Cartan formula $(Bg)^*(w_{8\ell})$ pulls back to the cross product in $H^*(S^8 \times \cdots \times S^8)$ of the generators in the ℓ copies of $H^8(S^8)$, which is nonzero. □

Lemma. *$\gamma^*(w_{2^j}) \neq 0$ for all $j \geq 3$.*

Proof. The Serre spectral sequence for the loop–path fibration over ΩS^9 shows that $\epsilon^* \colon H^*(\Omega S^9) \to H^*(\Sigma\Omega^2 S^9)$ is injective in degree 2^j for all $j \geq 3$. □

Proof of claim (2). By the corollary and the last lemma, Sq^{2^j} acts non-trivially from degree 0 in the cohomology of the mapping cone of Jg.

Now restrict Jg to the kth stable summand ($k = 2^{j-3}$, $j \geq 3$)

$$\Sigma^{7k} B(k) \simeq C_2(k)_+ \wedge_{\Sigma_k} S^{7k} \hookrightarrow \Sigma^\infty \Omega^2 S^9 \xrightarrow{Jg} S\,.$$

The mapping cone of the composite map $\Sigma^{7k} B(k) \to S$ has cohomology $\mathbb{Z}/2 \oplus \Sigma^{7k+1} M(k)$, concentrated in degrees 0, $7k+1, \dots, 8k$, and one can check that $Sq^{2^j} = Sq^{8k}$ acts non-trivially.

Take Spanier–Whitehead $8k$-duals, to get the desired map

$$g_j \colon S^{8k} \to \Sigma^{8k} D \Sigma^{7k} B(k) = T(k)\,.$$

Then Sq^{2^j} also acts non-trivially in its mapping cone C_{g_j}, from degree 1.

The inclusion map $i \colon S^1 \to T(k)$ on the bottom cell is $8k$-dual to the pinch map $\Sigma^{7k} B(k) \to S^{8k-1}$ to the top cell. Thus $g_j \neq 0$ is represented in the image under

$$\mathbb{Z}/2\{h_j\} = \mathrm{Ext}_A^{1,1}(\Sigma^1 \mathbb{Z}/2, \Sigma^{8k} \mathbb{Z}/2) \xrightarrow{i_*} \mathrm{Ext}_A^{1,1}(H^*(T(k)); \Sigma^{8k} \mathbb{Z}/2)\,,$$

i.e., by $i_*(h_j)$. □

References: Mahowald (Topology, 1977).

7. Construction of f_j

The Adams spectral sequence edge homomorphism can be continued

$$\begin{aligned} B(n)_n(Z) \cong [T(n), Z] &\to \mathrm{Hom}_A(H^*(Z), H^*(T(n)) = J(n)) \\ &\to \mathrm{Hom}_{\mathbb{Z}/2}(H^n(Z), H^n(T(n)) = \mathbb{Z}/2) \cong H_n(Z)\,, \end{aligned}$$

taking f first to the A-module homomorphism $H^*(f)$ and then to the $\mathbb{Z}/2$-module homomorphism $H^n(f)$. By the (second) Brown–Gitler property, the composite map is onto for all spaces Z.

Remark. In fact, $J(n)$ is the injective hull of $\Sigma^n\mathbb{Z}/2$ in the category $\mathcal{U}$ of unstable A-modules, so the second arrow is an isomorphism whenever $H^*(Z)$ is an unstable A-module, which it is for spaces Z.

Proof of claim (1). Take $n = k = 2^{j-3}$, $Z = \mathbb{R}P^\infty$ and let

$$f_j \colon T(k) \to \mathbb{R}P^\infty$$

be a map that goes to the generator $a_k \in H_k(\mathbb{R}P^\infty)$ under the composite map above. So $H^k(f_j)$ is an isomorphism. By naturality of the Steenrod operations, we have a commutative square

$$\begin{array}{ccc} H^k(T(k)) & \xleftarrow[\cong]{H^k(f_j)} & H^k(\mathbb{R}P^\infty) \\ {\scriptstyle Sq^{k/2}\cdots Sq^2Sq^1}\uparrow\cong & & \cong\uparrow{\scriptstyle Sq^{k/2}\cdots Sq^2Sq^1} \\ H^1(T(k)) & \xleftarrow[H^1(f_j)]{} & H^1(\mathbb{R}P^\infty)\,. \end{array}$$

Here $Sq^{k/2}\cdots Sq^2Sq^1$ is an isomorphism from H^1 in both $H^*(T(k))$ and $H^*(\mathbb{R}P^\infty)$, so also $H^1(f_j)$ is an isomorphism. $\square$

References: Schwartz (Unstable modules over the Steenrod algebra and Sullivan's fixed point set conjecture, 1994).

8. Brown–Gitler's approach to $B(n)$

9. Mahowald's approach to $B(n)$

10. Goerss–Lannes–Morel's approach to $B(n)$