TWO-PRIMARY ALGEBRAIC K-THEORY OF POINTED SPACES

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Introduction.

I wish to review the relationship between geometric topology and algebraic K-theory, and to discuss how results in algebraic K-theory from the 1990's give some new information about geometric topology.

Let M be a compact manifold.

Isometries.

If M comes equipped with a Riemannian metric, then we may consider the topological group $\mathrm{Isom}(M)$ of isometries $\phi\colon M \to M$. By a theorem of Myers and Steenrod, this is a compact Lie group. For example, $\mathrm{Isom}(D^n) = O(n)$ is the orthogonal group, when D^n has the usual metric.

There are interesting things to say about the homotopy type of $O(n)$.

Its homotopy groups $\pi_i O(n)$ are only partially known. Their determination is in general at least as hard as the computation of the homotopy groups of spheres, which is very hard.

But there is a stabilization map $O(n) \to O(n+1)$, which is $(n-1)$-connected. Letting $O = \mathrm{colim}_n\, O(n)$, the homotopy groups of $O(n)$ are the same up to degree $(n-1)$ as the homotopy groups of the stabilized group O.

By the Bott periodicity theorem, the homotopy groups of O are completely understood. There is a homotopy equivalence $\Omega^8 O \simeq O$, and $\pi_i O$ is 8-periodic, starting $\mathbb{Z}/2$, $\mathbb{Z}/2$, 0, $\mathbb{Z}$, 0, 0, 0, $\mathbb{Z}$ for $i = 0, \dots, 7$.

Thus for n large, we understand the homotopy groups of $\mathrm{Isom}(D^n)$ is a large range.

CAT **automorphisms.**

Suppose instead that M is a smooth or a topological manifold. Briefly, it is a CAT manifold, where CAT is DIFF or TOP.

In each case we may consider the collection of CAT isomorphisms $\phi\colon M \to M$, viewed as a topological group, and inquire about its homotopy type. Immediately much less is known, even for $M = D^n$.

A uniform way to view the CAT automorphisms of M as a space is to define it as a simplicial set. Let $\mathrm{CAT}(M)$ be the simplicial group with q-simplices $\mathrm{CAT}(M)_q$ the group of CAT automorphisms

$$\phi\colon M \times \Delta^q \xrightarrow{\cong} M \times \Delta^q$$

commuting with projection to Δ^q. Here one is of course obliged to allow manifolds with corners in the smooth case.

A second issue is what to do when M has a boundary. We write $\mathrm{CAT}(M)$ for the space of automorphisms fixing the boundary of M, and $\mathrm{CAT}(M, \partial M)$ for the larger space of automorphisms not necessarily fixing the boundary. If $N \subset \partial M$ has codimension 0, we write $\mathrm{CAT}(M, N)$ for the automorphisms of M fixing the closure of $\partial M \setminus N$.

Similarly, let $G(M)$ be the simplicial monoid of homotopy equivalences $M \to M$, fixing the boundary. For example, $G(D^n)$ consists of the two path components of $\Omega^n S^n$ of degree ± 1 maps.

Can we determine the homotopy groups of $\mathrm{CAT}(M)$ or $\mathrm{CAT}(M, \partial M)$ in a range of dimensions that increases to infinity with n ?

Surgery.

The surgery theory of Browder, Novikov, Sullivan and Wall addresses the classification of manifolds up to automorphism, but in its parametrized form it really considers manifolds up to a weaker form of automorphisms, namely block automorphisms.

Let $\widetilde{\mathrm{CAT}}(M)$ be the simplicial group with q-simplices $\widetilde{\mathrm{CAT}}(M)_q$ the group of CAT automorphisms

$$\psi \colon M \times \Delta^q \xrightarrow{\cong} M \times \Delta^q$$

taking $M \times F$ to $M \times F$ for each face $F \subset \Delta^q$.

The 0-simplices of $\mathrm{CAT}(M)$ and $\widetilde{\mathrm{CAT}}(M)$ are the same, but the 1-simplices of $\mathrm{CAT}(M)$ are isotopies, while the 1-simplices of $\widetilde{\mathrm{CAT}}(M)$ are concordances.

Surgery theory then gives a fiber sequence

$$G(M)/\widetilde{\mathrm{CAT}}(M) \to \mathrm{Map}(M, G/\mathrm{CAT}) \xrightarrow{\sigma} L^s(M)\,.$$

Here $L^s(M)$ is Quinn's surgery space, whose homotopy groups are Wall's L-groups for $w_1 \colon \pi_1(M) \to \mathbb{Z}/2$. These are known for a number of fundamental groups. The homotopy types of the spaces $G/\mathrm{DIFF} = G/O$ and G/TOP have also been carefully studied. Compare surveys by Brumfield, Madsen and Milgram. The mapping space consists of maps that take ∂M to the base point.

For example, $\pi_7(PL/O) \cong \pi_0(G(S^7)/\widetilde{\mathrm{DIFF}}(S^n)) \cong \mathbb{Z}/28$ is the group of concordance classes of smooth homotopy 7-spheres, i.e., the exotic 7-spheres.

Surgery theory thus gives a fairly good grasp on the block automorphism groups $\widetilde{\mathrm{CAT}}(M)$.

Concordance spaces.

The difference between $\mathrm{CAT}(M)$ and $\widetilde{\mathrm{CAT}}(M)$ is measured by the homogeneous space $\widetilde{\mathrm{CAT}}(M)/\mathrm{CAT}(M)$. Its homotopy type can be studied at the level of homotopy groups by a spectral sequence due to Hatcher, or approximated in a stable range by a space level interpretation of this spectral sequence due to Weiss and Williams.

In either case the needed input starts with the concordance space (or pseudoisotopy space)

$$C^{\mathrm{CAT}}(M) = \mathrm{CAT}(M \times I, M \times \{1\})$$

of CAT automorphisms of the cylinder $M \times I$, fixing the part $\partial M \times I \cup M \times \{0\}$ of the boundary.

There is a stabilization map

$$\sigma\colon C^{\mathrm{CAT}}(M) \to C^{\mathrm{CAT}}(M \times I)$$

which is known to be at least roughly $n/3$-connected, by a theorem of Igusa, where $n = \dim(M)$. Again we may form the colimit $\mathcal{C}^{\mathrm{CAT}}(M) = \operatorname{colim}_k C^{\mathrm{CAT}}(M \times I^k)$, called the stable concordance space. The map $C^{\mathrm{CAT}}(M) \to \mathcal{C}^{\mathrm{CAT}}(M)$ is at least roughly $n/3$-connected. This range of dimensions is called the concordance stable range. If we know the homotopy groups of the stable concordance space $\mathcal{C}^{\mathrm{CAT}}(M)$, then we also known the homotopy groups of the (unstable) concordance space $C^{\mathrm{CAT}}(M)$ in this range of dimensions.

Hatcher's spectral sequence.

For $s+t$ in the concordance stable range ($s+t \le n/3$ or so), Hatcher's spectral sequence is

$$\begin{aligned} E^2_{s,t} &= H_s(\mathbb{Z}/2, \pi_t \mathcal{C}^{\mathrm{CAT}}(M)) \\ &\Longrightarrow \pi_{s+t+1}(\widetilde{\mathrm{CAT}}(M)/\mathrm{CAT}(M)) \end{aligned}$$

for a suitable $\mathbb{Z}/2$-action (involution) arising from turning a concordance upside-down.

Weiss and Williams give a space level interpretation of this, using the CAT Whitehead spectrum $Wh^{\mathrm{CAT}}(M)$, as defined by Waldhausen (in a simple variant). Its underlying infinite loop space $\Omega^\infty Wh^{\mathrm{CAT}}(M)$ is a double delooping of the stable concordance space $\mathcal{C}^{\mathrm{CAT}}(M)$

$$\Omega^2\Omega^\infty Wh^{\mathrm{CAT}}(M) \simeq \mathcal{C}^{\mathrm{CAT}}(M)\,.$$

In either category $\pi_0 Wh^{\mathrm{CAT}}(M) = 0$, and $\pi_1 Wh^{\mathrm{CAT}}(M) \cong Wh_1(\pi_1(M))$ is the Whitehead group of $\pi_1(M)$. We write $Wh^{\mathrm{CAT}}_s(M))$ for the 1-connected cover.

Weiss and Williams construct a map

$$\widetilde{\mathrm{CAT}}(M)/\mathrm{CAT}(M) \to \Omega^\infty(E\mathbb{Z}/2_+ \wedge_{\mathbb{Z}/2} \Omega Wh^{\mathrm{CAT}}_s(M))$$

which is as highly connected as the concordance stable range for M. The homotopy orbit spectral sequence for the right hand side recovers the Hatcher spectral sequence in this stable range.

To summarize, if the homotopy type of the stable concordance space $\mathcal{C}^{\mathrm{CAT}}(M)$ is known, or equivalently that of the Whitehead spectrum $Wh^{\mathrm{CAT}}(M)$, including knowledge of the relevant involution, then we can in principle compute the homotopy of $\widetilde{\mathrm{CAT}}(M)/\mathrm{CAT}(M)$ in the concordance stable range. Together with knowledge of $\widetilde{\mathrm{CAT}}(M)$ from surgery theory, this gives us information about the homotopy of the CAT automorphism group $\mathrm{CAT}(M)$ in the same range of dimensions.

Discs and spheres.

In the case of diffeomorphisms of discs moving the boundary there is a direct approach. There is a fibration

$$\Delta\colon \mathrm{DIFF}(D^n, \partial D^n) \to E^n \times GL_n(\mathbb{R}) \simeq O(n)$$

with E^n the interior of D^n, taking a diffeomorphism $\phi\colon D^n \to D^n$ to its derivative at the origin. The fiber at the identity is the subgroup of diffeomorphisms of D^n that keep the origin tangentially fixed. This deformation retracts to the subgroup of diffeomorphisms of D^n keeping a smaller concentric disc $\epsilon D^n \subset D^n$ fixed, which equals the group of diffeomorphisms of the annulus $S^{n-1} \times I$ keeping one end fixed. This is $C^{\mathrm{DIFF}}(S^{n-1})$. Hence there is a fiber sequence $C^{\mathrm{DIFF}}(S^{n-1}) \to \mathrm{DIFF}(D^n, \partial D^n) \to O(n)$, which is split by the embedding of $O(n)$ into $\mathrm{DIFF}(D^n, \partial D^n)$ that views linear isometries as diffeomorphisms. Hence

$$\mathrm{DIFF}(D^n, \partial D^n) \simeq O(n) \times C^{\mathrm{DIFF}}(S^{n-1})$$

and the smooth concordance space $C^{\mathrm{DIFF}}(S^{n-1})$ measures the difference between the groups of linear isometries and diffeomorphisms of D^n.

Algebraic K-theory of spaces.

The fundamental link between geometric topology and algebraic K-theory goes by way of Waldhausen's algebraic K-theory of spaces. This is a homotopy functor $X \mapsto A(X)$, where X is a space and $A(X)$ is a spectrum called the algebraic K-theory of the space X.

In the smooth case, Waldhausen constructs a split cofiber sequence of spectra

$$\Sigma^\infty X_+ \to A(X) \to Wh^{\mathrm{DIFF}}(X)\,.$$

Here $\Sigma^\infty X_+$ is the suspension spectrum on X, whose underlying infinite loop space is $Q(X_+)$. Hence $Wh^{\mathrm{DIFF}}(X)$ is determined as a retract up to homotopy of $A(X)$.

In the topological case, which in this stable setting agrees with the PL case, there is also a cofiber sequence of spectra

$$A(*) \wedge X_+ \xrightarrow{\alpha} A(X) \to Wh^{\mathrm{TOP}}(X)\,,$$

but this does typically not split. Here α is the assembly map for A-theory. Hence $Wh^{\mathrm{TOP}}(X)$ is determined by an understanding of $A(X)$ and this assembly map.

For example, taking $X \simeq *$ to be contractible, $A(*) \simeq \Sigma^\infty S^0 \vee Wh^{\mathrm{DIFF}}(*)$ and $Wh^{\mathrm{TOP}}(*) \simeq *$.

$\mathbb{S}$-algebras.

Waldhausen's algebraic K-theory of spaces $A(X)$ can be subsumed into a wider framework that also includes Quillen's algebraic K-theory of rings $K(R)$, namely the algebraic K-theory of $\mathbb{S}$-algebras.

In recent years, several equivalent categories of spectra have been constructed that admit a coherently associative, unital and commutative smash product $\wedge$. These are the S-modules of Elmendorf, Kriz, Mandell and May, the symmetric spectra of Hovey, Shipley and Smith, and the Γ-spaces of Segal, as reconsidered by Lydakis. In each case the unit for the smash product is a version of the sphere spectrum, which we call $\mathbb{S}$.

A monoid in such a category is a spectrum A, equipped with a multiplication map $\mu\colon A \wedge A \to A$ and a unit map $\eta\colon \mathbb{S} \to A$, satisfying associativity and unit conditions. Such a spectrum is called an $\mathbb{S}$-algebra. If the multiplication is commutative, it is called a commutative $\mathbb{S}$-algebra.

These are then strict notions of ring spectra and commutative ring spectra, respectively, closely related to the older notions of A_∞ and E_∞ ring spectra.

A ring R, which is the same as a $\mathbb{Z}$-algebra, has an associated Eilenberg–Mac Lane spectrum HR representing cohomology with coefficients in R. Then there is a model for HR as an $\mathbb{S}$-algebra, which is commutative if R is a commutative ring. In this way $R \mapsto HR$ embeds the category of (commutative) rings into the category of (commutative) $\mathbb{S}$-algebras.

A connected, based space X has a Kan loop group $G = G(X)$, with $X \simeq BG$. Here G is a simplicial group and BG is its classifying space. The suspension spectrum $\Sigma^\infty(G_+)$ has a multiplication coming from the group product in G, and has a model $\mathbb{S}[G]$ as an $\mathbb{S}$-algebra, which we can think of as the group ring of G over the sphere spectrum $\mathbb{S}$. If G is commutative, $\mathbb{S}[G]$ is a commutative $\mathbb{S}$-algebra, but this only happens when $X \simeq BG$ is a product of (commutative) Eilenberg–Mac Lane spaces. In this way $X \mapsto \mathbb{S}[G]$ embeds the category of (connected, based) spaces into the category of $\mathbb{S}$-algebras.

Later we shall return to the simplest case $X = *$, $G = 1$, corresponding to the $\mathbb{S}$-algebra $\mathbb{S}$.

There are many other $\mathbb{S}$-algebras, such as the bordism spectra MO, MU and the topological K-theory spectra KO, KU, which do not come directly from either (simplicial) rings or spaces. In a sense these interpolate between these two extreme examples.

Algebraic K-theory.

To each $\mathbb{S}$-algebra A, one can form an algebraic K-theory spectrum $K(A)$. When $A = HR$ is an Eilenberg–Mac Lane spectrum, $K(HR) \simeq K(R)$ agrees with Quillen's algebraic K-theory of rings. When $A = \mathbb{S}[G]$ is a group ring over the sphere spectrum, $K(\mathbb{S}[G]) \simeq A(X)$ agrees with Waldhausen's algebraic K-theory of the space $X = BG$.

Thus for special classes of $\mathbb{S}$-algebras A, the algebraic K-theory $K(A)$ is known to have particular interpretations in terms of other mathematical fields. When $R = \mathcal{O}_F$ is the ring of integers in a number field F, the homotopy groups of $K(\mathcal{O}_F)$ are tightly linked to the algebraic number theory of F. When $X = M$ is a CAT manifold, $A(X)$ is tightly linked to the geometric topology of M. In most intermediate cases we do not yet know of a direct interpretation of the algebraic K-theory.

In the number theory case R is a commutative ring, and the framework of algebraic geometry involving étale and motivic cohomology appears to be essential for understanding $K(R)$. In the case of non-commutative rings, and equally so for non-commutative $\mathbb{S}$-algebras like a general $\mathbb{S}[G]$, it is not clear whether there exists a suitable non-commutative algebraic geometry that can explain $A(X)$. But for commutative $\mathbb{S}$-algebras, like $\mathbb{S}$ itself, the situation is relatively good.

Rational information.

For $X \simeq BG$, there is a linearization map of $\mathbb{S}$-algebras $L\colon \mathbb{S}[G] \to H\mathbb{Z}[G]$, where the target is the Eilenberg–Mac Lane $\mathbb{S}$-algebra associated to the simplicial group ring $\mathbb{Z}[G]$. The linearization map is a 1-connected rational equivalence, and can be viewed as induced by a change of ground "ring" from $\mathbb{S}$ to $\mathbb{Z}$. It induces a map of algebraic K-theory spectra

$$L\colon A(X) = K(\mathbb{S}[G]) \xrightarrow{L} K(H\mathbb{Z}[G]) = K(\mathbb{Z}[G])$$

which is always a 2-connected rational equivalence.

Let $\pi = \pi_1(X) = \pi_0(G)$. If the natural map $X \to B\pi$ is k-connected then the induced map $A(X) \to A(B\pi)$ is also at least k-connected, so the composite map

$$A(X) \to A(B\pi) \to K(\mathbb{Z}[\pi])$$

is rationally at least k-connected.

When $\pi = 1$ is trivial, the rational homotopy type of $K(\mathbb{Z})$ was found by Borel, and similarly for π a finite group the rational homotopy type of $K(\mathbb{Z}[\pi])$ has been described in terms of the representation theory of π by Jahren. Hence for CAT manifolds X with finite fundamental group and k-connected universal cover, the rational homotopy type of $A(X)$ is known up to dimension k or so.

For example, with $X = D^n$, $A(D^n) \simeq A(*)$ is rationally equivalent to $K(\mathbb{Z})$, whose rational homotopy has rank 1 in degrees 0 and $4m+1$ for $m \geq 1$, and zero otherwise. It follows that $\mathcal{C}^{\text{DIFF}}(*) \simeq \Omega^2 Wh^{\text{DIFF}}(*)$ rationally has rank 1 in degrees $4m+3$ for $m \geq 0$, and zero otherwise. Farrell and Hsiang use this to show that in the concordance stable range the rational homotopy groups of $\text{DIFF}(D^n)$ have rank 1 in degrees $4m+3$ for $m \geq 0$ and n odd, and zero otherwise.

p-primary information.

If we wish to understand the integral homotopy type of the CAT automorphism spaces of manifolds, we need to understand the integral homotopy type of $A(X)$, not just its rational homotopy type. For connected X with $\pi = \pi_1(X)$ finite, $A(X)$ has finite type (Dwyer), so we can do this one prime at a time, by studying the p-adic completion of $A(X)$ for each prime p.

The maps $A(X) \to K(\mathbb{Z}[G]) \to K(\mathbb{Z}[\pi])$ do typically not induce an equivalence after p-adic completion, but in recent years it has become possible to measure the difference between $A(X)$, $K(\mathbb{Z}[G])$ and $K(\mathbb{Z}[\pi])$, using a version of Connes' cyclic homology due to Bökstedt, Hsiang and Madsen, called topological cyclic homology.

Let p be a prime. Topological cyclic homology is a homotopy functor $A \mapsto TC(A; p)$ from $\mathbb{S}$-algebras to spectra. We briefly write $TC(X; p) = TC(\mathbb{S}[G]; p)$ for a space $X \simeq BG$, and $TC(R; p) = TC(HR; p)$ for a ring R.

Relative theorems.

There is a kind of Chern character map, called the cyclotomic trace map, which is a natural transformation $\text{trc}_A \colon K(A) \to TC(A; p)$.

Theorem (McCarthy, Dundas). *Let $\phi\colon A \to B$ be a map of connective $\mathbb{S}$-algebras, such that $\pi_0(\phi)\colon \pi_0(A) \to \pi_0(B)$ is a surjection of rings with nilpotent kernel. Then the commutative square*

$$\begin{array}{ccc} K(A) & \xrightarrow{\phi} & K(B) \\ \downarrow{\scriptstyle \text{trc}_A} & & \downarrow{\scriptstyle \text{trc}_B} \\ TC(A;p) & \xrightarrow{\phi} & TC(B;p) \end{array}$$

becomes homotopy Cartesian after p-adic completion.

Goodwillie first proved a similar theorem about rational algebraic K-theory and rational (negative) cyclic homology. Then McCarthy proved the above theorem when $\phi\colon A \to B$ arises from a map of simplicial rings. Dundas showed how to extend the result to the generality of connective $\mathbb{S}$-algebras.

The composite linearization map $\mathbb{S}[G] \to \mathbb{Z}[\pi]$ is an example of such a map of $\mathbb{S}$-algebras. The induced map on π_0 is an isomorphism, which certainly is a surjection with nilpotent kernel.

Corollary. *Let X be a connected based space with fundamental group $\pi = \pi_1(X)$. The commutative square*

$$\begin{array}{ccc} A(X) & \xrightarrow{L} & K(\mathbb{Z}[\pi]) \\ \downarrow{\scriptstyle \mathrm{trc}_X} & & \downarrow{\scriptstyle \mathrm{trc}_{\mathbb{Z}[\pi]}} \\ TC(X;p) & \xrightarrow{L} & TC(\mathbb{Z}[\pi];p) \end{array}$$

becomes homotopy Cartesian after p-adic completion.

Thus to understand the p-adic homotopy type of $A(X)$, we may study the p-adic homotopy types of $TC(X;p)$, $K(\mathbb{Z}[\pi])$ and $TC(\mathbb{Z}[\pi];p)$, as well as the maps between these spectra. The latter two depend only on the fundamental group of X, and the former is quite well understood.

Topological cyclic homology of spaces.

The topological cyclic homology of a space X was determined by Bökstedt, Hsiang and Madsen.

Theorem (Bökstedt, Hsiang and Madsen). *There is a homotopy Cartesian square*

$$\begin{array}{ccc} TC(X;p) & \xrightarrow{\alpha} & \Sigma^\infty(\Sigma(\Lambda X_{hS^1})_+) \\ \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \mathrm{trf}_{S^1}} \\ \Sigma^\infty(\Lambda X)_+ & \xrightarrow{1-\Delta_p} & \Sigma^\infty(\Lambda X)_+ \end{array}$$

after p-adic completion.

We now explain the notation: The free loop space $\Lambda X = \mathrm{Map}(S^1, X)$ has a natural S^1-action rotating the loops. The p-th power map $\Delta_p \colon \Lambda X \to \Lambda X$ winds a loop p times around itself.

There is a principal S^1-bundle $\Lambda X \simeq ES^1 \times \Lambda X \to ES^1 \times_{S^1} \Lambda X = \Lambda X_{hS^1}$, with an associated dimension-shifting S^1-transfer map of suspension spectra

$$\mathrm{trf}_{S^1} \colon \Sigma^\infty(\Sigma(\Lambda X_{hS^1})_+) \to \Sigma^\infty(\Lambda X)_+ \,.$$

Its homotopy fiber can be identified as

$$\mathrm{hofib}(\mathrm{trf}_{S^1}) \simeq \Sigma Th(-\gamma^1 \downarrow \Lambda X_{hS^1}) \,,$$

i.e., the suspended Thom spectrum of minus the canonical complex line bundle γ^1 over the S^1-homotopy orbit space (Borel construction) ΛX_{hS^1}.

For $X = *$ we have $\Lambda X = *$ and $\Lambda X_{hS^1} = BS^1 \simeq \mathbb{C}P^\infty$. Then $\gamma^1 \downarrow \mathbb{C}P^\infty$ is the Hopf line bundle, and the Thom spectrum of minus γ^1 is the stunted complex projective space $\mathbb{C}P^\infty_{-1}$, which has a model as a CW spectrum with one cell in each even dimension ≥ -2.

Two homotopy Cartesian squares.

In the case of smooth manifolds, one choice of a splitting map for Waldhausen's cofiber sequence

$$\Sigma^\infty X_+ \to A(X) \to Wh^{\mathrm{DIFF}}(X)$$

can be factored as

$$A(X) \xrightarrow{\mathrm{trc}_X} TC(X;p) \xrightarrow{\beta} \Sigma^\infty \Lambda X_+ \xrightarrow{e_1} \Sigma^\infty X_+ \,.$$

Here $e_1 \colon \Lambda X \to X$ evaluates a free loop at the base point $1 \in S^1$. Hence we can identify $Wh^{\mathrm{DIFF}}(X)$ with the homotopy fiber of this composite map.

Let $\widetilde{TC}(X;p)$ be the homotopy fiber of the composite map

$$TC(X;p) \xrightarrow{\beta} \Sigma^\infty \Lambda X_+ \xrightarrow{e_1} \Sigma^\infty X_+$$

and let $\widetilde{\mathrm{trc}}_X \colon Wh^{\mathrm{DIFF}}(X) \to \widetilde{TC}(X;p)$ be the induced map of homotopy fibers.

Proposition. *The two square diagrams*

$$\begin{array}{ccccc} Wh^{\mathrm{DIFF}}(X) & \longrightarrow & A(X) & \xrightarrow{L} & K(\mathbb{Z}[\pi]) \\ \downarrow{\scriptstyle \widetilde{\mathrm{trc}}_X} & & \downarrow{\scriptstyle \mathrm{trc}_X} & & \downarrow{\scriptstyle \mathrm{trc}_{\mathbb{Z}[\pi]}} \\ \widetilde{TC}(X;p) & \longrightarrow & TC(X;p) & \xrightarrow{L} & TC(\mathbb{Z}[\pi];p) \end{array}$$

are homotopy Cartesian after p-adic completion, thus inducing homotopy equivalences

$$\mathrm{hofib}(\widetilde{\mathrm{trc}}_X) \xrightarrow{\simeq} \mathrm{hofib}(\mathrm{trc}_X) \xrightarrow{\simeq} \mathrm{hofib}(\mathrm{trc}_{\mathbb{Z}[\pi]})$$

of the vertical homotopy fibers.

In particular the homotopy type of $\mathrm{hofib}(\mathrm{trc}_X)$ only depends on the fundamental group of X. When $\pi = \pi_1(X)$ is implicit, we can write hofib(trc) for either one of these homotopy fibers.

There is a cofiber sequence of spectra

$$\Sigma Th(-\gamma^1 \downarrow \Lambda X_{hS^1}) \to \widetilde{TC}(X;p) \to \Sigma^\infty(\Lambda X/X)\,.$$

Here X is embedded in ΛX as the constant loops. For $X = *$ the right hand term is contractible, so $\widetilde{TC}(*;p) \simeq \Sigma Th(-\gamma^1 \downarrow BS^1) \simeq \Sigma \mathbb{C}P^\infty_{-1}$.

Corollary. *For $\pi = 1$ there is a cofiber sequence of spectra*

$$\mathbb{C}P^\infty_{-1} \xrightarrow{i} \mathrm{hofib}(\mathrm{trc}) \xrightarrow{j} Wh^{\mathrm{DIFF}}(*)$$

after p-adic completion.

For 1-connected X Hess and Rognes can often give algebraic models for the mod p spectrum cohomology of $\widetilde{TC}(X;p)$ in terms of the negative cyclic homology of a mod p cochain algebra model for ΩX.

If $X \to B\pi$ is k-connected then $TC(X;p) \to TC(B\pi;p)$ is also k-connected, so if k is in the concordance stable range of a manifold X and we are only interested in $A(X)$ or $TC(X;p)$ up to dimension k, then we may as well study $A(B\pi)$ and $TC(B\pi;p)$.

The fiber of the cyclotomic trace map.

For X simply-connected we can determine $\mathrm{hofib}(\mathrm{trc}_X)$ as the homotopy fiber of the cyclotomic trace map

$$\mathrm{trc}_{\mathbb{Z}} \colon K(\mathbb{Z}) \to TC(\mathbb{Z}; p)\,.$$

Here $TC(\mathbb{Z}; p)$ is derived from the S^1-equivariant topological Hochschild homology spectrum $THH(\mathbb{Z})$. Its homotopy type was found for $p \neq 2$ by Bökstedt and Madsen and by Rognes for $p = 2$.

The Lichtenbaum–Quillen conjectures in algebraic K-theory predict the homotopy type of $K(\mathbb{Z})$ in terms of étale cohomology groups. For $p = 2$ these conjectures follow from Voevodsky's proof of the Milnor conjecture, using further work of Bloch, Lichtenbaum, Rognes and Weibel. For $p \neq 2$ the status of these conjectures is not quite clear, but appear to follow from results in preparation by Voevodsky and Rost.

For $p = 2$ the map $K(\mathbb{Z}) \to TC(\mathbb{Z}; p)$ is also sufficiently well understood to describe its homotopy fiber.

Theorem (Rognes). *For $\pi = 1$ and $p = 2$ there is a cofiber sequence of spectra*

$$\Sigma^3 ko \to \mathrm{hofib}(\mathrm{trc}) \to \Sigma^{-2} ku \xrightarrow{\delta} \Sigma^4 ko$$

after p-adic completion. The connecting map ∂ is determined by its K-localization, which is the composite map

$$\Sigma^{-2} KU \underset{\simeq}{\xleftarrow{\beta}} KU \xrightarrow{\psi^3 - 1} KU \underset{\simeq}{\xleftarrow{\beta^2}} \Sigma^4 KU \xrightarrow{\Sigma^4 r} \Sigma^4 KO\,.$$

Here β is the Bott periodicity map, ψ^k is the kth Adams operation, and r is the realification map.

For odd regular primes p (in the sense of number theory) we can give a similar result if we assume the Lichtenbaum–Quillen conjecture about $K(\mathbb{Z})$ at p.

Proposition (Rognes). *For $\pi = 1$ and $p \neq 2$ a regular prime for which the Lichtenbaum–Quillen conjecture holds for $\mathbb{Z}$, there is a cofiber sequence of spectra*

$$j \vee \Sigma^2 ko \to \mathrm{hofib}(\mathrm{trc}) \to \Sigma^{-2} H\mathbb{Z}$$

after p-adic completion.

Here j is the connective image of j spectrum at p.

For irregular primes p, the identification of the cyclotomic trace map $K(\mathbb{Z}) \to TC(\mathbb{Z}; p)$ amounts to the identification of the map induced by the completion homomorphism $\mathbb{Z} \to \mathbb{Z}_p$ on algebraic K-theory. This is conjectured to involve multiplication by values of a p-adic L-function on the torsion free parts, but generally seems to be a difficult number-theoretic problem.

For non-simply connected spaces X we are led to ask about the algebraic K-theory and topological cyclic homology of $\mathbb{Z}[\pi]$ with $\pi \neq 1$. I believe little is known in general about $K(\mathbb{Z}[\pi])$ at p for finite non-abelian π, but for $\pi = C_q$ cyclic of prime order q there is an excision result (Charney, Weibel) asserting that

$$\begin{array}{ccc} K(\mathbb{Z}[C_q]) & \longrightarrow & K(\mathbb{Z}[\zeta_q]) \\ \downarrow & & \downarrow \\ K(\mathbb{Z}) & \longrightarrow & K(\mathbb{F}_q) \end{array}$$

is homotopy Cartesian after localizing away from q (inverting q). Assuming the Lichtenbaum–Quillen conjecture at a prime p (which is a theorem for $p = 2$), it is possible to identify hofib(trc) for $\pi = C_q$ at p for primes $q \neq p$ that are number-theoretically simple at the prime p.

Cohomology of the smooth Whitehead spectrum of a disc.

In the initial case $X \simeq *$, $\pi = 1$, $p = 2$, we can assemble these results to determine the mod p spectrum cohomology of $Wh^{\mathrm{DIFF}}(X)$ as a module over the Steenrod algebra A.

We have a diagram of horizontal and vertical cofiber sequences

$$\begin{array}{ccccccc}
 & & \mathbb{C}P^{\infty}_{-1} & = & \mathbb{C}P^{\infty}_{-1} & & \\
 & & \downarrow i & & \downarrow \epsilon & & \\
\Sigma^3 ko & \longrightarrow & \mathrm{hofib}(\mathrm{trc}) & \longrightarrow & \Sigma^{-2} ku & \xrightarrow{\delta} & \Sigma^4 ko \\
\| & & \downarrow j & & \downarrow & & \| \\
\Sigma^3 ko & \longrightarrow & Wh^{\mathrm{DIFF}}(*) & \longrightarrow & \Sigma\,\mathrm{hofib}(\epsilon) & \longrightarrow & \Sigma^4 ko \\
 & & \downarrow & & \downarrow & & \\
 & & \Sigma\mathbb{C}P^{\infty}_{-1} & = & \Sigma\mathbb{C}P^{\infty}_{-1}\,. & &
\end{array}$$

The mod 2 spectrum cohomologies of $\mathbb{C}P^{\infty}_{-1}$, ko and ku are all cyclic as modules over the mod 2 Steenrod algebra A:

Proposition.

(1) $H^*(\mathbb{C}P^{\infty}_{-1}; \mathbb{F}_2) \cong \Sigma^{-2} A/C$ *where* $C \subset A$ *is the left ideal with basis all the admissible monomials* Sq^I *other than* Sq^k *with* $k \geq 0$ *even.*

(2) $H^*(ko; \mathbb{F}_2) \cong A/A\{Sq^1, Sq^2\}$.

(3) $H^*(ku; \mathbb{F}_2) \cong A/A\{Sq^1, Sq^3\}$.

Using only that $Wh^{\mathrm{DIFF}}(*)$ is connective, we find that δ induces the zero map on cohomology, while i and ϵ induce surjections on cohomology. Hence there are horizontal and vertical extensions of left A-modules:

$$\begin{array}{ccccc}
 & & \Sigma^{-2} A/C & = & \Sigma^{-2} A/C \\
 & & \uparrow & & \uparrow \\
\Sigma^3 A/A\{Sq^1, Sq^2\} & \longleftarrow & H^*(\mathrm{hofib}(\mathrm{trc}); \mathbb{F}_2) & \longleftarrow & \Sigma^{-2} A/A\{Sq^1, Sq^3\} \\
\| & & \uparrow & & \uparrow \\
\Sigma^3 A/A\{Sq^1, Sq^2\} & \longleftarrow & H^*(Wh^{\mathrm{DIFF}}(*); \mathbb{F}_2) & \longleftarrow & \Sigma^{-2} C/A\{Sq^1, Sq^3\}\,.
\end{array}$$

Using that $\pi_3 Wh^{\mathrm{DIFF}}(*) = \mathbb{Z}/2$ (Bökstedt–Waldhausen) we can identify the lower extension.

Theorem (Rognes). *There is an extension of A-modules*

$$0 \leftarrow \Sigma^3 A/A\{Sq^1, Sq^2\} \leftarrow H^*(Wh^{\mathrm{DIFF}}(*); \mathbb{F}_2) \leftarrow \Sigma^{-2} C/A\{Sq^1, Sq^3\} \leftarrow 0\,.$$

It is the unique non-trivial such extension.

Giving the cohomology as a module over the Steenrod algebra is about as complete a characterization of a spectrum of this form as one can expect to obtain in the language of algebraic topology.

This also determines the cohomology of $A(*) = K(\mathbb{S})$, since $A(*) \simeq \Sigma^\infty S^0 \vee Wh^{\mathrm{DIFF}}(*)$, so $H^*(A(*); \mathbb{F}_2) \cong \mathbb{F}_2 \oplus H^*(Wh^{\mathrm{DIFF}}(*); \mathbb{F}_2)$.

To obtain similar results about $H^*(Wh^{\mathrm{DIFF}}(*); \mathbb{F}_p)$ for odd regular primes p, one needs to identify the map induced on spectrum cohomology by $i\colon \mathbb{C}P^\infty_{-1} \to \mathrm{hofib}(\mathrm{trc})$. For $p = 2$ this was particularly easy, since $H^*(\mathbb{C}P^\infty_{-1}; \mathbb{F}_2)$ is cyclic as an A-module. For odd primes p this is not so, and a new idea would be needed.

Homotopy of the smooth Whitehead spectrum of a disc.

There is an Adams spectral sequence

$$E_2^{s,t} = \mathrm{Ext}_A^{s,t}(H^*(Wh^{\mathrm{DIFF}}(*); \mathbb{F}_2), \mathbb{F}_2) \Longrightarrow \pi_{t-s} Wh^{\mathrm{DIFF}}(*)^\wedge_2$$

converging to the 2-completed homotopy groups of $Wh^{\mathrm{DIFF}}(*)$.

By comparison with the Adams spectral sequences for $\mathbb{C}P^\infty_{-1}$ and hofib(trc), I have determined the differentials in this spectral sequence in degrees ≤ 21, yielding a calculation of $\pi_* Wh^{\mathrm{DIFF}}(*)^\wedge_2$ up to this degree.

Theorem (Rognes). *Modulo odd torsion groups:*

n	$\pi_n Wh^{\mathrm{DIFF}}(*)$	$\pi_n B(G/O)$
≤ 2	0	0
3	$\mathbb{Z}/2$	$\mathbb{Z}/2$
4	0	0
5	$\mathbb{Z}$	$\mathbb{Z}$
6	0	0
7	$\mathbb{Z}/2$	$\mathbb{Z}/2$
8	0	0
9	$\mathbb{Z}/2 \oplus \mathbb{Z}$	$\mathbb{Z}/2 \oplus \mathbb{Z}$
10	$(\mathbb{Z}/2)^2 \oplus \mathbb{Z}/8$	$\mathbb{Z}/2 \oplus \mathbb{Z}/2$
11	$\mathbb{Z}/2$	$\mathbb{Z}/2$
12	$\mathbb{Z}/4$	0
13	$\mathbb{Z}$	$\mathbb{Z}$
14	$\mathbb{Z}/4$	0
15	$(\mathbb{Z}/2)^2$	$(\mathbb{Z}/2)^2$
16	$\mathbb{Z}/2 \oplus \mathbb{Z}/8$	$\mathbb{Z}/2$
17	$(\mathbb{Z}/2)^2 \oplus \mathbb{Z}$	$\mathbb{Z}/2 \oplus \mathbb{Z}$
18	$(\mathbb{Z}/2)^2 \oplus \mathbb{Z}/32$	$(\mathbb{Z}/2)^3$

We have listed the 2-primary homotopy groups of $B(G/O)$ together with those of $Wh^{\mathrm{DIFF}}(*)$ for convenient comparison.

In general the complexity of the homotopy groups of $Wh^{\mathrm{DIFF}}(*)$ is at least as great as that of the sphere spectrum, so any such calculation can only be expected to be carried out in a finite range.

Diffeomorphisms of discs.

Recall the splitting

$$\mathrm{DIFF}(D^n, \partial D^n) \simeq O(n) \times C^{\mathrm{DIFF}}(S^{n-1})\,.$$

For k in the concordance stable range for $S^{n-1} = \partial D^n$, i.e., for $k \le n/3$ or so, there are isomorphisms

$$\pi_k C^{\mathrm{DIFF}}(S^{n-1}) \cong \pi_k \mathcal{C}^{\mathrm{DIFF}}(S^{n-1}) \cong \pi_{k+2} Wh^{\mathrm{DIFF}}(S^{n-1}) \cong \pi_{k+2} Wh^{\mathrm{DIFF}}(*)\,.$$

These are the groups computed modulo odd torsion above. Hence these groups measure the 2-primary difference between linear isometries and diffeomorphisms of discs in the concordance stable range.

For example, $\pi_7 Wh^{\mathrm{DIFF}}(*)^{\wedge}_2 = \mathbb{Z}/2$, implying that for $n \ge 19$ there is a (unique up to homotopy) nontrivial family of diffeomorphisms of D^n parametrized over S^5, modulo families of linear isometries.

Rigid tubes.

A smooth tube on D^n of index k is a codimension zero submanifold of $D^n \times [-1, 1]$ obtained by attaching a single handle of index k to $D^n \times [-1, 0]$. There is a space $T_k(D^n)$ of such tubes, and a stable tube space $\mathcal{T}(*)$ obtained by letting both the handle index and the coindex increase to infinity.

The Grassmannian $Gr_k(\mathbb{R}^n)$ of k-planes in $\mathbb{R}^n$ maps to $T_k(D^n)$ by taking a subspace $V^k \subset \mathbb{R}^n$ to a tube with handle core a hemisphere over the radius $1/2$ disc about the origin in V. As V moves about, so does the tube. After stabilizing, there is a map

$$rt\colon BO \to \mathcal{T}(*)$$

called the rigid tube map.

There are maps of horizontal fiber sequences of spaces

$$\begin{array}{ccccc}
G/O & \longrightarrow & BO & \xrightarrow{j} & BG \\
\downarrow & & \downarrow{\scriptstyle rt} & & \| \\
\Omega Wh^{\mathrm{DIFF}}(*) & \longrightarrow & \mathcal{T}(*) & \longrightarrow & BG \\
\| & & \downarrow & & \downarrow \\
\Omega Wh^{\mathrm{DIFF}}(*) & \longrightarrow & Q(S^0) & \longrightarrow & A(*)
\end{array}$$

based on Waldhausen's manifold models for $A(*)$. We call the map $hw\colon G/O \to \Omega Wh^{\mathrm{DIFF}}(*)$ the Hatcher–Waldhausen map. It is a rational equivalence (Bökstedt), and equally highly connected as the rigid tube map.

Proposition. *After 2-adic completion the rigid tube map $rt\colon BO \to \mathcal{T}(*)$ and Hatcher–Waldhausen map $G/O \to \Omega Wh^{\mathrm{DIFF}}(*)$ are precisely 8-connected.*

This can be reinterpreted in terms of which parametrized families of smooth tubes can be linearized to come from rigid tubes.