LECTURES ON THE STABLE PARAMETRIZED H-COBORDISM THEOREM

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We first present work of Hatcher, Waldhausen and others, on the relation of automorphism groups of manifolds to concordance theory, h-cobordism spaces and algebraic K-theory of spaces. See [Ha78], [Wa87b] and [WW01] for more detailed surveys. Then we proceed to present the relation of the algebraic K-theory of spaces to algebraic K-theory of rings and to topological cyclic homology.

These are informal lecture notes. Corrections and comments are welcome.

1. High-dimensional geometric topology

1.1. Automorphisms of manifolds.

We consider the three geometric categories of manifolds, CAT = TOP (topological), PL (piecewise linear) or DIFF (smooth). There is a forgetful functor from DIFF to PD (piecewise differentiable), which is equivalent to PL, and another one from PL to TOP.

We are interested in the CAT **automorphism groups** $CAT(M) = TOP(M)$, $PL(M)$ or $DIFF(M)$ of compact manifolds. When M is not closed, we assume that the automorphism is the identity on the boundary. These groups can be interpreted as a compact-open mapping space, a simplicial group and a C^∞ mapping space, respectively, or uniformly as simplicial groups. The simplicial group $[q] \mapsto CAT_q(M)$ has q-simplices the set of diagrams

$$\begin{array}{ccc} M \times \Delta^q & \xrightarrow{\ \phi\ } & M \times \Delta^q \\ & \searrow \quad \swarrow & \\ & \Delta^q & \end{array}$$

where ϕ is a CAT isomorphism. So $CAT_0(M)$ is the set of CAT automorphisms of M, while $CAT_1(M)$ is the set of CAT isotopies between the automorphisms at the two ends. Let $G(M)$ be the (simplicial) grouplike monoid of self-homotopy equivalences of M. We get homomorphisms

$$DIFF(M) \to PL(M) \to TOP(M) \to G(M)\,,$$

starting at the category where M is a manifold.

We are interested in the **moduli space** of all CAT manifolds, which classifies (locally trivial) CAT manifold bundles. It is homotopy equivalent to

$$\coprod_{[M]} BCAT(M)$$

where M ranges over the isomorphism classes of CAT manifolds. Compare with how the Grassmann space of all finite-dimensional real vector spaces is homotopy equivalent to $\coprod_{n\geq 0} BO(n)$.

1.2. Surgery.

Surgery theory classifies CAT manifolds of a given homotopy type, focusing on a fixed manifold dimension n, but in a weaker sense than the moduli space. Let $\widetilde{CAT}(M)$ be the CAT **block automorphism group**. It is the simplicial group with q-simplices the CAT isomorphisms

$$M \times \Delta^q \xrightarrow{\phi} M \times \Delta^q$$

that restrict to a map (and thus a CAT isomorphism) $\phi|\colon M \times F \to M \times F$ for each simplicial face $F \subset \Delta^q$. So $\widetilde{CAT}_0(M)$ is still the set of CAT automorphisms of M, but $\widetilde{CAT}_1(M)$ is the set of CAT **concordances**, or **pseudoisotopies**, between the automorphisms at the two ends. There is an inclusion $CAT(M) \subset \widetilde{CAT}(M)$. In the case $G(M) \subset \widetilde{G}(M)$, it is a homotopy equivalence. Surgery theory classifies manifolds up to block isomorphism, rather than up to isomorphism

Let $CAT(n) = CAT(\mathbb{R}^n, 0)$ be the simplicial group of (germs of) CAT automorphisms of $\mathbb{R}^n$ fixing the origin. Let $G(n)$ be the simplicial grouplike monoid of homotopy equivalences of $\mathbb{R}^n \setminus 0$, or equivalently, of the unit sphere S^{n-1}. There are group homomorphisms

$$DIFF(n) \to PL(n) \to TOP(n) \to G(n)\,,$$

where the last map restricts a homeomorphism to the complement of 0. There are stabilization maps $CAT(n) \to CAT(n+1)$, taking an automorphisms to its product with the identity on $\mathbb{R}$, and we let $CAT = \operatorname{colim}_n CAT(n)$. The inclusions $O(n) \subset DIFF(n)$ and $O \subset DIFF$ are homotopy equivalences. Stably, G is homotopy equivalent to the ± 1 components of $Q(S^0) = \Omega^\infty(S) = \operatorname{colim}_n \Omega^n S^n$.

Let G/CAT be the homotopy fiber of $BCAT \to BG$. For each CAT manifold M with stable normal (micro-)bundle $\nu\colon M \to BCAT$ the set of possible lifts ν' of the underlying stable spherical fibration $M \to BG$ to a CAT (micro-)bundle is in bijection with the set of homotopy classes $[M, G/CAT]$. The parametrized form of this set of surgery problems is the mapping space $\mathrm{Map}(M, G/CAT)$. The surgery obstruction to realizing ν' as the stable normal bundle of a CAT n-manifold, up to normal cobordism, is in the Wall L-group $L^s_n(M) = L^s_n(\pi, w)$, which only depends on the fundamental group $\pi = \pi_1(M)$ and the orientation character $w = w_1 \colon \pi \to \mathbb{Z}/2$ of M. In its parametrized form, it is a map θ to Quinn's surgery space $L^s(M)$.

The Browder–Novikov–Sullivan–Wall surgery exact sequence, see [Wl70, 17.A], is then the long exact sequence associated to a fiber sequence

$$\frac{G(M)}{\widetilde{CAT}(M)} \to \mathrm{Map}(M, G/CAT) \xrightarrow{\theta} L^s(M)\,,$$

where the left hand term is defined as the fiber of $B\widetilde{CAT}(M) \to B\widetilde{G}(M) \simeq BG(M)$. Here G/CAT and the L-groups are quite well understood. For example, by Sullivan and Kirby–Siebenmann [KS77], G/TOP is a product of Eilenberg–Mac Lane spaces when localized at 2 and homotopy equivalent to BSO when localized at an odd prime. For simply-connected M, the L-groups $L_n^s(1)$ are $\mathbb{Z}$ for $n = 4k$, $\mathbb{Z}/2$ for $n = 4k+2$ and 0 otherwise, and these are also the homotopy groups of G/TOP. In this sense, surgery theory computes the space $G(M)/\widetilde{CAT}(M)$ of manifolds homotopy equivalent to M, up to concordance, or in other words, up to block isomorphism.

When the homotopy type of M is well understood, e.g., if M is aspherical (= a $K(\pi, 1)$), the homotopy type of $BG(M)$ can also be worked out. In view of the fiber sequence

$$\frac{G(M)}{\widetilde{CAT}(M)} \to B\widetilde{CAT}(M) \to BG(M)$$

we will also grant that surgery theory identifies the component of M in the **block moduli space**

$$\coprod_{[M]} B\widetilde{CAT}(M)\,.$$

To get at the honest moduli space, or the honest automorphisms groups of M, we wish to identify the following homotopy fiber

$$\frac{\widetilde{CAT}(M)}{CAT(M)} \to BCAT(M) \to B\widetilde{CAT}(M)\,,$$

which measures the difference between blocked and unblocked automorphisms. For example, the difference between the isotopies in $CAT_1(M)$ and the pseudoisotopies in $\widetilde{CAT}_1(M)$ contributes to the lowest homotopy group of the fiber, namely π_1.

1.3. Concordances.

This difference is controlled by the **concordance space** $C(M) = C^{CAT}(M)$, also known as the **pseudoisotopy space**. It has various homotopy equivalent definitions. One is

$$C(M) = CAT(M \times I \ \text{ rel } \ M \times 0)\,,$$

i.e., the CAT automorphisms ψ of $M \times I$ that restrict to the identity on (a neighborhood of) $M \times 0$. A homotopy equivalent definition is

$$C(M) = CAT(M \times I \ \text{ rel } \ M \times 0 \cup \partial M \times I)\,.$$

When M is not closed, the latter is properly contained in the former, but the two groups are homotopy equivalent, by conjugation with a chosen CAT isomorphism of pairs

$$(M \times I, M \times 0) \cong (M \times I, M \times 0 \cup \partial M \times I)\,.$$

A third definition is

$$C(M) = \frac{\widetilde{CAT}_1(M)}{CAT_1(M)}$$

where the free pseudoisotopies $\widetilde{CAT}_1(M) = CAT(M \times I, M \times 0, M \times 1)$ consists of the automorphisms of $M \times I$ that restrict to automorphisms of $M \times 0$ and $M \times 1$, respectively, and the free isotopies $CAT_1(M) = CAT(M \times I \downarrow I)$ consists of the automorphisms of $M \times I$ that fiber over I, i.e., that commute with the projection to I. (Note: In this formulation of the definition, $CAT_1(M)$ and $\widetilde{CAT}_1(M)$ should be interpreted as the CAT automorphism spaces indicated, not as discrete sets.) The first definition maps isomorphically to $\widetilde{CAT}_1(M)/CAT(M)$, and $CAT(M) \subset CAT_1(M)$ is a homotopy equivalence.

By triangulation theory and Kirby–Siebenmann's result that $TOP(n)/PL(n) \simeq TOP/PL$ for $n \geq 5$, the inclusion $C^{PL}(M) \subset C^{TOP}(M)$ is a homotopy equivalence, for M a PL n-manifold for $n \geq 5$. However, $C^{DIFF}(M)$ is usually different from the other two concordance spaces.

There is a **Hatcher spectral sequence**

$$E^1_{s,t} = \pi_{t-1}(C^{CAT}(M \times I^s)) \Longrightarrow \pi_{s+t}\left(\frac{\widetilde{CAT}(M)}{CAT(M)}\right),$$

which is concentrated in the region $s \geq 0$, $t \geq 1$, see [Ha78]. We shall say more about its E^2-term below. Modulo differentials and extensions in this spectral sequence, the concordance spaces $C(M \times I^s)$ determine the difference between $CAT(M)$ and $\widetilde{CAT}(M)$.

We briefly explain the filtration. The group

$$\pi_n(\widetilde{CAT}(M)/CAT(M)) \cong \pi_n(\widetilde{CAT}(M), CAT(M))$$

is realized as the CAT automorphisms ψ of $M \times I^n$ that restrict to automorphisms of $M \times \partial I^n$ that fiber over ∂I^n, modulo those that fiber over I^n. Let $\bar{A}_n$ be this space. It is contained in the larger group $\bar{C}_{n-1}$ of CAT automorphisms ψ that are only required to fiber over $I^{n-1} \times 0 \cup \partial I^{n-1} \times I$. This larger group is homotopy equivalent to the concordance space $C(M \times I^{n-1})$, in view of the third definition above. Restriction to $M \times I^{n-1} \times 1 \cong M \times I^{n-1}$ identifies the quotient space $\bar{C}_{n-1}/\bar{A}_n$ with (some path components of) $\bar{A}_{n-1}$, so there is a Puppe fiber sequence

$$\Omega \bar{A}_{n-1} \to \bar{A}_n \to \bar{C}_{n-1} \to \bar{A}_{n-1}.$$

Roughly, we can identify loops in $\bar{A}_{n-1}$ with elements in $\bar{A}_n$ that fiber over I in the sense that they commute with the composite projection to $I^n \to I^n/I^{n-1} \cong I$. So these are the block automorphisms that can be "unblocked" in one of the parameters. Varying n one gets an exact couple and a spectral sequence, essentially filtering $\bar{A}_n$ by the extent to which ψ fibers over a quotient $I^n/I^s \cong I^{n-s}$.

There is a **stabilization** (= suspension) map

$$\sigma \colon C(M) \to C(M \times J),$$

where $J = [0, 1]$, like I, which in terms of the first definition of $C(M)$ takes an automorphism ψ of $M \times I$ that is the identity on $M \times 0$ to $\psi \times id$ on $M \times I \times J$, now fixing $M \times 0 \times J$. Identifying $M \times I \times J \cong M \times J \times I$, we have an element of $C(M \times J)$. In terms of the second definition of $C(M)$, the isomorphism of pairs mentioned yields a more curved image.

Iterating the stabilization, we obtain the **stable concordance space** $\mathcal{C}(M) = \mathcal{C}^{CAT}(M)$, defined as the homotopy colimit

$$\mathcal{C}(M) = \operatorname*{colim}_s C(M \times J^s).$$

In the smooth category, $CAT = DIFF$, Igusa's **stability theorem** for smooth pseudoisotopies (= concordances) asserts that the stabilization map $\sigma\colon C(M) \to C(M \times J)$ is k-connected if

$$\dim(M) \geq \max(2k + 7, 3k + 4)\,.$$

See [Ig88]. In other words, if $n = \dim(M)$ then σ is at least $[(n-4)/3]$-connected, for $n \geq 10$, and at least $[(n-7)/2]$-connected for $n < 10$. It follows that the stabilization map $\sigma^\infty\colon C(M) \to \mathcal{C}(M)$ is k-connected if $\dim(M) \geq \max(2k+7, 3k+4)$. We call the range of degrees in which σ^∞ is an isomorphism the **concordance stable range** of M.

It was shown by Burghelea and Lashof [BL77] that a PL stability theorem for concordances implies a DIFF stability theorem, with the known stable range doubled. This was relevant, as Hatcher was thought to have proved such a PL stability theorem as part of his parametrized h-cobordism theorem, in [Ha75]. Without Hatcher's result, presumably it is possible to turn the argument around and deduce a PL stability theorem from Igusa's DIFF theorem, but there does not seem to be a reference for this.

There is a canonical **involution** χ on $C(M)$, which is relevant to the spectral sequence above. In terms of the third definition, it conjugates an isomorphism in $CAT(M \times I, M \times 0, M \times 1)$ by the flip

$$\rho\colon M \times I \xrightarrow{\cong} M \times I$$

that reflects I about its mid-point. In symbols, $\chi(\psi) = \rho\psi\rho^{-1}$. This preserves the subgroup $CAT(M \times I \downarrow I)$.

In terms of the second definition, it maps ψ in $CAT(M \times I \text{ rel } M \times 0 \cup \partial M \times I)$ to its conjugate with the same flip ρ, followed by $\phi^{-1} \times id$, where ϕ is the image of ψ under the map r that restricts an automorphism to $M \times 1 \cong M$, and which appears in the fiber sequence

$$CAT(M \times I) \to C^{CAT}(M) \xrightarrow{r} CAT(M)\,.$$

The involution anti-commutes with stabilization, up to homotopy, so it induces an involution on $\mathcal{C}(M)$, at least at the level of homotopy groups, given as the colimit of $(-1)^s$ times the involution on $C(M \times I^s)$. So $C(M) \to \mathcal{C}(M)$ respects the involution.

((Describe $d^1_{s,t}$ on an element in $E^1_{s,t} = \pi_{t-1}C(M \times I^s)$ of the form $\sigma(\psi)$. It equals $\psi + \chi_{s-1}(\psi)$, where χ_{s-1} denotes the involution in $C(M \times I^{s-1})$. In the stable range, $E^1_{s,t} = \pi_{t-1}\mathcal{C}(M)$ and $d^1_{s,t}$ takes $\sigma(\psi)$ to $\psi + (-1)^{s-1}\chi(\psi)$, which stably equals $\sigma(\psi) + (-1)^s\chi(\sigma(\psi))$. So the d^1-differential takes x to $x + (-1)^s\chi(x)$ for $x \in E^1_{s,t}$.))

In the concordance stable range ($3t+4 \leq \dim(M)+s$ or so, for $CAT = DIFF$),

$$E^2_{s,t} = H^{gp}_s(\mathbb{Z}/2; \pi_{t-1}\mathcal{C}(M))$$

where $\mathbb{Z}/2$ acts on $\pi_*\mathcal{C}(M)$ via **minus** the given involution. ((Should double-check the signs!))

If we localize away from 2, i.e., study homotopy groups with 2 inverted, then the spectral sequence collapses to the vertical axis, in the stable range, with

$$E^2_{0,t} = \pi_{t-1}(\mathcal{C}^{CAT}(M))[\frac{1}{2}]^- \cong \pi_t(\frac{\widetilde{CAT}(M)}{CAT(M)})[\frac{1}{2}] .$$

Here $(-)^-$ indicates the coinvariants for minus the canonical involution, i.e., the (-1)-eigenspace for the involution.

1.4. h-Cobordisms.

A cobordism on a closed n-manifold M is a compact $(n+1)$-manifold W with $\partial W = M \sqcup M'$, a disjoint union. It is an **h-cobordism** if the two inclusions

$$M \subset W \supset M'$$

are both homotopy equivalences. For example, $W = M \times I$ is a trivial h-cobordism.

((Discuss the non-closed case.))

For connected M with fundamental group $\pi = \pi_1(M)$, the Whitehead group $\mathrm{Wh}(\pi) = \mathrm{Wh}_1(\pi)$ is defined as the quotient group

$$\mathrm{Wh}_1(\pi) = K_1(\mathbb{Z}[\pi])/(\pm\pi)$$

of the first algebraic K-group

$$K_1(R) = GL_\infty(R)^{ab} = \frac{GL_\infty(R)}{[GL_\infty(R), GL_\infty(R)]}$$

of the integral group ring $R = \mathbb{Z}[\pi]$. The Whitehead torsion of (W, M) is an element $\tau(W, M) \in \mathrm{Wh}_1(\pi)$, see [Mi66], and the s-cobordism theorem says that the pair (W, M) is isomorphic to the trivial h-cobordism $(M \times I, M \times 0)$ if and only if the Whitehead torsion $\tau(W, M)$ is zero, in which case we say that W is an s-cobordism. See [Mi65] for the simply-connected DIFF case.

In dimensions $n \geq 5$ all elements of the group $\mathrm{Wh}_1(\pi)$ are realized by the torsion of h-cobordisms, so given an h-cobordism W from M to some M', as above, there exists an inverse h-cobordism W' from M' to some M'' with the negative torsion, so that the combined h-cobordism $W \cup_{M'} W'$ is an s-cobordism, and thus isomorphic to $M \times I$. So W can be embedded as a codimension 0 submanifold of $M \times I$, taking M to $M \times 0$.

Let $H(M) = H^{CAT}(M)$ be the **space of h-cobordisms** on M. As a set, we can let this consist of the set of codimension 0 submanifolds W of $M \times I$ that are h-cobordisms on $M \times 0$. Instead of giving this a topology, we realize it as a simplicial set, with q-simplices $H_q(M)$ the set of CAT bundles of h-cobordisms over Δ^q, contained as codimension 0 submanifolds of $M \times I \times \Delta^q$. By a CAT bundle we mean a family that is locally trivial in the CAT sense.

There is a fiber sequence

$$C(M) \to K(M) \xrightarrow{\pi} H(M)$$

where $K(M)$ is the **space of collars** on $M \times 0$ in $M \times I$, i.e., of embeddings $k \colon M \times I \to M \times I$ relative to $M \times 0$, and π takes a collar k to its image $W =$

$k(M \times I)$, viewed as an (s-)cobordism on M. The fiber at the trivial h-cobordism is the group of automorphisms of $M \times I$ rel $M \times 0$, i.e., the concordance group $C(M)$.

The existence and essential uniqueness of collars (like tubular neighborhoods), asserts that the space of collars $K(M)$ is contractible. Hence there is a homotopy equivalence

$$C(M) \simeq \Omega H(M)$$

and the connected delooping $BC(M)$, given by the bar construction on the simplicial group, realizes the base point component $H^s(M)$ of $H(M)$, i.e., the space of s-cobordisms on M, at least for $n \geq 5$.

Again, there is a **stabilization** map

$$\sigma \colon H(M) \to H(M \times J)$$

taking $W \subset M \times I$ to something like

$$W \times J \subset (M \times I) \times J \cong (M \times J) \times I\,.$$

To be more precise, $M \times J$ is here a non-closed manifold, so some adjustment is needed to arrange that $W \times J$ is made standard near $M \times \partial J \times I$. The details are presented in Waldhausen's "manifold approach" paper, [Wa82].

Iterating the stabilization, we obtain the **stable h-cobordism space** $\mathcal{H}(M) = \mathcal{H}^{CAT}(M)$, defined as the homotopy colimit

$$\mathcal{H}(M) = \operatorname*{colim}_s H(M \times J^s)\,.$$

The whole fiber sequence above stabilizes, so there is a homotopy equivalence

$$\mathcal{C}(M) \simeq \Omega\mathcal{H}(M)$$

and $B\mathcal{C}(M)$ is a path component $\mathcal{H}^s(M)$ of $\mathcal{H}(M)$. Igusa's DIFF stability theorem now says that the stabilization map $\sigma \colon H(M) \to H(M \times J)$ is k-connected if

$$\dim(M) \geq \max(2k+5, 3k+1)\,,$$

and likewise for $\sigma^\infty \colon H(M) \to \mathcal{H}(M)$.

The involution on $C(M)$ carries over to one on $BC(M)$, and on all of $H(M)$, and corresponds very roughly to turning an h-cobordism up-side down.

The Hatcher spectral sequence can be rewritten as

$$E^1_{s,t} = \pi_t H^s(M \times I^s) \Longrightarrow \pi_{s+t}\big(\frac{\widetilde{CAT}(M)}{CAT(M)}\big)$$

for $s \geq 0$, $t \geq 1$, with

$$E^2_{s,t} = H^{gp}_s(\mathbb{Z}/2; \pi_t\mathcal{H}^s(M))$$

in the stable range, where $\mathbb{Z}/2$ acts on $\pi_*\mathcal{H}(M)$ by minus the canonical involution. Note that the groups $\pi_0 H(M \times I^s) \cong \mathrm{Wh}_1(\pi)$ do not appear in the spectral sequence. (The decoration s is not related to the spectral sequence index s.) Inverting 2, the spectral sequence again collapses to the isomorphism

$$\pi_*(\mathcal{H}^s(M))[\frac{1}{2}]^- \cong \pi_*\big(\frac{\widetilde{CAT}(M)}{CAT(M)}\big)[\frac{1}{2}]\,,$$

in the concordance stable range.

Weiss and Williams [WW88] show that there is a CAT Whitehead spectrum $\mathrm{Wh}^s(M)$ with $\mathbb{Z}/2$-action, such that $\Omega\,\mathrm{Wh}^s(M)$ has underlying infinite loop space $\mathcal{H}^s(M)$, and a map

$$\frac{\widetilde{CAT}(M)}{CAT(M)} \to \Omega^\infty(E\mathbb{Z}/2_+ \wedge_{\mathbb{Z}/2} \Omega\,\mathrm{Wh}^s(M))$$

that is as highly connected as the CAT concordance stable range, plus 1. The homotopy orbit spectral sequence for the target, with

$$E^2_{s,t} = H^{gp}_s(\mathbb{Z}/2; \pi_t \Omega\,\mathrm{Wh}^s(M))\,,$$

agrees with the Hatcher spectral sequence in the stable range. We shall return to the Whitehead spaces in the following section.

2. The stable parametrized h-cobordism theorem

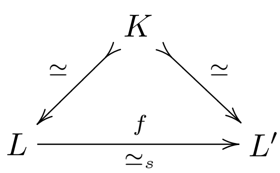
2.1. Simple maps of polyhedra.

A **compact polyhedron** K is the underlying topological space $|X|$ of a finite simplicial complex X. The affine linear structure on each simplex determines a piecewise linear (= PL) structure on the polyhedron, i.e., a notion of what constitutes a PL map on K, and any subdivision of the simplicial complex determines the same PL structure, by definition. We call X a **triangulation** of K. More precisely, a polyhedron is a topological space equipped with such a PL structure.

A map $f\colon K \to L$ of compact polyhedra is piecewise linear (= PL) if there exist triangulations X and Y of K and L, respectively, in their PL structures, and a simplicial map $X \to Y$ whose geometric realization is f. A PL map $f\colon K \to L$ of compact polyhedra is **simple** if its point inverses are contractible, i.e., if for each $p \in L$ the preimage $f^{-1}(p)$ has the homotopy type of a point.

When $L' = L \cup_{e^n} e^{n+1}$ is obtained from L by attaching a PL $(n+1)$-cell along an n-cell in its boundary, there is by definition an **elementary expansion** from L to L' and an **elementary collapse** from L' to L. Two compact polyhedra are said to be **simple homotopy equivalent**, in the sense of J.H.C. Whitehead, if they can be connected by a finite chain of elementary collapses and expansions. This is, in general, a more restrictive relation than that of homotopy equivalence. The elementary collapse can be realized by a simple map $L' \to L$, contracting e^{n+1} onto the e^n in its boundary. So simple/simply homotopy equivalent polyhedra can be connected by a chain of simple maps. Conversely, Marshall Cohen [Co67] proved that any simple map is a simple homotopy equivalence. Hence simple maps and elementary collapses/expansions generate the same equivalence relation on compact polyhedra, namely simple homotopy equivalence.

Let $\mathcal{E}$ be the category of compact polyhedra L and PL maps $f\colon L \to L'$. More generally, for a fixed compact polyhedron K let $\mathcal{E}(K)$ be the category of compact polyhedra L containing K as a subcomplex, and morphisms from L to L' the PL maps $f\colon L \to L'$ that restrict to the identity on K. Let $s\mathcal{E}^h(K) \subset \mathcal{E}(K)$ be the subcategory with objects such that the inclusion $K \subset L$ is a homotopy equivalence, and morphisms the simple maps $f\colon L \to L'$ fixing K.



The set of simple homotopy types of compact polyhedra L containing K as a deformation retract can now be expressed as $\pi_0(s\mathcal{E}^h(K))$. The operation $(L, L') \mapsto L \cup_K L'$, union along K, makes this set an abelian group. The basic result of simple homotopy theory is that this group is isomorphic to the Whitehead group of the fundamental group of K, for connected K, so

$$\pi_0(s\mathcal{E}^h(K)) \cong \mathrm{Wh}_1(\pi_1(K)) \,.$$

Note that when M is a compact PL manifold, each PL h-cobordism (W, M) defines an object of $s\mathcal{E}^h(M)$, and each PL homeomorphism $W \to W'$ of h-cobordisms on M defines a morphism in this category. This transformation forgets the PL manifold structure, and only retains the underlying polyhedron. The Whitehead torsion of (W, M) is the image in $\mathrm{Wh}_1(\pi_1(M))$ of the path component of this polyhedral object, under the isomorphism above (for $M = K$). This invariant suffices for the classical h- or s-cobordism theorem. To obtain a parametrized result, we wish to accommodate for the simplicial direction in the h-cobordism space $H(M) = H^{PL}(M)$, so we also define simplicial categories $\mathcal{E}_\bullet$, $\mathcal{E}_\bullet(K)$ and $s\mathcal{E}^h_\bullet(K)$.

In simplicial degree q, the objects of $\mathcal{E}_q$ are PL bundles $p\colon P \downarrow \Delta^q$ of compact PL manifolds over Δ^q. Again, by a PL bundle we mean a family that is locally trivial in the PL sense. The morphisms of $\mathcal{E}_q$ are PL maps of bundles over Δ^q.

$$\begin{array}{ccc} P & \longrightarrow & P' \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle p'} & \\ & \Delta^q & \end{array}$$

The objects of $\mathcal{E}_q(K)$ are compact PL manifold bundles $p\colon P \downarrow \Delta^q$ that contain the trivial bundle $K \times \Delta^q \downarrow \Delta^q$ as a PL subbundle. So these are commutative diagrams

$$\begin{array}{ccc} K \times \Delta^q & \rightarrowtail & P \\ & \searrow \quad \swarrow{\scriptstyle p} & \\ & \Delta^q & \end{array}$$

such that the pair $(P, K \times \Delta^q)$ is locally trivial over Δ^q in the PL sense. The morphisms are PL bundle maps $f\colon P \to P'$ over Δ^q that restrict to the identity on $K \times \Delta^q$. Let $s\mathcal{E}^h_q(K) \subset \mathcal{E}_q(K)$ be the subcategory where the objects satisfy that the inclusion $K \times \Delta^q \subset P$ is a homotopy equivalence, and the morphisms satisfy that $f\colon P \to P'$ is a simple map.

$$\begin{array}{ccc} & K \times \Delta^q & \\ {\scriptstyle \simeq}\swarrow & & \searrow{\scriptstyle \simeq} \\ P & \xrightarrow[\simeq_s]{f} & P' \\ {\scriptstyle p}\searrow & & \swarrow{\scriptstyle p'} \\ & \Delta^q & \end{array}$$

The composite map $K \times \Delta^q \to \Delta^q$ is the projection.

Here is then the manifold part of the stable parametrized h-cobordism theorem, in its PL form. As noted above, $\mathcal{H}^{PL}(M) \simeq \mathcal{H}^{TOP}(M)$ for any PL manifold M. We always think of a category, like $s\mathcal{E}^h_\bullet(M)$, as a space by way of its nerve.

Theorem. *For each compact PL manifold M there is a homotopy equivalence*

$$\mathcal{H}^{PL}(M) \simeq s\mathcal{E}^h_\bullet(M)\,.$$

More precisely, the forgetful maps $H^{PL}(M \times J^s) \to s\mathcal{E}^h_\bullet(M \times J^s)$ induce a natural chain of homotopy equivalences

$$\mathcal{H}^{PL}(M) = \operatorname*{colim}_s H^{PL}(M \times J^s) \xrightarrow{\simeq} \operatorname*{colim}_s s\mathcal{E}^h_\bullet(M \times J^s) \xleftarrow{\simeq} s\mathcal{E}^h_\bullet(M)\,.$$

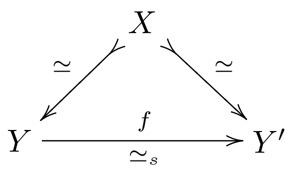
We shall present a proof of this theorem in the second part of these lectures. In the published version of this theorem [Ha75] there is no mention of a simplicial direction in $s\mathcal{E}^h(M)$, but we do not know if such a claim can be proved.

2.2. Simple maps of simplicial sets.

Let X be a finite simplicial set. Its geometric realization can be triangulated, i.e., it is homeomorphic to the compact polyhedron underlying some finite simplicial complex, but not in any natural way. Conversely, each compact polyhedron can be triangulated by some finite simplicial complex, which is a finite simplicial set, but again, the association cannot be made natural. Some work is therefore required to continue the passage from spaces of PL manifolds to categories of polyhedra across to categories of simplicial sets.

A map $f\colon X \to Y$ of finite simplicial sets is a **weak homotopy equivalence** if the induced map $|f|\colon |X| \to |Y|$ of geometric realizations is a homotopy equivalence, and it is a **simple map** if the induced map $|f|\colon |X| \to |Y|$ has contractible point inverses, so that $|f|^{-1}(p)$ is contractible for each $p \in |Y|$. Simple maps are weak homotopy equivalences.

Fix a finite simplicial set X. Let $\mathcal{C}(X)$ be the category of finite cofibrations $X \subset Y$, i.e., finite simplicial sets Y that contain X, and let the morphisms from Y to Y' be the simplicial maps $f\colon Y \to Y'$ that are the identity on X. Let $s\mathcal{C}^h(X) \subset \mathcal{C}(X)$ be the subcategory where the objects satisfy that $X \subset Y$ is a weak homotopy equivalence, and the morphisms are the simple maps.



The non-manifold part of the stable parametrized h-cobordism theorem follows.

Theorem. *Let X be a finite simplicial complex. There is a homotopy equivalence*

$$s\mathcal{C}^h(X) \simeq s\mathcal{E}^h_\bullet(|X|)$$

induced by a natural chain of homotopy equivalences

$$s\mathcal{C}^h(X) \xleftarrow{\simeq} s\mathcal{D}^h(X) \xrightarrow{\simeq} s\mathcal{D}^h_\bullet(X) \xrightarrow{\simeq} s\mathcal{E}^h_\bullet(|X|)\,.$$

Here $s\mathcal{D}^h(X) \subset s\mathcal{C}^h(X)$ denotes a full subcategory of non-singular finite simplicial sets Y containing X, and $s\mathcal{D}^h_\bullet(X)$ is a simplicial version involving PL bundles.

A simplicial set is **non-singular** if each non-degenerate simplex is embedded (not just its interior), and the geometric realization of such a simplicial set has a canonical PL structure.

We shall discuss its (nearly complete) proof in the third part of these lectures.

The union along X defines a sum operation $(Y_1, Y_2) \mapsto Y_1 \cup_X Y_2$ on $\mathcal{C}(X)$ and the subcategory $s\mathcal{C}^h(X)$. Following [Se74] and [Wa85, §1.8], one can deloop the nerve (= classifying space) of such categories by forming a simplicial category $N_\bullet\mathcal{C}(X)$ (resp. $sN_\bullet\mathcal{C}^h(X)$) of **sum diagrams**. In simplicial degree q, $N_q\mathcal{C}(X)$ is the category of q-tuples

$$(Y_1, Y_2, \dots, Y_q)$$

in $\mathcal{C}(X)$, together with suitably compatible choices of sums

$$Y_{\{i_1,\dots,i_s\}} \cong Y_{i_1} \cup_X \cdots \cup_X Y_{i_s}$$

for each subset $\{i_1, \dots, i_s\} \subset \{1, \dots, q\}$. For $0 < i < q$ the i-th face map replaces Y_i and Y_{i+1} with their sum $Y_i \cup_X Y_{i+1}$. For $i = 0$ it deletes Y_1, and for $i = q$ it deletes Y_q. Likewise, $sN_q\mathcal{C}^h(X)$ has as objects q-tuples of objects in $\mathcal{C}^h(X)$, plus sums, and the morphisms are q-tuples of simple maps $Y_i \to Y_i'$, for $1 \le i \le q$.

In low degrees, $N_0\mathcal{C}(X) = *$ is the trivial category with one object and one morphism, and $N_1\mathcal{C}(X) \cong \mathcal{C}(X)$. So each object Y of $\mathcal{C}(X)$ corresponds to a 1-simplex in (the geometric realization of) $N_\bullet\mathcal{C}(X)$ with ends at the base point $*$, i.e., it corresponds to a closed loop $[Y]$. There is a canonical map

$$\iota\colon s\mathcal{C}^h(X) \to \Omega sN_\bullet\mathcal{C}^h(X)$$

and Segal proves that it is a group completion map, making the monoid π_0 a group. But $\pi_0(s\mathcal{C}^h(X)) \cong \mathrm{Wh}_1(\pi_1(X))$, the Whitehead group, and the sum operation $\cup_X$ corresponds to the group operation in the Whitehead group, so $s\mathcal{C}^h(X)$ is already group complete, and this ι is a homotopy equivalence [Wa85, 3.1.1].

Definition. Let X be a finite simplicial set. Its **PL Whitehead space** is defined as

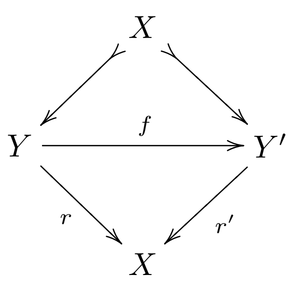
$$\mathrm{Wh}^{PL}(X) = sN_\bullet\mathcal{C}^h(X)\,.$$

When X is a finite combinatorial manifold, so $|X|$ is a compact PL manifold, we have obtained a chain of homotopy equivalences

$$\mathcal{H}^{PL}(|X|) \simeq \Omega\,\mathrm{Wh}^{PL}(X)\,.$$

2.3. Algebraic K-theory of spaces.

With X a finite simplicial set, as above, let $\mathcal{R}(X)$ be the category of **finite retractive spaces** over X, i.e., finite simplicial sets Y containing X that come equipped with a structural retraction $r\colon Y \to X$. A morphism from (Y, r) to (Y', r') is a map $f\colon Y \to Y'$ that is the identity on X and commutes with the retractions: $r'f = r$.



The composite map $X \to X$ is the identity.

For example, when $X = *$, $\mathcal{R}(*)$ is the category of finite based simplicial sets. For infinite X, the correct generalization is to ask that $X \subset Y$ is a finite cofibration, i.e., Y contains only finitely many non-degenerate simplices that are not in X.

As usual, we are mostly interested in subcategories of $\mathcal{R}(X)$. Let $\mathcal{R}^h(X)$ be the full subcategory with objects (Y, r) such that the inclusion $X \subset Y$ is a weak homotopy equivalence. Let $h\mathcal{R}(X)$ be the subcategory with morphisms f that are weak homotopy equivalences. Let $s\mathcal{R}(X)$ be the subcategory with morphisms f that are simple maps. Let $s\mathcal{R}^h(X) = s\mathcal{R}(X) \cap \mathcal{R}^h(X)$.

Forgetting the structural retractions defines a functor $\mathcal{R}(X) \to \mathcal{C}(X)$. In a sufficiently fibrant situation, the restricted functor $\mathcal{R}^h(X) \to \mathcal{C}^h(X)$ is a homotopy equivalence, because the space of possible retractions r in a pair $X \subset Y$ is the fiber at id_X of the fibration $\mathrm{Map}(Y, X) \to \mathrm{Map}(X, X)$. When $X \simeq Y$, the fibration is a homotopy equivalence so the fiber is contractible. However, in the present simplicial setting, this argument does not immediately apply.

A morphism $f \colon Y \to Y'$ is a **cofibration** if it is injective, and cofibrations are closed under base change (pushout), so this makes $\mathcal{R}(X)$ a category with cofibrations in the sense of [Wa85, §1.1]. Similarly for $\mathcal{R}^h(X)$. The morphisms of the subcategory $h\mathcal{R}(X)$ of $\mathcal{R}(X)$ satisfy a gluing lemma for base change along a cofibration, so $h\mathcal{R}(X) \subset \mathcal{R}(X)$ is a **category with cofibrations and weak equivalences** in the sense of [Wa85, §1.2]. Similarly for $s\mathcal{R}(X) \subset \mathcal{R}(X)$ and $s\mathcal{R}^h(X) \subset \mathcal{R}^h(X)$.

The **$S_\bullet$-construction** $S_\bullet\mathcal{R}(X)$ of [Wa85, §1.3] is then defined as a simplicial category. In degree q, the category $S_q\mathcal{R}(X)$ has as objects the sequences of cofibrations

$$X \rightarrowtail Y_1 \rightarrowtail \cdots \rightarrowtail Y_q$$

in $\mathcal{R}(X)$, together with choices of pushouts $Y_{i,j} = X \cup_{Y_i} Y_j$ for all $1 \le i < j \le q$. Here $r_i \colon Y_i \to X$ is the structural retraction, and $Y_i \to Y_j$ is a composite of cofibrations from the sequence above. For $0 < i \le q$ the i-th simplicial face map deletes Y_i from the sequence. For $i = 0$, the face is

$$X \rightarrowtail Y_{1,2} \rightarrowtail \cdots \rightarrowtail Y_{1,q} \,.$$

In low degrees, $S_0\mathcal{R}(X) = *$ is the trivial category with one object and one morphism, and $S_1\mathcal{R}(X) \cong \mathcal{R}(X)$. Next, $S_2\mathcal{R}(X)$ is the category of diagrams

$$\begin{array}{ccccc} X & \rightarrowtail & Y_1 & \rightarrowtail & Y_2 \\ & & \downarrow & & \downarrow \\ & & X & \rightarrowtail & Y_{1,2} \\ & & & & \downarrow \\ & & & & X \end{array}$$

where the square is a pushout and each composite maps $X \to X$ is the identity. So each object (Y, r) in $\mathcal{R}(X)$ produces a closed loop $[Y]$ in (the geometric realization of) $S_\bullet\mathcal{R}(X)$, since it is a 1-simplex with both ends at the base point. Furthermore, each extension $Y_1 \to Y_2 \to Y_{1,2}$, in the sense of retractive spaces over X, produces a planar triangle in $S_\bullet\mathcal{R}(X)$ that connects the loops associated to Y_1, Y_2 and $Y_{1,2}$. So up to homotopy, the loop $[Y_2]$ is homotopic to the loop sum of $[Y_1]$ and $[Y_{1,2}]$.

Let $hS_\bullet\mathcal{R}(X)$ be the simplicial subcategory of $S_\bullet\mathcal{R}(X)$ with morphisms in simplicial degree q the commutative diagrams

$$\begin{array}{ccccccc} X & \rightarrowtail & Y_1 & \rightarrowtail & \cdots & \rightarrowtail & Y_q \\ \downarrow{=} & & \downarrow{\simeq} & & & & \downarrow{\simeq} \\ X & \rightarrowtail & Y_1' & \rightarrowtail & \cdots & \rightarrowtail & Y_q' \end{array}$$

in $\mathcal{R}(X)$, where the vertical maps are weak homotopy equivalences. If $f\colon Y \to Y'$ is a weak homotopy equivalence in $\mathcal{R}(X)$, then the loops $[Y]$ and $[Y']$ are homotopic in $hS_\bullet\mathcal{R}(X)$.

The **algebraic K-theory of the space** X, denoted $A(X)$, is now defined [Wa85, §2.1] as the loop space of the geometric realization of the simplicial category $hS_\bullet\mathcal{R}(X)$:

$$A(X) = \Omega hS_\bullet\mathcal{R}(X)\,.$$

The identification $h\mathcal{R}(X) \cong hS_1\mathcal{R}(X)$ induces a map

$$\iota\colon h\mathcal{R}(X) \to A(X)$$

that takes the point corresponding to an object Y to the loop $[Y]$. This map converts each extension in $\mathcal{R}(X)$ to a loop sum in $A(X)$, up to preferred homotopy. In this sense, $A(X)$ is a suitably universal receptacle for homotopy invariants of finite retractive spaces over X, that are homotopy additive in the sense that extensions go to loop sums. To make this more precise would involve discussing coherences for the preferred homotopies, which essentially comes down to recovering the definition of the $S_\bullet$-construction.

2.4. The fiber sequence relating $A(X)$ and $\mathrm{Wh}^{PL}(X)$.

The right fiber at $X = Y$ of the inclusion functor $s\mathcal{R}(X) \to h\mathcal{R}(X)$ is the subcategory $s\mathcal{R}^h(X)$. The composite functor

$$s\mathcal{R}^h(X) \to s\mathcal{R}(X) \to h\mathcal{R}(X)$$

is naturally equivalent to the constant functor at $X = Y$, but this diagram is not in general a homotopy fiber sequence. However, the generic fibration theorem [Wa85, §1.6] asserts that after applying the $S_\bullet$-construction, there is indeed a homotopy fiber sequence

$$sS_\bullet\mathcal{R}^h(X) \to sS_\bullet\mathcal{R}(X) \to hS_\bullet\mathcal{R}(X)\,.$$

Only the right hand term is a homotopy functor in X, so we apply this degreewise to the simplicial space $X^{\Delta^\bullet}$, given in simplicial degree q as $X^{\Delta^q} = \mathrm{Map}(\Delta^q, X)$. Then there is a homotopy fiber sequence

$$(*) \qquad sS_\bullet\mathcal{R}^h(X^{\Delta^\bullet}) \to sS_\bullet\mathcal{R}(X^{\Delta^\bullet}) \to hS_\bullet\mathcal{R}(X^{\Delta^\bullet})\,.$$

At the left of $(*)$, Waldhausen shows [Wa85, 3.1.7] that there is a chain of homotopy equivalences

$$sN_\bullet\mathcal{C}^h(X) \xrightarrow{\simeq} sN_\bullet\mathcal{C}^h(X^{\Delta^\bullet}) \xleftarrow{\simeq} sN_\bullet\mathcal{R}^h(X^{\Delta^\bullet}) \xrightarrow{\simeq} sS_\bullet\mathcal{R}^h(X^{\Delta^\bullet})\,.$$

The first equivalence uses that $s\mathcal{C}^h(X)$ is a homotopy functor in X. The second equivalence forgets the structural retractions. The third equivalence takes a q-tuple $(Y_1, Y_2, \dots, Y_q)$ in the $N_\bullet$-construction to the the cofibration sequence

$$X \rightarrowtail Y_1 \rightarrowtail Y_1 \cup_X Y_2 \rightarrowtail \cdots \rightarrowtail Y_1 \cup_X \cdots \cup_X Y_q$$

in the $S_\bullet$-construction.

In the middle of $(*)$, Waldhausen shows [Wa85, 3.2.1] that the functor

$$X \mapsto sS_\bullet\mathcal{R}(X^{\Delta^\bullet})$$

is a homology theory. The excision property follows from the hereditary behavior of simple maps, under restriction to subspaces of the target. It is nontrivial to verify that this property is preserved by the construction imposing homotopy invariance. There are many other incarnations of this functor in the theory, but the other contexts do not appear to be flexible enough to allow a proof of this very important property.

At the right of $(*)$, the homotopy invariance of $A(X)$ shows that there is a homotopy equivalence

$$hS_\bullet\mathcal{R}(X) \xrightarrow{\simeq} hS_\bullet\mathcal{R}(X^{\Delta^\bullet}) .$$

In the special case $X = *$, it is easy to show that $sS_\bullet\mathcal{R}^h(*) \simeq *$, so the coefficient spectrum of the middle homology theory is $sS_\bullet\mathcal{R}(*) \simeq hS_\bullet\mathcal{R}(*)$. Hence there is a homotopy fiber sequence [Wa85, §3.3]

$$\Omega\,\mathrm{Wh}^{PL}(X) \to h(X; A(*)) \xrightarrow{\alpha} A(X) \to \mathrm{Wh}^{PL}(X)$$

where $h(X; A(*))$ denotes the unreduced homology theory of X, with coefficient spectrum $A(*)$.

Theorem. *For compact PL manifolds M there is a natural fiber sequence*

$$\mathcal{H}^{PL}(M) \to h(M; A(*)) \xrightarrow{\alpha} A(M) ,$$

where $h(M; A(*)) = \Omega^\infty(A(*) \wedge M_+)$ and α is the assembly map.

There are involutions on $A(M)$ and $A(*)$, depending on the tangent (micro-)bundle on M, so that the maps in this fiber sequence commute with the involutions. See [Vo85]. In this sense, the study of the canonical involution on $C(M)$ or $H(M)$, in the stable range, is translated into a problem about the involution in the algebraic K-theory of spaces.

2.5. A manifold approach.

In [Wa82], Waldhausen constructs a fiber sequence

$$\mathcal{H}^{CAT}(M) \to h^{CAT}(M) \to A(M) \to \mathrm{Wh}^{CAT}(M)$$

in each geometric category, which is homotopy equivalent to the one above in the PL case.

Let $\mathcal{P}(M) = \mathcal{P}^{CAT}(M)$ be the simplicial set of CAT **partitions** of $M \times I$, i.e., codimension 0 submanifolds $W \subset M \times I$ that contain $M \times 0$, but do not meet $M \times 1$.

If M has boundary, W is to be a product near ∂M. As usual, a q-simplex is a CAT bundle of such partitions over Δ^q. Let $h\mathcal{P}(M)$ be the simplicial category with morphisms $W \to W'$ the codimension 0 inclusions such that W' is obtained from W by adjoining an h-cobordism. (The precise definition is a little more flexible, allowing for identity morphisms.)

For each $m \geq 0$, let $h\mathcal{P}^m(M) \subset h\mathcal{P}(M)$ be the connected component containing the object W obtained from $M \times [0, \epsilon]$ by attaching finitely many trivial m-handles, i.e., unknotted handles of index m, and let $\mathcal{P}^m(M)$ be the simplicial set of objects in $h\mathcal{P}^m(M)$. It is possible to define **lower stabilization** maps

$$\underline{\sigma}\colon h\mathcal{P}^m(M) \to h\mathcal{P}^m(M \times J)$$

that increase the manifold dimension, and **upper stabilization** maps

$$\overline{\sigma}\colon h\mathcal{P}^m(M) \to h\mathcal{P}^{m+1}(M \times J)$$

that also increase the handle index. There is a homotopy fiber sequence

$$\operatorname*{colim}_{m,s} H^{CAT}(M \times J^s) \to \operatorname*{colim}_{m,s} \mathcal{P}^m(M \times J^s) \to \operatorname*{colim}_{m,s} h\mathcal{P}^m(M \times J^s)$$

which after group completion (in the middle and on the right) yields the fiber sequence

$$\mathcal{H}^{CAT}(M) \to h^{CAT}(M) \to A(M)\,.$$

In this model, $\mathrm{Wh}^{CAT}(M)$ is defined as a delooping of $\mathcal{H}(M)$, with respect to the (partially defined) sum of h-cobordisms. By the fiber sequence in Section 2.4, we know that $h^{PL}(M)$ is the homology theory $h(M; A(*))$.

The forgetful map from DIFF to PL induces a map of horizontal fiber sequences

$$\begin{array}{ccccccc}
\mathcal{H}^{DIFF}(M) & \longrightarrow & h^{DIFF}(M) & \xrightarrow{\eta} & A(M) & \longrightarrow & \mathrm{Wh}^{DIFF}(M) \\
\downarrow & & \downarrow & & \downarrow \simeq & & \downarrow \\
\mathcal{H}^{PL}(M) & \longrightarrow & h(M; A(*)) & \xrightarrow{\alpha} & A(M) & \longrightarrow & \mathrm{Wh}^{PL}(M)
\end{array}$$

where we now elucidate on the structure of $h^{DIFF}(M)$. By smoothing theory the fiber of $\mathcal{H}^{DIFF}(M) \to \mathcal{H}^{PL}(M)$ is a homology theory in M, as is $h(M; A(*))$, hence so is $h^{DIFF}(M)$. The stabilization (= Goodwillie derivative at $*$) of $\mathcal{H}^{DIFF}(M)$ is trivial, by Morlet's disjunction lemma and the lamp-bulb trick [Ha78], while the stabilization of $A(M)$ is $Q(M_+)$, by the "vanishing of the mystery homology theory" [Wa87a], or alternatively by "relative K-theory equals relative TC" [Du97] and a calculation of the stabilization of TC, so in fact the homology theory $h^{DIFF}(M)$, which equals its stabilization, must be $Q(M_+) = h(M; S) = \Omega^\infty \Sigma^\infty (M_+)$.

The stabilization map $A(M) \to Q(M_+)$, or the trace map [Wa79]

$$A(M) \xrightarrow{tr} Q(\Lambda M_+) \to Q(M_+)\,,$$

splits η, so in the DIFF case the fiber sequence becomes a product decomposition.

Theorem. *For compact DIFF manifolds M there is a natural fiber sequence*

$$\mathcal{H}^{DIFF}(M) \to Q(M_+) \xrightarrow{\eta} A(M)\,,$$

where the unit map η is split injective, and a homotopy equivalence

$$A(M) \simeq Q(M_+) \times \mathrm{Wh}^{DIFF}(M)\,.$$

Sverre Lunøe-Nielsen's Master's thesis [LN00] provides a definition of these manifold models that is strictly involutive, so that the stabilized fiber sequences are $\mathbb{Z}/2$-equivariant.

3. Algebraic K-theory of structured ring spectra

3.1. A-theory as K-theory.

Suppose that X is a based and connected simplicial set. Let $G = \Omega X$ be the Kan loop group of X, a simplicial group with $BG \simeq X$, see [Wa96]. Let $\mathcal{R}(*, G)$ be the category with cofibrations of finite G-free based simplicial sets, i.e., those simplicial sets with G-action Z that can be obtained from $*$ by attaching $G_+ \wedge \Delta^n$ along a G-map from $G_+ \wedge \partial\Delta^n$, finitely often. Let $h\mathcal{R}(*, G)$ be the subcategory of weak homotopy equivalences, i.e., based G-maps $Z \to Z'$ that are weak homotopy equivalences when considered as non-equivariant maps.

There is a pair of exact functors relating $h\mathcal{R}(X)$ and $h\mathcal{R}(*, G)$, which induce a homotopy equivalence

$$A(X) = \Omega hS_\bullet\mathcal{R}(X) \simeq \Omega hS_\bullet\mathcal{R}(*, G) = A(*, G)\,.$$

See [Wa85, 2.1.3]. If $G \to EG \to BG$ is a principal fiber bundle, with EG weakly contractible and $X \simeq BG$, then one functor takes a retractive space (Y, r) over BG to the pullback $EG \times_X Y$, viewed as a G-free retractive space over EG, and then collapses EG, to view $(EG \times_X Y)/EG$ as a G-free retractive space over $*$, i.e., a G-free based space. The other functor takes a G-free based space Z to the Borel construction $EG \times^G Z$ (often denoted $EG \times_G Z$), viewed as a retractive space over $BG \simeq X$. More care than usual is required about the finiteness conditions. See also [KR02, §8] for some variations.

The three functors id, C and Σ in the natural cofiber sequence

$$Z \to CZ \to \Sigma Z$$

induce self-maps id, C and Σ of $A(*, G)$, and by the additivity theorem [Wa85, 1.4], $id + \Sigma \simeq C$ (loop sum). The natural homotopy equivalence $CZ \to *$ makes C null-homotopic, so the suspension functor Σ induces minus the identity on $A(*, G)$. A similar argument can be made for $A(X)$, interpreting cone and suspension in the category $\mathcal{R}(X)$. Thus $A(*, G)$ is homotopy equivalent to the colimit (telescope) of

$$A(*, G) \xrightarrow{\Sigma} A(*, G) \xrightarrow{\Sigma} A(*, G) \xrightarrow{\Sigma} \dots$$

The latter is also the algebraic K-theory of the colimit of the categories with cofibrations and weak equivalences

$$h\mathcal{R}(*, G) \xrightarrow{\Sigma} h\mathcal{R}(*, G) \xrightarrow{\Sigma} h\mathcal{R}(*, G) \xrightarrow{\Sigma} \dots\,.$$

This is essentially the localization with respect to suspension of the category of finite G-CW complexes, i.e., the category of finite G-CW spectra. Let $S[G] = \Sigma^\infty(G_+)$ be the suspension spectrum on G_+. The multiplication on G induces a pairing $S[G] \wedge S[G] \to S[G]$. If $S[G]$ is interpreted in a structured sense, e.g. as a symmetric spectrum, then this pairing makes it a symmetric ring spectrum.

The colimit of categories above is then equivalent to the category $\mathcal{C}_{S[G]}$ of finite cell $S[G]$-module spectra, i.e., the $S[G]$-module spectra that can be obtained from $*$ by attaching $S[G] \wedge D^n$ along an $S[G]$-module map from $S[G] \wedge S^{n-1}$, finitely many times. More generally, for any symmetric ring spectrum (or S-algebra) B there is such a category $\mathcal{C}_B$ of **finite cell B-modules**. This is a category with cofibrations (the cellular inclusions) and weak equivalences (the stable equivalences, h), and the algebraic K-theory of B is defined as

$$K(B) = \Omega h S_\bullet \mathcal{C}_B \,.$$

Mandell [EKMM, VI] verifies that this definition agrees with the plus construction

$$K(B) \simeq \mathbb{Z} \times BGL_\infty(B)^+$$

and $K(HR) = K(R)$, where HR is the Eilenberg–Mac Lane spectrum of a ring R, and $K(R)$ is Quillen's algebraic K-theory.

Returning to the case $B = S[\Omega X]$, the approximation theorem [Wa85, 1.6] proves that

$$A(X) \simeq A(*, \Omega X) \simeq K(S[\Omega X])\,.$$

This is explained in [Wa84]. For example, with $X = *$ we have $A(*) \simeq K(S)$, the arithmetic of the sphere spectrum.

For DIFF manifolds M, Tore A. Kro's Ph.D. thesis [Kr05, 4.3.13] constructs a strict involution on $S[\Omega M]$, as a map of structured ring spectra, that takes the contribution from the tangent bundle of M into account. Presumably Vogell's involution on $A(M)$ and the involution on $K(S[\Omega M])$ induced by Kro's involution agree.

3.2. The linearization map and rational calculations.

From the plus construction models, it follows that a k-connected map $A \to B$ of S-algebras, such that $\pi_0(A) \cong \pi_0(B)$, induces $(k+1)$-connected maps $BGL_\infty(A) \to BGL_\infty(B)$ and $K(A) \to K(B)$. So for X based and connected, with fundamental group $\pi = \pi_1(X)$, the 2-connected map $X \to B\pi$ induces a 1-connected map $S[\Omega X] \to \mathbb{Z}[\pi]$, so the **linearization map**

$$L \colon A(X) = K(S[\Omega X]) \to K(\mathbb{Z}[\pi])$$

is 2-connected.

When $X = B\pi$ is aspherical, so that the universal cover $\widetilde{X}$ is contractible, then the map $S[\Omega X] \to \mathbb{Z}[\pi]$ is a π_0-isomorphism and a rational equivalence, from which it follows that $L \colon A(B\pi) \to K(\mathbb{Z}[\pi])$ is also a rational equivalence.

In the simply-connected case,

$$L \colon A(*) = K(S) \to K(\mathbb{Z})$$

is a rational equivalence. By Borel's rational calculation [Bo74]

$$K_i(\mathbb{Z}) \otimes \mathbb{Q} \cong \begin{cases} \mathbb{Q} & \text{if } i = 0 \text{ or } i = 4k+1,\ k \geq 1 \\ 0 & \text{otherwise,} \end{cases}$$

so the same formula gives $\pi_i A(*) \otimes \mathbb{Q}$. Thus

$$\pi_i H^{DIFF}(D^s) \otimes \mathbb{Q} \cong \begin{cases} \mathbb{Q} & \text{if } i = 4k,\ k \geq 1 \\ 0 & \text{otherwise} \end{cases}$$

for i in the DIFF concordance stable range for D^s, or one better. In this case $\widetilde{DIFF}(D^s)$ is rationally trivial, and the involution on $\pi_i \mathcal{H}^{DIFF}(*) \otimes \mathbb{Q}$ has eigenvalue $+1$, so Farrell and Hsiang [FH78] could deduce that

$$\pi_i DIFF(D^s) \otimes \mathbb{Q} \cong \begin{cases} \mathbb{Q} & \text{if } i = 4k-1,\ k \geq 1 \text{ and } s \text{ odd} \\ 0 & \text{otherwise} \end{cases}$$

for i in the DIFF concordance stable range for D^s i.e., for $s \geq \max(2i+7, 3i+4)$ or so. This is notably different from the TOP and PL cases, since $TOP(D^s) \simeq PL(D^s) \simeq *$ for all s, by the Alexander (coning, or conning) trick.

Hsiang and Jahren [HJ82], [HJ83] obtain similar results for $\pi_i DIFF(M) \otimes \mathbb{Q}$ when M is a lens space, or more generally, a spherical space form, in which case the maps $M \to B\pi$ and $A(M) \to A(B\pi)$ are as highly connected as the dimension of M.

By [Dw80], [Be86], $A(X)$ has finite type (each $\pi_i A(X)$ is a finitely generated abelian group) if X has finite type and $\pi_1(X)$ is finite.

3.3. The cyclotomic trace map.

For each S-algebra B and prime p, there is a **topological cyclic homology** spectrum $TC(B;p)$ and a natural map

$$trc_B \colon K(B) \to TC(B;p)$$

called the **cyclotomic trace map** [BHM93]. Very briefly, $TC(B;p)$ is built as a homotopy limit from the topological Hochschild homology $THH(B)$ and its cyclic fixed point spectra $THH(B)^{C_{p^n}}$, over the category generated by restriction and Frobenius maps

$$R, F \colon THH(B)^{C_{p^n}} \to THH(B)^{C_{p^{n-1}}} .$$

The trace map $tr \colon K(B) \to THH(B)$ constructed by Bökstedt admits a preferred lift to the homotopy limit $TC(B;p)$.

This target is a very fine relative invariant of algebraic K-theory, by [Du97] (building on previous work of Goodwillie and McCarthy). For any map $A \to B$ of connective S-algebras such that $\pi_0(A) \to \pi_0(B)$ is a nilpotent extension, i.e., a surjection with nilpotent kernel, the commutative square

$$\begin{array}{ccc} K(A) & \longrightarrow & K(B) \\ \downarrow & & \downarrow \\ TC(A;p) & \longrightarrow & TC(B;p) \end{array}$$

becomes homotopy Cartesian after p-adic completion. In fact there is an integral model for TC that makes the square homotopy Cartesian before completion; this is the main result of [BGM].

For example, this result applies with $A = S[\Omega X]$ and $B = \mathbb{Z}[\pi]$, with $\pi = \pi_1(X)$ as above, so that there is a homotopy Cartesian square

$$\begin{array}{ccc} A(X) & \xrightarrow{L} & K(\mathbb{Z}[\pi]) \\ \downarrow{\scriptstyle trc_X} & & \downarrow{\scriptstyle trc_{\mathbb{Z}[\pi]}} \\ TC(X;p) & \xrightarrow{L} & TC(\mathbb{Z}[\pi];p) \end{array}$$

after p-adic completion. Here $TC(X;p)$ is notation for $TC(S[\Omega X];p)$.

Writing $THH(X)$ for $THH(S[\Omega X])$, there is a homotopy equivalence

$$THH(X) \simeq \Sigma^\infty(\Lambda X)_+ \,,$$

where $\Lambda X = \mathrm{Map}(S^1, X)$ is the free loop space on X, now thought of as a topological space rather than as a simplicial set. The circle S^1 acts on both sides, by the Connes cyclic structure on the topological Hochschild construction, and by rotating the loops in ΛX. The homotopy equivalence above is C_{p^n}-equivariant for each finite subgroup $C_{p^n} \subset S^1$, when Σ^∞ is formed in a complete S^1-universe. From this, [BHM93, §5] (see also [Ro02, 1.16]) deduce that there is a homotopy Cartesian square

$$\begin{array}{ccc} TC(X;p) & \xrightarrow{\alpha} & \Sigma^\infty\Sigma_+(ES^1 \times_{S^1} \Lambda X) \\ \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle trf_{S^1}} \\ \Sigma^\infty(\Lambda X)_+ & \xrightarrow{1-\Delta_p} & \Sigma^\infty(\Lambda X)_+ \end{array}$$

after p-adic completion. Here trf_{S^1} is the S^1-transfer map for the S^1-bundle $ES^1 \times \Lambda X \to ES^1 \times_{S^1} \Lambda X$ (which is only defined up to homotopy), and $\Delta_p \colon \Lambda X \to \Lambda X$ winds a free loop p times around itself. (More precisely, given a choice of map trf_{S^1} there exists a commuting homotopy for the square so that the induced map of vertical fibers is a homotopy equivalence.) The composite $\beta \circ trc$ is the trace map $A(X) \to \Sigma^\infty(\Lambda X)_+$, previously constructed by Waldhausen [Wa79].

In the special case $X = *$, the map β has the section $S \to A(*) \to TC(*;p)$, and the homotopy fiber of $trf_{S^1} \colon \Sigma^\infty\Sigma_+\mathbb{C}P^\infty \to S$ is the "stunted projective" spectrum $\Sigma\mathbb{C}P^\infty_{-1}$, so there is a splitting

$$TC(*;p) \simeq S \vee \Sigma\mathbb{C}P^\infty_{-1} \,.$$

after p-adic completion. Similarly, there is a splitting

$$A(*) \simeq S \vee \mathrm{Wh}^{DIFF}(*) \,.$$

Hence there are two homotopy Cartesian squares

$$\begin{array}{ccccc} \mathrm{Wh}^{DIFF}(*) & \longleftarrow & A(*) & \xrightarrow{L} & K(\mathbb{Z}) \\ \downarrow{\scriptstyle \widetilde{trc}} & & \downarrow{\scriptstyle trc_*} & & \downarrow{\scriptstyle trc_{\mathbb{Z}}} \\ \Sigma\mathbb{C}P^\infty_{-1} & \xleftarrow{\alpha} & TC(*;p) & \xrightarrow{L} & TC(\mathbb{Z};p) \end{array}$$

and a (co-)fiber sequence

$$\mathrm{hofib}(trc_{\mathbb{Z}}) \to \mathrm{Wh}^{DIFF}(*) \xrightarrow{\widetilde{trc}} \Sigma\mathbb{C}P^\infty_{-1} \,.$$

3.4. Primary calculations.

We now analyze the outer corners of the last square, completed at a prime p. We deal with the cases $p = 2$, and p an odd regular prime.

The homotopy type of $\mathbb{C}P^\infty_{-1}$ is about as complicated as that of the sphere spectrum, but some detailed information is available. In particular, its mod p cohomology is easily described as a module over the Steenrod algebra A.

$$H^*(\mathbb{C}P^\infty_{-1}; \mathbb{Z}/p) = \mathbb{Z}/p\{y^n \mid i \geq -1\}$$

with $P^i(y^n) = \binom{n}{i} y^{n+(p-1)i}$ and $\beta(y^n) = 0$. For $p = 2$, we read P^i as Sq^{2i} and β as Sq^1, as usual. In this case the cohomology is a cyclic A-module:

$$H^*(\mathbb{C}P^\infty_{-1}; \mathbb{Z}/2) \cong \Sigma^{-2} A/C$$

where $C \subset A$ is the ideal generated by the admissible Sq^I of length ≥ 2, together with the Sq^{2i+1} of odd degree.

The homotopy type of $K(\mathbb{Z})$ is predicted by the Lichtenbaum–Quillen conjecture, in the strong form given by Dwyer and Friedlander [DF85], comparing algebraic K-theory to étale K-theory. These conjectures were verified at $p = 2$ in [RW00], using Voevodsky's proof of the Milnor conjecture and the Bloch–Lichtenbaum spectral sequence. At odd primes these conjectures follow from the Bloch–Kato conjecture, which is now (apparently) proven by Rost and Voevodsky.

Completed at $p = 2$, there is a well-understood fiber sequence

$$\begin{array}{ccc} \Sigma bo & \longrightarrow & K(\mathbb{Z}) \\ & \nwarrow & \downarrow \\ & & ju \end{array}$$

where ju is the connective complex image-of-J spectrum and bo is the connected real K-theory spectrum.

Recall that a prime p is **regular** if it does not divide the order h_p of the ideal class group $Cl(\mathbb{Q}(\zeta_p))$ of the p-th cyclotomic field $\mathbb{Q}(\zeta_p)$, or equivalently, if it does not divide the numerator of any divided Bernoulli number B_n/n. For irregular primes p, the p-torsion in $Cl(\mathbb{Q}(\zeta_p))$ is quite well understood by Iwasawa's Main Conjecture, which was proved in this case by Mazur and Wiles, but the precise homotopy type of the étale K-theory of $\mathbb{Z}[1/p]$ is not yet explicitly understood (as far as I know), so the same goes for the p-completed homotopy type of $K(\mathbb{Z})$.

Completed at an odd regular prime p the situation is easier; there is a splitting

$$K(\mathbb{Z}) \simeq j \vee \Sigma^5 ko$$

where j is the connective image-of-J spectrum and ko the connective real K-theory spectrum.

The homotopy type of $TC(\mathbb{Z}; p)$ was computed for p odd by Bökstedt and Madsen in [BM94] and [BM95], and for $p = 2$ in [Ro99a] and [Ro99b]. Its connective cover is homotopy equivalent to $K(\mathbb{Z}_p)_p$. For $p = 2$ there is a tower of maximally non-trivial fiber sequences

$$\begin{array}{ccccc} \Sigma ju & \longrightarrow & K^{red}(\mathbb{Z}_2) & \longrightarrow & K(\mathbb{Z}_2) \\ & \nwarrow & \downarrow & \nwarrow & \downarrow \\ & & \Sigma^3 ku & & ju \end{array}$$

where ku is the connective complex K-theory spectrum. At odd primes there is a splitting

$$K(\mathbb{Z}_p) \simeq j \vee \Sigma j \vee \Sigma^3 ku \,.$$

For $p = 2$, the calculation [Ro99b] contains enough information to compute $trc\colon K(\mathbb{Z}) \to TC(\mathbb{Z}; 2)$ and its homotopy fiber. There is an explicit fiber sequence

$$\begin{array}{ccc} \Sigma^3 ko & \longrightarrow & \mathrm{hofib}(trc_{\mathbb{Z}}) \\ & \nwarrow & \downarrow \\ & & \Sigma^{-2} ku \end{array}$$

obtained in [Ro02, 3.13]. At odd regular primes, [Ro03, 3.3] uses Tate–Poitou duality to identify $trc\colon K(\mathbb{Z}) \to TC(\mathbb{Z}; p)$ and its homotopy fiber, which splits as

$$\mathrm{hofib}(trc_{\mathbb{Z}}) \simeq j \vee \Sigma^{-2} ko \,.$$

For $p = 2$, we can proceed to determine the mod 2 cohomology of $\mathrm{Wh}^{DIFF}(*)$ as a module over the Steenrod algebra A. Recall the left ideal $C \subset A$ from above. It contains the ideal $A\{Sq^1, Sq^3\} \subset A$.

Theorem [Ro02, 4.5]**.** *The mod 2 spectrum cohomology of* $\mathrm{Wh}^{DIFF}(*)$ *is the unique non-trivial extension of A-modules*

$$0 \to \Sigma^{-2} C/A\{Sq^1, Sq^3\} \to H^*(\mathrm{Wh}^{DIFF}(*); \mathbb{Z}/2) \to \Sigma^3 A/A\{Sq^1, Sq^2\} \to 0 \,.$$

In [Ro02, §5] the Adams spectral sequence

$$E_2^{s,t} = \mathrm{Ext}_A^{s,t}(H^*(\mathrm{Wh}^{DIFF}(*); \mathbb{Z}/2), \mathbb{Z}/2) \Longrightarrow \pi_{t-s}\,\mathrm{Wh}^{DIFF}(*)_2$$

is determined in degrees $t - s \le 21$, up to some extensions in degrees ≥ 19, in part by comparison with the Adams spectral sequence for $\mathbb{C}P^\infty_{-1}$.

As a corollary, a geometrically defined map of spaces $G/O \to \Omega\,\mathrm{Wh}^{DIFF}(*)$ is found to be precisely 8-connected, after 2-completion.

For odd regular primes p, we show that there is a commutative diagram

$$\begin{array}{ccc} \Sigma c & \xrightarrow{\Sigma f} & S^1 \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle i_1} \\ \mathrm{Wh}^{DIFF}(*) & \longrightarrow & \Sigma \mathbb{C}P^\infty_{-1} \end{array}$$

where i_1 is inclusion of the 1-cell in $\Sigma\mathbb{C}P^\infty_{-1}$, and c is the cokernel-of-J spectrum defined by the fiber sequence

$$c \xrightarrow{f} S \xrightarrow{e} j$$

where e induces the Adams e-invariant. The lift g is split injective:

$$\mathrm{Wh}^{DIFF}(*) \simeq \Sigma c \vee (\mathrm{Wh}^{DIFF}(*)/\Sigma c)$$

Let $t\colon \Sigma\mathbb{C}P^\infty \to S$ be the **restricted S^1-transfer**, i.e., the composite of $\Sigma\mathbb{C}P^\infty \subset \Sigma\mathbb{C}P^\infty_+$ and trf_{S^1}. Splitting the suspended $c \to S \to j$ fiber sequence off from that containing $\widetilde{trc}$ leaves a fiber sequence

$$\Sigma^{-2} ko \to (\mathrm{Wh}^{DIFF}(*)/\Sigma c) \to \mathrm{hofib}(t) \,.$$

Here the right hand map induces an isomorphism from the p-torsion in the homotopy of $(\mathrm{Wh}^{DIFF}(*)/\Sigma c)$ to the p-torsion in the homotopy of $\mathrm{hofib}(t)$.

Theorem [Ro03, 3.8]**.** *When completed at an odd regular prime p,* $\mathrm{Wh}^{\mathit{DIFF}}(*)$ *splits off a suspended copy of the cokernel-of-J spectrum, and the torsion homotopy of the remainder is isomorphic to the torsion homotopy of the fiber of the restricted* S^1*-transfer map*

$$t\colon \Sigma\mathbb{C}P^\infty \to S\,.$$

In symbols,

$$\pi_*(\mathrm{Wh}^{\mathit{DIFF}}(*)) \cong \pi_*(\Sigma c) \oplus \pi_*(\mathrm{Wh}^{\mathit{DIFF}}(*)/\Sigma c)$$

and

$$\operatorname{tors}\pi_*(\mathrm{Wh}^{\mathit{DIFF}}(*)/\Sigma c) \cong \operatorname{tors}\pi_*(\mathrm{hofib}(t))\,.$$

In fact, we can also split the suspended infinite quaternionic projective space $\Sigma\mathbb{H}P^\infty$ off from $(\mathrm{Wh}^{\mathit{DIFF}}(*)/\Sigma c)$, but this makes for a more complicated statement.

Corollary [Ro03, 4.9]**.** *The p-torsion in* $\pi_* \mathrm{Wh}^{\mathit{DIFF}}(*)$ *begins:*

For $p = 3$*,* $\mathbb{Z}/3\{\Sigma\beta_1\}$ *in degree* 11 *and* $\mathbb{Z}/3\{\Sigma\alpha_1\beta_1\}$ *in degree* 14*.*

For $p \geq 5$ *regular, a* $\mathbb{Z}/p$ *in degrees* $* = 2n$ *for* $m(p-1) < n < mp$ *and* $1 < m < p$*, except in degree* $2p^2 - 2p - 2$*, followed by* $\mathbb{Z}/p\{\Sigma\beta_1\}$ *in degree* $2p^2 - 2p - 1$ *and a group of order* p^2 *in degree* $2p^2 - 2p + 2$*.*

Corollary [Ro03, 6.4]**.**

For $p = 3$ *and* $s \geq 34$*,* $\pi_{10}H^{\mathit{DIFF}}(D^s)$ *and either* $\pi_9\mathit{DIFF}(D^s)$ *or* $\pi_9\mathit{DIFF}(D^{s+1})$ *contain a copy of* $\mathbb{Z}/p$*.*

For $p \geq 5$ *regular and* $s \geq 12p - 5$*,* $\pi_{4p-3}H^{\mathit{DIFF}}(D^s)$ *and either* $\pi_{4p-4}\mathit{DIFF}(D^s)$ *or* $\pi_{4p-4}\mathit{DIFF}(D^{s+1})$ *contain a copy of* $\mathbb{Z}/p$*.*

A more complete calculation, incorporating the results from surgery theory on $\widetilde{\mathit{DIFF}}(D^s)$, should be possible.

One can also make a cohomological analysis. For $p = 3$ there is a splitting of A-modules

$$H^*(\mathrm{Wh}^{\mathit{DIFF}}(*); \mathbb{Z}/3) \cong H^*(\Sigma c; \mathbb{Z}/3) \oplus H^*(\Sigma\mathbb{H}P^\infty; \mathbb{Z}/3) \oplus \Sigma^{-2}C/A\{\beta, Q_1\}$$

where $C \subset A$ is the annihilator ideal of y^{-1}, spanned by all admissible monomials in A except 1 and the P^i for $i \geq 1$.

For $p \geq 5$ there is a non-trivial extension with more terms. See [Ro03, §5.]

4. Some open or unresolved problems

(1) Establish a PL concordance stable range (Burghelea).
(2) Study concordance spaces in a meta-stable range (Goodwillie, Meng, Weiss), i.e., study Ω^s of the homotopy fiber of fiber of $\sigma\colon C(M \times J^s) \to C(M \times J^{s+1})$ as s grows.
(3) Compare Vogell's involution on $A(M)$ with Kro's involution on $K(S[\Omega M])$, [Vo85], [Kr05].
(4) Identify the involution on $TC(X)$, in the model from [BHM93], making $A(X) \to TC(X)$ involutive (Kro, Dundas).
(5) Can one recover or improve on the explicit results of Hatcher–Wagoner on $\pi_0 C(M)$ and Igusa on $\pi_1 C(M)$ in the DIFF category, derived using Cerf theory, see [Ha78, §3], using $TC(X; p)$?

(6) Compute $A(S^1) = K(S[t, t^{-1}])$ (Hüttemann, Klein, Vogell, Waldhausen, Williams, Madsen, Hesselholt, Schlichtkrull). By Farrell–Jones [FJ91], this determines $A(M)$ for all compact Riemannian M with everywhere non-positive sectional curvature.

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