HIGHER FIXED POINTS OF TOPOLOGICAL HOCHSCHILD HOMOLOGY OF THE INTEGERS

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The spectral sequences.

Let p be an odd prime. The Tate spectral sequence for C_{p^n} acting on $THH(\mathbb{Z})$ with $\mathbb{F}_p$ coefficients has E^2 term

$$\begin{aligned}\hat{E}^2_{**}(C_{p^n}) &= \hat{H}^{-*}(C_{p^n}; \pi_*(THH(\mathbb{Z}); \mathbb{F}_p)) \\ &= E(u_n) \otimes P(t, t^{-1}) \otimes E(e) \otimes P(f)\end{aligned}$$

and converges to $\pi_*(\hat{\mathbb{H}}(C_{p^n}, THH(\mathbb{Z})); \mathbb{F}_p)$. The bidegrees are: $\deg(u_n) = (-1, 0)$, $\deg(t) = (-2, 0)$, $\deg(e) = (0, 2p-1)$ and $\deg(f) = (0, 2p)$.

Restricting to the second quadrant, we have the homotopy fixed point spectral sequence for C_{p^n} acting on $THH(\mathbb{Z})$ with $\mathbb{F}_p$ coefficients:

$$\begin{aligned}E^2_{**}(C_{p^n}) &= H^{-*}(C_{p^n}; \pi_*(THH(\mathbb{Z}); \mathbb{F}_p)) \\ &= E(u_n) \otimes P(t) \otimes E(e) \otimes P(f)\,.\end{aligned}$$

It converges to $\pi_*(THH(\mathbb{Z})^{hC_{p^n}}; \mathbb{F}_p)$. Both spectral sequences are algebra spectral sequences.

The first case.

For $n = 1$, there is a nonzero differential

$$d^{2p}(t^{1-p}) = te \tag{1}$$

(up to a unit in $\mathbb{F}_p$, which we hereafter omit to mention). Also e and tf are infinite cycles and $d^{2p}(u_1) = 0$, so

$$\hat{E}^{2p+1}_{**}(C_p) = E(u_1) \otimes P(t^p, t^{-p}) \otimes E(e) \otimes P(tf)\,.$$

There is also a nonzero differential

$$d^{2p+1}(u_1 t^{-p}) = tf\,. \tag{2}$$

Thus

$$\hat{E}^{2p+2}_{**}(C_p) = P(t^p, t^{-p}) \otimes E(e)$$

which is also the E^∞ term.

It follows that $\hat{\Gamma}_1 \colon THH(\mathbb{Z}) \to \hat{\mathbb{H}}(C_p, THH(\mathbb{Z}))$ induces an isomorphism on homotopy with $\mathbb{F}_p$ coefficients in all nonnegative dimensions. Hence the maps $\Gamma_n \colon THH(\mathbb{Z})^{C_{p^n}} \to THH(\mathbb{Z})^{hC_{p^n}}$ and $\hat{\Gamma}_n \colon THH(\mathbb{Z})^{C_{p^{n-1}}} \to \hat{\mathbb{H}}(C_{p^n}, THH(\mathbb{Z}))$ induce isomorphisms on homotopy with $\mathbb{F}_p$ coefficients in all nonnegative dimensions.

Restricting to the second quadrant, the first differential $d^{2p}(t) = t^{p+1}e$ in the homotopy fixed point spectral sequence yields

$$\begin{aligned} E^{2p+1}_{**}(C_p) &= E(u_1) \otimes E(e) \otimes P(tf) \otimes P(f^p, f^{-p}) \\ &\oplus E(u_1) \otimes P_p(tf)\{ef^j \mid v_p(j) = 0\} \end{aligned}$$

in nonnegative total degrees. Here $P_h(x) = P(x)/(x^h = 0)$ is the truncated polynomial algebra of height h, and $v_p(j)$ is the p-adic valuation of j. The second differential $d^{2p+1}(u_1) = t^{p+1}f$ yields

$$\begin{aligned} E^{2p+2}_{**}(C_p) &= E(e) \otimes P_{p+1}(tf) \otimes P(f^p, f^{-p}) \\ &\oplus E(u_1) \otimes P_p(tf)\{ef^j \mid v_p(j) = 0\} \end{aligned}$$

in nonnegative total degrees, which is also the E^∞ term.

This computes $\pi_*(THH(\mathbb{Z})^{C_p}; \mathbb{F}_p)$, giving

$$\dim_{\mathbb{F}_p}(\pi_k(THH(\mathbb{Z})^{C_p}; \mathbb{F}_p)) = \begin{cases} 1 & k \not\equiv 0, -1 \bmod 2p^2 \\ 2 & k \equiv 0, -1 \bmod 2p^2 \, . \end{cases}$$

The second case.

Now consider the Tate spectral sequence for $n = 2$. By naturality with respect to the Frobenius and Verschiebung maps, its E^{2p+1} term agrees with that for $n = 1$, replacing u_1 by u_2, and furthermore $d^{2p+1}(u_2) = 0$. Thus:

$$\hat{E}^{2p+2}_{**}(C_{p^2}) = E(u_2) \otimes P(t^p, t^{-p}) \otimes E(e) \otimes P(tf) \, .$$

All classes of the form $t^i e f^j$ with $j \geq 0$ are in the image of the Tate spectral sequence for S^1 acting on $THH(\mathbb{Z})$ with $\mathbb{Z}_p$ coefficients, under the natural map induced by restriction over $C_{p^n} \subset S^1$ and the coefficient homomorphism $\mathbb{Z}_p \to \mathbb{F}_p$. For bidegree reasons all classes above the horizontal axis in the latter spectral sequence are infinite cycles. Thus so are the image classes $t^i e f^j$ in $\hat{E}^2_{**}(C_{p^n})$ for all $n \geq 1$.

We will show that there is a nonzero differential of the form

$$d^r(t^{p-p^2}) = t^i \cdot e \cdot f^j$$

with $r \geq 2p+2$. The class $t^i e f^j$ lies in $\hat{E}^{2p+2}_{**}(C_{p^2})$ when $i \equiv j \bmod p$. It is in the total degree of $d^r(t^{p-p^2})$ when $i + p^2 = 2p + pj$. Of these classes, the two with least j are $t^{2p-p^2} \cdot e$ and $t^{2p} \cdot e \cdot f^p = t^p \cdot e \cdot (tf)^p$.

The first cannot be hit by a d^r-differential with $r \geq 2p+2$, hence survives to E^∞. It is in total degree $2p^2 - 2p - 1 \not\equiv 0, -1 \bmod 2p^2$, so there is only one permanent cycle in this total degree. Hence the second infinite cycle must be a boundary. We claim the differential hitting it is

$$d^{2p^2+2p}(t^{p-p^2}) = t^p \cdot e \cdot (tf)^p \,. \tag{3}$$

For the classes in total degree $2p^2-2p$ are $t^{pi}\cdot(tf)^j\cdot t^{p-p^2}$ or $t^{pi}\cdot u_2e\cdot(tf)^{j-1}\cdot t^{p-p^2}$ with $pi=(p-1)j$. The only possible sources for a differential are t^{p-p^2}, $(tf)^p$ and $u_2e\cdot(tf)^{p-1}$. Since $r\geq 2p+2$ the latter two are excluded. This proves the claimed formula (3).

We will also show that there is a nonzero differential of the form

$$d^r(u_2e\cdot t^{-p^2}) = t^i\cdot e\cdot f^j$$

with $r\geq 2p+2$. The class $t^i\cdot e\cdot f^j$ is in the total degree of $d^r(u_2e\cdot t^{-p^2})$ when $i+p^2=1+pj$. Of these classes with $i\equiv j \bmod p$, the two with the least j are $t^{-p^2+p+1}\cdot e\cdot f=(t^p)^{1-p}\cdot e\cdot(tf)$ and $e\cdot(tf)^{p+1}$.

The first cannot be hit by a differential of length $r\geq 2p+2$, hence survives to E^∞. It is in total degree $2p^2+2p-3\not\equiv 0,-1 \bmod 2p^2$, so there is only one permanent cycle in this total degree. Hence the second infinite cycle must be a boundary. We claim the differential hitting it is

$$d^{2p^2+2p+1}(u_2e\cdot t^{-p^2}) = e\cdot(tf)^{p+1}\,.$$

For the only possible differentials of length $r\geq 2p+2$ hitting $e\cdot(tf)^{p+1}$ are $t^{-p^2}\cdot tf$, with $r=2p^2+2p$ and $u_2e\cdot t^{-p^2}$ with $r=2p^2+2p+1$. But $d^{2p^2+2p}(t^{-p^2}\cdot tf)=0$ because t^{-p} survives to the E^{2p^2+2p} term and tf is an infinite cycle. This proves the claim.

Canceling the e's, we deduce the differential

$$d^{2p^2+2p+1}(u_2t^{-p^2}) = (tf)^{p+1}\,. \tag{4}$$

Hence t^p and t^{-p} survive to the E^{2p^2+2p} term, while u_2 survives to the E^{2p^2+2p+1} term. Thus there are no earlier differentials; the first odd differential in $\hat{E}^*_{**}(C_{p^2})$ is of length $2p^2+2p+1$. We compute:

$$\begin{aligned}\hat{E}^{2p^2+2p+1}_{**}(C_{p^2}) &= E(u_2)\otimes P(t^{p^2},t^{-p^2})\otimes E(e)\otimes P(tf)\\ &\oplus E(u_2)\otimes P_p(tf)\{t^ie \mid v_p(i)=1\}\end{aligned}$$

and

$$\begin{aligned}\hat{E}^{2p^2+2p+2}_{**}(C_{p^2}) &= P(t^{p^2},t^{-p^2})\otimes E(e)\otimes P_{p+1}(tf)\\ &\oplus E(u_2)\otimes P_p(tf)\{t^ie \mid v_p(i)=1\}\end{aligned}$$

which is also the E^∞ term.

Now restrict to the second quadrant, considering the homotopy fixed point spectral sequence $E^*_{**}(C_{p^2})$. The first differential $d^{2p}(t)=t^{p+1}e$ yields

$$\begin{aligned}E^{2p+1}_{**}(C_{p^2}) &= E(u_2)\otimes E(e)\otimes P(tf)\otimes P(f^p,f^{-p})\\ &\oplus E(u_2)\otimes P_p(tf)\{ef^j \mid v_p(j)=0\}\end{aligned}$$

in nonnegative total degrees. The second differential $d^{2p^2+2p}(t^p) = t^{p^2+p} \cdot e \cdot (tf)^p$ yields

$$\begin{aligned} E^{2p^2+2p+1}_{**}(C_{p^2}) &= E(u_2) \otimes E(e) \otimes P(tf) \otimes P(f^{p^2}, f^{-p^2}) \\ &\oplus E(u_2) \otimes P_p(tf)\{ef^j \mid v_p(j) = 0\} \\ &\oplus E(u_2) \otimes P_{p^2+p}(tf)\{ef^j \mid v_p(j) = 1\} \end{aligned}$$

in nonnegative total degrees. The third differential $d^{2p^2+2p+1}(u_2) = t^{p^2}(tf)^{p+1}$ yields

$$\begin{aligned} E^{2p^2+2p+2}_{**}(C_{p^2}) &= E(e) \otimes P_{p^2+p+1}(tf) \otimes P(f^{p^2}, f^{-p^2}) \\ &\oplus E(u_2) \otimes P_p(tf)\{ef^j \mid v_p(j) = 0\} \\ &\oplus E(u_2) \otimes P_{p^2+p}(tf)\{ef^j \mid v_p(j) = 1\} \end{aligned}$$

in nonnegative total degrees, which is also the E^∞ term.

This computes $\pi_*(THH(\mathbb{Z})^{C_{p^2}}; \mathbb{F}_p)$, giving

$$\dim_{\mathbb{F}_p}(\pi_k(THH(\mathbb{Z})^{C_{p^2}}; \mathbb{F}_p)) = \begin{cases} 2 & k \not\equiv 0, -1 \bmod 2p^3 \\ 3 & k \equiv 0, -1 \bmod 2p^3 \, . \end{cases}$$

The general case.

Inductively, suppose that in the Tate spectral sequence $\hat{E}^*_{**}(C_{p^n})$ there are differentials

$$d^r(t^{p^k - p^{k+1}}) = t^{p^k} \cdot e \cdot (tf)^{p^k + \cdots + p} \tag{5}$$

with $r = 2(p^{k+1} + \cdots + p)$ for all $0 \le k < n$, and a differential

$$d^r(u_n t^{-p^n}) = (tf)^{p^{n-1} + \cdots + 1} \tag{6}$$

with $r = 2(p^n + \cdots + p) + 1$. Also suppose that t^{p^n} survives to the E^r term, with $r = 2(p^n + \cdots + 1)$. In these formulas, each expression $p^a + \cdots + p^b$ means the partial geometric series $\sum_{a \ge i \ge b} p^i$.

The inductive hypothesis completely determines the differential structure in the algebra spectral sequence $\hat{E}^*_{**}(C_{p^n})$. The intermediate terms are, for $0 < k \le n$:

$$\begin{aligned} \hat{E}^r_{**}(C_{p^n}) &= E(u_n) \otimes P(t^{p^k}, t^{-p^k}) \otimes E(e) \otimes P(tf) \\ &\oplus \bigoplus_{m = v_p(i) < k} E(u_n) \otimes P_{p^m + \cdots + p}(tf)\{t^i e \mid v_p(i) = m\} \end{aligned}$$

with $r = 2(p^k + \cdots + p) + 1$. The final (odd) differential leaves the E^r term with $r = 2(p^n + \cdots + 1)$, which equals the E^∞ term since all remaining classes are close to the horizontal axis:

$$\begin{aligned} \hat{E}^\infty_{**}(C_{p^n}) &= P(t^{p^n}, t^{-p^n}) \otimes E(e) \otimes P_{p^{n-1} + \cdots + 1}(tf) \\ &\oplus \bigoplus_{m = v_p(i) < n} E(u_n) \otimes P_{p^m + \cdots + p}(tf)\{t^i e \mid v_p(i) = m\} \, . \end{aligned}$$

By restriction to the second quadrant, the homotopy fixed point spectral sequence $E^*_{**}(C_{p^n})$ has differentials

$$d^r(t^{p^k}) = t^{p^{k+1}+p^k} \cdot e \cdot (tf)^{p^k+\cdots+p} \tag{7}$$

with $r = 2(p^{k+1} + \cdots + p)$ for all $0 \le k < n$, and a differential

$$d^r(u_n) = (tf)^{p^n+\cdots+1} \tag{8}$$

with $r = 2(p^n + \cdots + p) + 1$.

This completely determines the differential structure of the spectral sequence $E^*_{**}(C_{p^n})$. The intermediate terms are

$$\begin{aligned} E^r_{**}(C_{p^n}) &= E(u_n) \otimes E(e) \otimes P(tf) \otimes P(f^{p^k}, f^{-p^k}) \\ &\oplus \bigoplus_{m=v_p(j)<k} E(u_n) \otimes P_{p^{m+1}+\cdots+p}(tf)\{ef^j \mid v_p(j) = m\} \end{aligned}$$

in nonnegative total degrees, for $0 < k \le n$ with $r = 2(p^k + \cdots + p) + 1$. The final (odd) differential leaves the E^r term with $r = 2(p^n + \cdots + 1)$, which again equals the E^∞ term for degree reasons:

$$\begin{aligned} E^\infty_{**}(C_{p^n}) &= E(e) \otimes P_{p^n+\cdots+1}(tf) \otimes P(f^{p^n}, f^{-p^n}) \\ &\oplus \bigoplus_{m=v_p(j)<n} E(u_n) \otimes P_{p^{m+1}+\cdots+p}(tf)\{ef^j \mid v_p(j) = m\} \end{aligned}$$

in nonnegative total degrees.

Counting classes, one obtains:

$$\dim_{\mathbb{F}_p}(\pi_k(THH(\mathbb{Z})^{C_{p^n}}; \mathbb{F}_p)) = \begin{cases} n & k \not\equiv 0, -1 \bmod 2p^{n+1} \\ n+1 & k \equiv 0, -1 \bmod 2p^{n+1} . \end{cases}$$

Since $\hat{\Gamma}_{n+1}$ induces an isomorphism on homotopy with $\mathbb{F}_p$ coefficients in nonnegative dimensions, the same formula computes $\dim_{\mathbb{F}_p}(\pi_k(\hat{\mathbb{H}}(C_{p^{n+1}}, THH(\mathbb{Z})); \mathbb{F}_p))$. We thus know the number of permanent cycles in the next Tate spectral sequence $\hat{E}^*_{**}(C_{p^{n+1}})$.

By naturality with respect to the Frobenius and Verschiebung maps, the E^r terms of $\hat{E}^*_{**}(C_{p^{n+1}})$ are isomorphic to those of $\hat{E}^*_{**}(C_{p^n})$ for $r \le 2(p^n + \cdots + p) + 1$, replacing u_n by u_{n+1}, and $d^r = 0$ for $r = 2(p^n + \cdots + p) + 1$. Thus

$$\begin{aligned} \hat{E}^r_{**}(C_{p^{n+1}}) &= E(u_{n+1}) \otimes P(t^{p^n}, t^{-p^n}) \otimes E(e) \otimes P(tf) \\ &\oplus \bigoplus_{m=v_p(i)<n} E(u_{n+1}) \otimes P_{p^m+\cdots+p}(tf)\{t^i e \mid v_p(i) = m\} \end{aligned}$$

for $r = 2(p^n + \cdots + 1)$.

We will show that there is a differential

$$d^r(t^{p^n - p^{n+1}}) = t^{p^n} \cdot e \cdot (tf)^{p^n+\cdots+p} \tag{9}$$

with $r = 2(p^{n+1} + \cdots + p)$. The target is in total degree $2p^{n+1} - 2p^n - 1 \not\equiv 0, -1 \bmod 2p^{n+1}$, where there are a total of n permanent cycles. These can be seen to be of the form $t^i \cdot e \cdot f^j$ with

$$\begin{aligned} i &= p^n - p^{n+1} + p^{k+1} + \cdots + p \\ j &= p^k + \cdots + p \,, \end{aligned}$$

for $0 \le k < n$. Thus the next infinite cycle $t^{p^n} \cdot e \cdot (tf)^{p^n + \cdots + p}$, which is the case $k = n$ of the formulas above, must be a boundary. Checking bidegrees, the only possible source of the differential hitting it is $t^{p^n - p^{n+1}}$. This establishes the claimed differential (9).

We also claim there is a differential

$$d^r(u_{n+1} e \cdot t^{-p^{n+1}}) = e \cdot (tf)^{p^n + \cdots + 1}$$

with $r = 2(p^{n+1} + \cdots + p) + 1$, which upon canceling the permanent cycle e yields the desired formula

$$d^r(u_{n+1} \cdot t^{-p^{n+1}}) = (tf)^{p^n + \cdots + 1} \,. \tag{10}$$

The target class $e \cdot (tf)^{p^n + \cdots + 1}$ is in total degree $2p^{n+1} + 2p - 3 \not\equiv 0, -1 \bmod 2p^{n+1}$, so there are precisely n permanent cycles in this total degree. These are the classes $t^i \cdot e \cdot f^j$ with

$$\begin{aligned} i &= -p^{n+1} + p^{k+1} + \cdots + 1 \\ j &= p^k + \cdots + 1 \,, \end{aligned}$$

for $0 \le k < n$. Thus the next infinite cycle $e \cdot (tf)^{p^n + \cdots + 1}$ is a boundary, and for bidegree reasons the only possible differential hitting it is given by the claimed formula (10).

These counting arguments also show that $t^{p^{n+1}}$ survives past the last odd differential, thus establishing the inductive hypothesis for $n + 1$. This completes the determination of the differentials in the Tate- and homotopy fixed point spectral sequences for the groups C_{p^n} acting on $THH(\mathbb{Z})$ with coefficient in $\mathbb{F}_p$.