ALGEBRAIC K-THEORY OF SYMMETRIC MONOIDAL $\Gamma\mathcal{S}_*$-CATEGORIES WITH WEAK EQUIVALENCES

John Rognes

October 22–29 1998

Abstract. These are notes for the third talk on Dundas' preprint "The cyclotomic trace for symmetric monoidal categories".

1. More about Γ-spaces

1.1 Special and very special Γ-spaces. A Γ-space is a functor $F\colon \Gamma^{op} \to \mathcal{S}_*$ such that $F(S^0) = *$. We say that F is *special* if the natural map

$$F(X \vee Y) \to F(X) \times F(Y)$$

is a homotopy equivalence for all (finite) pointed sets X and Y. Then applying π_0 to the diagram

$$F(S^0) \times F(S^0) \xleftarrow{\simeq} F(S^0 \vee S^0) \xrightarrow{\nabla} F(S^0)$$

induces a commutative monoid structure on $\pi_0 F(S^0)$. We say that F is *very special* if this abelian monoid is in fact a group, i.e., has inverses.

If F is a special Γ-space, then the associated prespectrum $\{n \mapsto F(S^n)\}$ is a semi-Ω-spectrum, in the sense that the adjoint structure maps

$$F(S^n) \to \Omega F(S^{n+1})$$

are homotopy equivalences for all $n \geq 1$. For a proof, see [Segal] or [Bousfield–Friedlander]. The idea is to show that the cofiber sequence $X \to CX \to \Sigma X$ induces a simplicial fiber sequence, whose realization $F(X) \to F(CX) \to F(\Sigma X)$ is also a fiber sequence when $F(X)$ is connected. Since $F(CX)$ is contractible, this gives the result for $X = S^n$, $n \geq 1$.

If F is a very special Γ-space, then the associated prespectrum $\{n \mapsto F(S^n)\}$ is a Ω-spectrum, in the sense that the adjoint structure maps

$$F(S^n) \to \Omega F(S^{n+1})$$

are homotopy equivalences for all $n \geq 0$. The proof for $n = 0$ uses that $F(S^0)$ is a grouplike monoid.

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

1.2 Closed model category structures on $\Gamma\mathcal{S}_*$. A *closed model category* is a category $\mathcal{C}$ equipped with three subcategories $c\mathcal{C}$, $f\mathcal{C}$ and $w\mathcal{C}$ whose morphisms are called *cofibrations*, *fibrations* and *weak equivalences* in $\mathcal{C}$, respectively, subject to axioms reminiscent of the properties of cofibrations, fibrations and homotopy equivalences in the category of spaces and maps. In particular $\mathcal{C}$ has initial and terminal objects $\emptyset$ and $*$, respectively. An object $c \in ob\mathcal{C}$ is called *cofibrant* if the unique morphism $\emptyset \to c$ is a cofibration, and c is called *fibrant* if the unique morphism $c \to *$ is a fibration.

The category $\mathcal{S}_*$ of pointed simplicial sets (= spaces) is a closed model category, with degreewise injections as cofibrations, Kan fibrations as fibrations, and homotopy equivalences as weak equivalences. A Kan fibration $E \to B$ is a map with the right lifting property with respect to all inclusions $\Lambda^n_k \to \Delta^n$ for $0 \leq k \leq n$. Here $\Lambda^n_k \subset \partial\Delta^n$ is the *kth horn* given by omitting the kth face from the boundary of Δ^n. All spaces are cofibrant, while the fibrant spaces are the Kan complexes, i.e., the spaces having the extension property with respect to all inclusions $\Lambda^n_k \to \Delta^n$.

The category $\Gamma\mathcal{S}_*$ has several useful closed model category structures. We now describe two: the *strict Q-structure* and the *stable Q-structure*. In both cases the cofibrations are the Q-cofibrations, i.e., the maps $A \to X$ having the left lifting property with respect to all pointwise fibrations $E \to B$ which are pointwise equivalences. This means that for each $k_+ \in ob\Gamma^{op}$ the map of spaces $E(k_+) \to B(k_+)$ is a fibration and a homotopy equivalence.

In the strict Q-structure the weak equivalences are the pointwise equivalences. In the stable Q-structure the weak equivalences are the stable equivalences, i.e., the maps inducing isomorphisms between the homotopy groups of the associated (pre-)spectra.

In the strict Q-structure, the fibrations are the pointwise fibrations which are pointwise equivalences. These have the right lifting property with respect to all maps which are (strict) Q-cofibrations and strict (= pointwise) weak equivalences. The strictly fibrant objects are those which are pointwise fibrant, i.e., Kan.

In the stable Q-structure the fibrations are the maps having the right lifting property with respect to all maps which are (stable) Q-cofibrations and stable weak equivalences. In this case the stably fibrant objects are the very special Γ-spaces which are pointwise fibrant.

2. Algebraic K-theory

2.1 Segal's construction. Let $(\mathcal{C}, \Box, e)$ be a symmetric monoidal category. It comes equipped with isomorphisms

$$\begin{aligned} a_{xyz} &: (x\Box y)\Box z \cong x\Box(y\Box z) \\ c_{xy} &: x\Box y \cong y\Box x \\ l_y &: e\Box y \cong y \\ r_x &: x\Box e \cong x \end{aligned}$$

for all $x, y, z \in ob\mathcal{C}$, which satisfy a number of coherence diagrams. It is often convenient to suppress the direction of these isomorphisms, by using the same notation for a_{xyz} and its inverse, etc.

Building on an idea of Segal, we now define a functor $\bar{H}\mathcal{C}$ from Γ^{op} taking values in (symmetric monoidal) categories, i.e., a so-called Γ-category.

For $k_+ \in ob\Gamma^{op}$ we let $(\bar{H}\mathcal{C})(k_+)$ be a category of certain k-dimensional cubical diagrams in $\mathcal{C}$, and maps between such cubical diagrams. Precisely, let $\mathcal{P}k_+$ be the (discrete) category of pointed subsets of k_+, with only the identity morphisms. An object (c,γ) of $(\bar{H}\mathcal{C})(k_+)$ is a functor $c\colon \mathcal{P}k_+ \to \mathcal{C}$ such that $c(*) = e$, equipped with coherent isomorphisms

$$\gamma_{A,B}\colon c(A\vee B) \overset{\cong}{\longrightarrow} c(A)\Box c(B)$$

in $\mathcal{C}$ for all pointed subsets $A, B \subseteq k_+$ with $A\cap B = *$. Coherence means that the following diagrams commute:

$$\begin{array}{ccccc} c(A\vee B\vee C) & \xrightarrow[\cong]{\gamma_{A\vee B,C}} & c(A\vee B)\Box c(C) & \xrightarrow[\cong]{\gamma_{A,B}\Box id} & (c(A)\Box c(B))\Box c(C) \\ \| & & & & \cong\big\downarrow a \\ c(A\vee B\vee C) & \xrightarrow[\cong]{\gamma_{A,B\vee C}} & c(A)\Box c(B\vee C) & \xrightarrow[\cong]{id\Box\gamma_{B,C}} & c(A)\Box(c(B)\Box c(C))\,. \end{array}$$

Here $a = a_{c(A),c(B),c(C)}$.

$$\begin{array}{ccc} c(A\vee B) & \xrightarrow[\cong]{\gamma_{A,B}} & c(A)\Box c(B) \\ \| & & \cong\big\downarrow c \\ c(B\vee A) & \xrightarrow[\cong]{\gamma_{B,A}} & c(B)\Box c(A)\,. \end{array}$$

Here $c = c_{c(A),c(B)}$.

$$\begin{array}{ccc} c(*\vee B) & \xrightarrow[\cong]{\gamma_{*,B}} & c(*)\Box c(B) \\ \| & & \| \\ c(B) & \xrightarrow[\cong]{l} & e\Box c(B)\,. \end{array}$$

Here $l = l_{c(B)}$ (or its inverse).

$$\begin{array}{ccc} c(A\vee *) & \xrightarrow[\cong]{\gamma_{A,*}} & c(A)\Box c(*) \\ \| & & \| \\ c(A) & \xrightarrow[\cong]{r} & c(A)\Box e\,. \end{array}$$

Here $r = r_{c(A)}$ (or its inverse).

A morphism $f\colon (c,\gamma) \to (d,\delta)$ in $(\bar{H}\mathcal{C})(k_+)$ is a natural transformation $f\colon c \to d$ of functors $\mathcal{P}k_+ \to \mathcal{C}$ which is compatible with the structural isomorphisms, i.e., such that the diagram

$$\begin{array}{ccc} c(A\vee B) & \xrightarrow[\cong]{\gamma_{A,B}} & c(A)\Box c(B) \\ \big\downarrow f_{A\vee B} & & \big\downarrow f_A\Box f_B \\ d(A\vee B) & \xrightarrow[\cong]{\delta_{A,B}} & d(A)\Box d(B) \end{array}$$

commutes for all pointed subsets $A, B \subseteq k_+$ with $A \cap B = *$.

We think of the objects $c(A)$ for pointed subsets $A \subseteq k_+$ as located at the corners of a k-dimensional cube. Corresponding to the k subsets $\{0, i\} \subset k_+$ for $1 \le i \le k$ there are k objects

$$c(\{0,1\}), \dots, c(\{0,k\})$$

at the corners adjacent to the minimal one. If $A = \{i_1, \dots, i_n\}_+$ there is an isomorphism

$$c(A) \cong c(\{0,i_1\}) \Box \dots \Box c(\{0,i_n\})$$

induced by the structural isomorphisms γ. Hence the k objects displayed above determine the cube $c\colon \mathcal{P}k_+ \to \mathcal{C}$ up to isomorphism. Also a morphism $f = \{f_A \mid A \in ob\mathcal{P}k_+\}$ is determined by its value on these corners, since $f_{A \vee B} = \delta^{-1}_{A,B}(f_A \Box f_B)\gamma_{A,B}$ allows us to recover each f_A from the $f_{\{0,i\}}$ for $1 \le i \le k$.

When $k = 0$ note that $(\bar{H}\mathcal{C})(0_+)$ is the trivial category (one object and one morphism), while when $k = 1$ there is an isomorphism of categories $(\bar{H}\mathcal{C})(1_+) \cong \mathcal{C}$.

Next we explain how $k_+ \mapsto (\bar{H}\mathcal{C})(k_+)$ defines a functor from Γ^{op}. Given a pointed function $f\colon k_+ \to l_+$, we obtain a functor

$$(\bar{H}\mathcal{C})(f)\colon (\bar{H}\mathcal{C})(k_+) \to (\bar{H}\mathcal{C})(l_+)$$

taking a k-dimensional cubical diagram (c, γ) to an l-dimensional cubical diagram (d, δ), given as follows. For pointed subsets $B \subseteq l_+$, let

$$f^!(B) = f^{-1}(B \setminus *) \cup * \subseteq k_+ .$$

Then $d(B) = c(f^!(B))$ and $\delta_{A,B} = \gamma_{f^!(A), f^!(B)}$ defines $(d, \delta) = (\bar{H}\mathcal{C})(f)(c, \gamma)$.

There are k projection maps $p_i\colon k_+ \to 1_+$ given by $p_i(j) = 0$ for $j \ne i$ and $p_i(i) = 1$. They induce functors $(\bar{H}\mathcal{C})(p_i)\colon (\bar{H}\mathcal{C})(k_+) \to (\bar{H}\mathcal{C})(1_+)$ for $1 \le i \le k$.

Lemma. *$\bar{H}\mathcal{C}$ is a special Γ-category, i.e., the product of functors*

$$\prod_{i=1}^{k} (\bar{H}\mathcal{C})(p_i)\colon (\bar{H}\mathcal{C})(k_+) \to \prod_{i=1}^{k} (\bar{H}\mathcal{C})(1_+)$$

is an equivalence of categories.

2.2 Algebraic K-theory of a symmetric monoidal category.

Let $\mathcal{C}$ be a (small) category. Its *nerve category* $N\mathcal{C}$ is the simplicial category

$$[q] \mapsto N_q\mathcal{C}$$

which in degree q is the category of functors $[q] \to \mathcal{C}$ and natural transformations between these. Taking object sets degreewise defines its traditional nerve as a simplicial set, here denoted $obN\mathcal{C}$.

Let $(\mathcal{C}, \Box, e)$ be a (small) symmetric monoidal category. The *algebraic K-theory* of $\mathcal{C}$ is the special Γ-space

$$K(\mathcal{C}) = obN\bar{H}\mathcal{C} .$$

Example. Let R be an associative ring, and let $i\mathcal{P}(R)$ be the category of finitely generated projective R-modules and R-module isomorphisms. Its algebraic K-theory

$$K(R) = obN\bar{H}i\mathcal{P}(R)$$

is a special Γ-space with underlying space

$$K(R)(S^0) = obN(\bar{H}i\mathcal{P}(R))(1_+) \cong obNi\mathcal{P}(R) \simeq \coprod_P B\,\mathrm{Aut}_R(P)\,,$$

i.e., the traditional nerve of $i\mathcal{P}(R)$ as a simplicial set. Its underlying infinite loop space is

$$\Omega^\infty K(R) \xleftarrow{\simeq} \Omega K(R)(S^1) \xleftarrow{\iota} K(R)(S^0)\,,$$

i.e., the group completion of this underlying space.

Example. Let M be an abelian monoid, viewed as a symmetric monoidal category $\mathcal{M}$ with $ob\mathcal{M} = M$ and only identity morphisms. Then

$$(\bar{H}\mathcal{M})(k_+) \cong \mathcal{M} \times \cdots \times \mathcal{M}$$

(k factors on the right), and $ob\bar{H}\mathcal{M}$ is isomorphic to the Eilenberg–Mac Lane Γ-space HM.

2.3 Enrichment. Each category $(\bar{H}\mathcal{C})(k_+)$ is again a symmetric monoidal category, with operation $\bar{\Box}$ acting pointwise on cubical diagrams as in $(c\bar{\Box}d)(A) = c(A)\Box d(A)$ for all $A \in ob\mathcal{P}k_+$. Hence the construction $\bar{H}$ acts on symmetric monoidal categories, and may be iterated.

Also, if $\mathcal{C}$ is enriched, say in $\Gamma\mathcal{S}_*$, then so is each category $(\bar{H}\mathcal{C})(k_+)$, and $\bar{H}\mathcal{C}$ is a functor from Γ^{op} to (symmetric monoidal) $\Gamma\mathcal{S}_*$-categories. This does not affect the objects of $(\bar{H}\mathcal{C})(k_+)$, but replaces the morphism sets $(\bar{H}\mathcal{C})(k_+)(c,d)$ by morphism Γ-spaces.

When $\mathcal{C}$ is a $\Gamma\mathcal{S}_*$-category we let $U\mathcal{C}$ denote its underlying $\mathcal{E}ns$-category, with the same object class $obU\mathcal{C} = ob\mathcal{C}$, but with morphism sets

$$U\mathcal{C}(c,d) = \Gamma\mathcal{S}_*(\mathbb{S}, \underline{\mathcal{C}}(c,d))_0 \cong \underline{\mathcal{C}}(c,d)(S^0)_0\,.$$

Then $\mathcal{C}$ defines a functor

$$\underline{\mathcal{C}} \colon U\mathcal{C}^{op} \times U\mathcal{C} \to \Gamma\mathcal{S}_*$$

given on objects by $\underline{\mathcal{C}} \colon (c,d) \mapsto \underline{\mathcal{C}}(c,d)$. Morphisms $f \colon c' \to c$ and $g \colon d \to d'$ in $U\mathcal{C}$ are represented by maps $f \colon \mathbb{S} \to \underline{\mathcal{C}}(c',c)$ and $g \colon \mathbb{S} \to \underline{\mathcal{C}}(d,d')$ in $\Gamma\mathcal{S}_*$, and $\underline{\mathcal{C}}$ takes (f,g) to the composite

$$f^*g_* = g_*f^* \colon \underline{\mathcal{C}}(c,d) \cong \mathbb{S} \wedge \underline{\mathcal{C}}(c,d) \wedge \mathbb{S} \xrightarrow{f\wedge 1\wedge g} \underline{\mathcal{C}}(c',c) \wedge \underline{\mathcal{C}}(c,d) \wedge \underline{\mathcal{C}}(d,d') \xrightarrow{\circ} \underline{\mathcal{C}}(c',d')\,.$$

3. Nerve categories

3.1 The nerve of a $\Gamma\mathcal{S}_*$-category. Let $\mathcal{C}$ be a $\Gamma\mathcal{S}_*$-category. We will define its *nerve category* $N\mathcal{C}$ as a simplicial $\Gamma\mathcal{S}_*$-category:

$$N\mathcal{C}\colon [q] \mapsto N_q\mathcal{C}\,.$$

In degree q, $N_q\mathcal{C}$ has as objects the functors $c\colon [q] \to U\mathcal{C}$, i.e., the diagrams

$$c_0 \to c_1 \to \cdots \to c_q$$

in the underlying $\mathcal{E}ns$-category $U\mathcal{C}$. Thus $obN_q\mathcal{C} = obN_qU\mathcal{C}$. The problem is to define the morphism Γ-spaces $N_q\underline{\mathcal{C}}(c,d)$.

Let $c, d\colon [q] \to U\mathcal{C}$ be objects in $N_q\mathcal{C}$. The morphism Γ-space should be the Γ-space of diagrams

$$\begin{array}{ccccccc}
c_0 & \longrightarrow & c_1 & \longrightarrow & \cdots & \longrightarrow & c_q \\
\downarrow{\scriptstyle f_0} & & \downarrow{\scriptstyle f_1} & & & & \downarrow{\scriptstyle f_q} \\
d_0 & \longrightarrow & d_1 & \longrightarrow & \cdots & \longrightarrow & d_q
\end{array}$$

in some appropriate sense. Here the horizontal arrows are morphisms in $U\mathcal{C}$, while the vertical arrows belong to $\mathcal{C}$. Their composites must be interpreted in the sense of section 2.3, namely using the associated functor $\underline{\mathcal{C}}\colon U\mathcal{C}^{op} \times U\mathcal{C} \to \Gamma\mathcal{S}_*$. For each arrow $i \to j$ in $[q]$ we then want the various composite maps $c_i \to d_j$ to agree, as elements in $\underline{\mathcal{C}}(c_i, d_j)$.

To make this precise, consider the following diagram in $\Gamma\mathcal{S}_*$:

(∗)

$$\begin{array}{ccccccc}
 & & & & & & \underline{\mathcal{C}}(c_0,d_0) \\
 & & & & & & \downarrow{\scriptstyle d(0\to1)_*} \\
 & & & & \underline{\mathcal{C}}(c_1,d_1) & \xrightarrow{c(0\to1)^*} & \underline{\mathcal{C}}(c_0,d_1) \\
 & & & & \downarrow{\scriptstyle d(1\to2)^*} & & \downarrow{\scriptstyle d(1\to2)_*} \\
 & & & & \vdots & & \vdots \\
 & & & & \downarrow{\scriptstyle d(q-1\to q)_*} & & \downarrow{\scriptstyle d(q-1\to q)_*} \\
\underline{\mathcal{C}}(c_q,d_q) & \xrightarrow{c(q-1\to q)^*} & \cdots & \xrightarrow{c(1\to2)^*} & \underline{\mathcal{C}}(c_1,d_q) & \xrightarrow{c(0\to1)^*} & \underline{\mathcal{C}}(c_0,d_q)\,.
\end{array}$$

Its categorical limit in $\Gamma\mathcal{S}_*$ is contained in the product $\prod_{i=0}^q \underline{\mathcal{C}}(c_i, d_i)$, given pointwise as the subspace of tuples $(f_i)_{i=0}^q$ such that $c(i \to j)^*(f_j) = d(i \to j)_*(f_i)$ in $\underline{\mathcal{C}}(c_i, d_j)$. This is the morphism Γ-space we want.

Hence we define

$$N_q\underline{\mathcal{C}}(c,d) = \lim_{i\to j\in obT[q]} \underline{\mathcal{C}}(c_i,d_j)$$

pointwise as a Γ-space. Here $T[q]$ is the twisted arrow category of $[q]$, appearing as the category indexing the diagram $(*)$ above. This defines the $\Gamma\mathcal{S}_*$-category $N_q\mathcal{C}$, and the association $[q] \mapsto N_q\mathcal{C}$ is contravariantly functorial in $[q]$ as usual, defining the *nerve category* of $\mathcal{C}$ as a simplicial $\Gamma\mathcal{S}_*$-category $N\mathcal{C}$.

We have $obN\mathcal{C} = obNU\mathcal{C}$ as simplicial sets.

3.2 The homotopy nerve of a $\Gamma\mathcal{S}_*$-category. The categorical limit used in defining the nerve category $N\mathcal{C}$ does in general not preserve homotopy equivalences. This will be a problem later, and one way to alleviate the difficulty is to replace the categorical limit by a homotopy limit, in the sense of [Bousfield–Kan].

We define the *homotopy nerve category* $hoN\mathcal{C}$ of a $\Gamma\mathcal{S}_*$-category $\mathcal{C}$ as the simplicial $\Gamma\mathcal{S}_*$-category

$$[q] \mapsto hoN_q\mathcal{C}$$

where $ob\, hoN_q\mathcal{C} = obN_qU\mathcal{C}$ is the set of functors $c\colon [q] \to U\mathcal{C}$, with morphism Γ-spaces

$$hoN_q\underline{\mathcal{C}}(c,d) = \operatorname*{holim}_{i\to j\in obT[q]} \underline{\mathcal{C}}(c_i,d_j)\,.$$

There is a canonical map

$$\kappa\colon \lim_{i\to j\in obT[q]} \underline{\mathcal{C}}(c_i,d_j) \to \operatorname*{holim}_{i\to j\in obT[q]} \underline{\mathcal{C}}(c_i,d_j)$$

inducing a $\Gamma\mathcal{S}_*$-functor $\kappa\colon N\mathcal{C} \to hoN\mathcal{C}$. If $\mathcal{C}$ is such that the diagram $(*)$ consists of pointwise equivalences of pointwise fibrant Γ-spaces, then κ is a pointwise equivalence, and the nerve category and homotopy nerve categories will be pointwise equivalent as $\Gamma\mathcal{S}_*$-categories.

3.3 Nerve categories of $\Gamma\mathcal{S}_*$-categories with weak equivalences. Let $wU\mathcal{C}$ be a subcategory of $U\mathcal{C}$. We call the morphisms in $wU\mathcal{C}$ *weak equivalences*. Let $N^w\mathcal{C}$ be the full simplicial $\Gamma\mathcal{S}_*$-subcategory of $N\mathcal{C}$ with objects in degree q the functors $c\colon [q] \to U\mathcal{C}$ that factorize through the subcategory $wU\mathcal{C}$. Likewise let $hoN^w\mathcal{C}$ be the full simplicial $\Gamma\mathcal{S}_*$-subcategory of $hoN\mathcal{C}$ with objects in degree q the functors $c\colon [q] \to U\mathcal{C}$ that factorize through the subcategory $wU\mathcal{C}$.

The nerve and homotopy nerve constructions discussed in this chapter are functorial with respect to $\Gamma\mathcal{S}_*$-functors, and $\Gamma\mathcal{S}_*$-natural transformations induce simplicial homotopies, much as for the traditional nerve construction.

4. Enriched algebraic K-theory

4.1 The algebraic K-theory category. A *symmetric monoidal $\Gamma\mathcal{S}_*$-category with weak equivalences* is a symmetric monoidal $\Gamma\mathcal{S}_*$-category $(\mathcal{C}, \Box, e)$ together with a subcategory $wU\mathcal{C} \subseteq U\mathcal{C}$ which is closed under $\Box$, and contains all isomorphisms.

(Thus $(wU\mathcal{C}, \Box, e)$ is a symmetric monoidal subcategory of $(U\mathcal{C}, \Box, e)$ with the $\Box$-operation induced from $\mathcal{C}$.)

We define the *algebraic K-theory* of $\mathcal{C}$ with respect to $wU\mathcal{C}$ to be the objects of the simplicial $\Gamma\mathcal{S}_*$-category

$$K(\mathcal{C}, w) = hoN^w\bar{H}\mathcal{C}\,.$$

Thus $obK(\mathcal{C}, w)$ is a simplicial special Γ-space. In degree $[q]$ it is the Γ-space of functors $c\colon [q] \to U\bar{H}\mathcal{C} \cong \bar{H}U\mathcal{C}$ that factor through the subcategory $w\bar{H}U\mathcal{C} = \bar{H}wU\mathcal{C}$. Its underlying space $obK(\mathcal{C}, w)(S^0)$ is the traditional nerve of the category $(\bar{H}wU\mathcal{C})(1_+) \cong wU\mathcal{C}$. Hence there is a natural map

$$\iota\colon obNwU\mathcal{C} \cong obK(\mathcal{C}, w)(S^0) \to \Omega obK(\mathcal{C}, w)(S^1) \simeq \Omega^\infty obK(\mathcal{C}, w)$$

to the underlying infinite loop space of $obK(\mathcal{C}, w)$.

4.2 The Dennis trace map. For any $\Gamma\mathcal{S}_*$-category $\mathcal{C}$ we have defined its topological Hochschild homology $THH(\mathcal{C})$ as a simplicial Γ-space, and there is a Dennis trace map

$$ob\mathcal{C} \to THH(\mathcal{C})(S^0)_0 \subset THH(\mathcal{C})(S^0)\,.$$

Now suppose $\mathcal{C}$ is a symmetric monoidal $\Gamma\mathcal{S}_*$-category with weak equivalences $wU\mathcal{C}$. Then $K(\mathcal{C}, w)$ is a simplicial $\Gamma\mathcal{S}_*$-category, so we may apply the Dennis trace map construction degreewise. This defines a natural transformation

$$tr\colon obK(\mathcal{C}, w) \to THH(K(\mathcal{C}, w))(S^0)_0 \subset THH(K(\mathcal{C}, w))(S^0)$$

from the algebraic K-theory of $\mathcal{C}$ with respect to w to the topological Hochschild homology of $K(\mathcal{C}, w)$.

5. Strong equivalences

5.1 Topological Hochschild homology and strong equivalences. Let $\mathcal{C}$ be a symmetric monoidal $\Gamma\mathcal{S}_*$-category with weak equivalences $wU\mathcal{C}$. We say that $\mathcal{C}$ is a symmetric monoidal $\Gamma\mathcal{S}_*$-category with *strong equivalences* if each morphism $f\colon c \to d$ in $wU\mathcal{C}$ induces pointwise equivalences of pointwise fibrant Γ-spaces

$$f_*\colon \underline{\mathcal{C}}(b, c) \to \underline{\mathcal{C}}(b, d)$$

for all $b \in obU\mathcal{C}$.

Proposition. [Dundas, 4.7] *Let $\mathcal{C}$ be a symmetric monoidal $\Gamma\mathcal{S}_*$-category with strong equivalences $wU\mathcal{C}$. Then*

$$THH(\mathcal{C}) \xrightarrow{\simeq} THH(hoN^w\mathcal{C})\,.$$

Here $\mathcal{C}$ is identified with the 0-simplices of $hoN^w\mathcal{C}$.

The proof uses that all vertical maps in $(*)$ are pointwise equivalences, so the homotopy limit defining $hoN^w\underline{\mathcal{C}}(c, d)$ is homotopy equivalent to the bottom left Γ-space $\underline{\mathcal{C}}(c_q, d_q)$. This makes it possible to replace the morphism Γ-spaces in $hoN^w\mathcal{C}$ by the morphism Γ-spaces in $\mathcal{C}$, and then an explicit simplicial homotopy at the level of THH replaces the underlying $\mathcal{E}ns$-category of $hoN^w\mathcal{C}$ by the underlying $\mathcal{E}ns$-category of $\mathcal{C}$.

Proposition. [Dundas, 4.9] *Let $\mathcal{C}$ be a symmetric monoidal $\Gamma\mathcal{S}_*$-category with weak equivalences $wU\mathcal{C}$. Then*

$$\prod_{i=1}^{k} (p_i)_*\colon THH((\bar{H}\mathcal{C})(k_+)) \xrightarrow{\simeq} \prod_{i=1}^{k} THH(\mathcal{C})$$

is a pointwise equivalence, and $THH(\bar{H}\mathcal{C})$ *is a very special* Γ*-space (in both directions).*

Here $THH(\bar{H}\mathcal{C})$ is a *bi*Γ-space, i.e., a functor $\Gamma^{op} \times \Gamma^{op} \to \mathcal{S}_*$, with one direction coming from THH and the $\Gamma\mathcal{S}_*$-enrichment in $\mathcal{C}$, the other direction coming from Segal's construction $\bar{H}$ and the symmetric monoidal structure on $\mathcal{C}$.

The proof uses the equivalence $(\bar{H}\mathcal{C})(k_+) \to \mathcal{C}^{\times k}$ of $\Gamma\mathcal{S}_*$-categories and the (not so obvious) fact that THH preserves products, to factorize the map above into the two pointwise equivalences

$$THH((\bar{H}\mathcal{C})(k_+)) \xrightarrow{\simeq} THH(\mathcal{C}^{\times k})$$

and

$$THH(\mathcal{C}^{\times k}) \xrightarrow{\simeq} THH(\mathcal{C})^{\times k}.$$

Let $k_+, l_+ \in ob\Gamma^{op}$. For a fixed k_+ the assembly map

$$THH(\mathcal{C})(l_+) \wedge k_+ \to THH((\bar{H}\mathcal{C})(k_+))(l_+)$$

is a stable equivalence viewed as a map of Γ-spaces in l_+, because $THH((\bar{H}\mathcal{C})(k_+))$ is special in k_+. We can express this as:

Corollary. *There is a stable equivalence*

$$(\Sigma^\infty THH(\mathcal{C}))(X) \xrightarrow{\simeq} THH((\bar{H}\mathcal{C})(X))$$

for each $X \in \Gamma^{op}$.

To combine these two results we can think of $\bar{H}\mathcal{C}$ as a (symmetric monoidal) $\Gamma\mathcal{S}_*$-category with strong equivalences.

Corollary. [Dundas, 4.10] *Let* $\mathcal{C}$ *be a symmetric monoidal* $\Gamma\mathcal{S}_*$*-category with strong equivalences* $wU\mathcal{C}$*. Then the composite*

$$(\Sigma^\infty THH(\mathcal{C}))(X) \to THH((\bar{H}\mathcal{C})(X)) \to THH((K(\mathcal{C}, w))(X))$$

is a stable equivalence for each $X \in \Gamma^{op}$.

5.2 The trace map and strong equivalences. Let $\mathcal{C}$ be a symmetric monoidal $\Gamma\mathcal{S}_*$-category with strong equivalences $wU\mathcal{C}$. The results above combine to define a trace map of infinite loop spaces

$$\begin{aligned}\Omega^\infty obK(\mathcal{C}, w) &\xrightarrow{tr} \Omega^\infty THH(K(\mathcal{C}, w)) \\ &\xleftarrow{\simeq} \Omega^\infty THH(\bar{H}\mathcal{C}) \\ &\xleftarrow{\simeq} \Omega^\infty THH(\mathcal{C})\end{aligned}$$

where the 'wrong way' maps are homotopy equivalences. The diagram is natural in $\mathcal{C}$ and w.

Given an $\mathbb{S}$-algebra A, i.e., a monoid in $(\Gamma\mathcal{S}_*, \wedge, \mathbb{S})$, it remains to construct a symmetric monoidal $\Gamma\mathcal{S}_*$-category $\mathcal{F}_A$ with strong equivalences such that

$$\Omega obK(\mathcal{F}_A, w)(S^1) \simeq K_0^f(\pi_0(A)) \times B\widehat{GL}(A)^+$$

and such that $THH(A) \simeq THH(\mathcal{F}_A)$. We leave this for another lecture.