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$K_4(\mathbb{Z})$ is the trivial group

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Abstract

We prove that the fourth algebraic K-group of the integers is the trivial group, i.e., that $K_4(\mathbb{Z}) = 0$. The argument uses rank-, poset- and component filtrations of the algebraic K-theory spectrum $K(\mathbb{Z})$ from Rognes (Topology 31 (1992) 813–845; K-Theory 7 (1993) 175–200), and a group homology computation of $H_1(SL_4(\mathbb{Z}); St_4)$ from Soulé, to compute the odd primary spectrum homology of $K(\mathbb{Z})$ in degrees $\leqslant 4$. This shows that the odd torsion in $K_4(\mathbb{Z})$ is trivial. The 2-torsion in $K_4(\mathbb{Z})$ was shown to be trivial in Rognes and Weibel (J. Amer. Math. Soc., to appear). © 1999 Elsevier Science Ltd. All rights reserved.

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We prove that $K_4(\mathbb{Z})$, the fourth algebraic K-group of the integers, is the trivial group. The argument uses the spectrum level rank filtration from Rognes [6], preliminary computations from Rognes [7], and Soulé's calculation of $H_1(SL_4(\mathbb{Z}); St_4)$ from Soulé [11], to deduce that the finite Abelian group $K_4(\mathbb{Z})$ contains no odd torsion.

This result can be combined with the computation by the author and Weibel [8, 15] of the 2-primary algebraic K-theory of the integers, to deduce that $K_4(\mathbb{Z})$ contains no 2-torsion. That work uses Voevodsky's proof of the Milnor conjecture [12], as well as the Bloch–Lichtenbaum spectral sequence [1] from motivic cohomology to algebraic K-theory. Hence we can state the following theorem:

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Theorem 0.1. $K_4(\mathbb{Z}) = 0$.

This result improves on that of Lee and Szczarba [4], asserting that $K_4(\mathbb{Z})$ is 2-, 3- and 5-torsion, and of Soulé [9, 10], asserting that $K_4(\mathbb{Z})$ is 2- and 3-torsion, with 3-torsion either 0 or $\mathbb{Z}/3$.

The paper is organized as follows. Sections 1–4 review the algebraic K-theory spectrum, rank and poset filtrations, stable buildings and stable apartments from Rognes [6], with a view towards the applications in this paper. The reader may refer to [6, Sections 1–9] for further discussion. Section 7 introduces the new component filtration of the stable buildings, and Proposition 7.7 describes its filtration subquotients in terms of homotopy types that were introduced and analyzed in Sections 5 and 6. This leads to the component filtration spectral sequence in Theorem 7.9. In the final section 8, calculations are made with this spectral sequence, and the rank filtration spectral sequence (8.3). The argument concludes in Theorem 8.5 by citing results from Soulé [11] and Rognes [7] to compute the spectrum homology of $K(\mathbb{Z})$ in degrees $\leqslant 4$.

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1. The rank filtration

Let R be an associative, unital ring. The category $\mathscr{P}(R)$ of finitely generated projective R-modules admits the structure of a category with cofibrations and weak equivalences in the sense of Waldhausen [13]. The nth space of the algebraic K-theory spectrum $K(R)$ can be obtained by applying the $S_\bullet$-construction of loc.cit. n times to this category, restricting to the isomorphism subcategory, and taking geometric realization:

$$K(R)_n = |iS^n_\bullet \mathscr{P}(R)|.$$

The simplices of $K(R)_n$ are suitable diagrams in $\mathscr{P}(R)$. The homotopy groups of the spectrum $K(R)$ are Quillen's higher algebraic K-groups [5] of the ring R.

Definition 1.1. By a *spectrum* X we mean a sequence of simplicial sets X_n, equipped with simplicial structure maps $\Sigma X_n = X_n \wedge S^1 \to X_{n+1}$ for all $n \geqslant 0$. A *subspectrum* $Y \subseteq X$ is a sequence of simplicial subsets $Y_n \subseteq X_n$ preserved by the structure maps.

Now suppose R is such that R^i split injects into R^j as an R-module only if $i \leqslant j$. This certainly holds if R is commutative. Then each finitely generated free R-module has a well-defined rank, and there is a sequence of subspectra

$$* \simeq F_0K(R) \subset F_1K(R) \subset \cdots \subset F_kK(R) \subset \cdots \subset K(R) \tag{1.2}$$

called the *spectrum level rank filtration* [6, Section 3]. Here $F_kK(R) \subset K(R)$ is the subspectrum with nth space built from the simplices of $K(R)_n$ that only involve free modules in $\mathscr{P}(R)$ of rank $\leqslant k$. When all finitely generated projective R-modules are free the rank filtration exhausts $K(R)$. This holds when R is a PID. To describe the filtration subquotients

$$\bar{F}_k K(R) = F_k K(R)/F_{k-1} K(R)$$

for $k \geqslant 1$ we need to introduce some terminology, and the stable buildings $D(R^k)$.

2. Stable buildings

Sets equipped with a partial quasi-ordering (= posets) can be thought of as small categories, with a unique morphism $a \to b$ if and only if $a \leqslant b$ in the partial quasi-ordering. Conversely, small categories with at most one morphism between any pair of objects correspond to such posets. We shall use both points of view. Also, a functor from a small category can be thought of as a (commutative) diagram indexed on that category. We will then call the objects of the indexing category the *vertices* of the diagram.

Let $[q] = \{0 \to 1 \to \dots \to q\}$ and let $[q]^n$ be its n-fold Cartesian power. Viewed as posets, $[q]$ is linearly ordered while $[q]^n$ has the product partial ordering. Let $Sub(R^k)$ be the subcategory of $\mathscr{P}(R)$ of submodules of R^k and inclusions among these. By a *split injection* of R-modules we mean an R-module homomorphism that admits a left inverse. A choice of a left inverse is *not* part of the structure.

Definition 2.1. Let $k \geqslant 1$. Define $D(R^k)_n$ as the based simplicial set with q-simplices (the base point $*_q$ and) the diagrams

$$X : [q]^n \to Sub(R^k)$$

satisfying the following three axioms:

(1) $X(i_1, \dots, i_n) = 0$ if some $i_s = 0$,
(2) $X(q, \dots, q) = R^k$, and
(3) for each pair of vertices $u < v$ in $[q]^n$ the induced map

$$\operatorname*{colim}_{u \leqslant w < v} X(w) \to X(v)$$

is a split injection of R-modules.

The simplicial structure maps d_i and s_j act on X by precomposition, except d_0 which takes any nondegenerate q-simplex to $*_{q-1}$. Its value on a degenerate simplex is determined by the simplicial identities. The spaces $D(R^k)_n$ for $n \geqslant 0$ assemble to a spectrum $D(R^k)$, called the kth *stable building* of R. (See also [6, Section 3].)

These axioms are called the *lattice conditions*, and a diagram $X : [q]^n \to Sub(R^k)$ satisfying them is called a *lattice*. In a lattice, each $X(v)$ is submodule of R^k, and each map $X(u) \to X(v)$ is a split injection. By induction each colimit $\operatorname{colim}_{u \leqslant w < v} X(w)$ equals the internal sum in $X(v)$ of the submodules $X(w)$ for the w satisfying $u \leqslant w < v$.

The following result is [6, 3.8].

Proposition 2.2. $\bar{F}_k K(R) \simeq EGL_k(R)_+ \wedge_{GL_k(R)} D(R^k)$.

Here the general linear group $GL_k(R)$ acts on $D(R^k)$ via its natural action on the category of submodules of R^k. It remains to determine the *naively* $GL_k(R)$-equivariant homotopy type of $D(R^k)$, in the sense where a $GL_k(R)$-equivariant map which is a nonequivariant homotopy equivalence is considered to be an equivalence. Such maps preserve the homotopy type of the corresponding $GL_k(R)$-homotopy orbit spectra.

3. The poset filtration

Fix a lattice $X : [q]^n \to Sub(R^k)$. The vertices $v \in [q]^n$ where the induced map

$$\operatorname*{colim}_{w<v} X(w) \to X(v)$$

is *not* an isomorphism will be called *special vertices*, and the corresponding submodules $X(v) \subseteq R^k$ will be called the *special modules* of X. (These were called 'pick sites' and 'the submodule configuration' in [6, Section 5].) Count each special vertex v with a multiplicity equal to the rank of the cokernel of the map above. Then by [6, 5.5] there are precisely k special vertices in $[q]^n$, and we can number these as $v_1, \dots, v_k$.

The special vertices and the corresponding special modules uniquely determine the lattice X. At any vertex $w \in [q]^n$ the module $X(w)$ is the internal sum in R^k of the special modules $X(v)$ indexed by the special vertices v with $v \leqslant w$.

Define a partial quasi-ordering $\leqslant$ on $\{1, \dots, k\}$ by setting $i \leqslant j$ if and only if $v_i \leqslant v_j$ in the product partial ordering on $[q]^n$, or equivalently, if $X(v_i) \subseteq X(v_j)$. This gives $\{1, \dots, k\}$ the structure of a poset, which we denote as $\omega(X)$. A different choice of numbering of the k special vertices $v_1, \dots, v_k$ may well give rise to a different poset structure on $\{1, \dots, k\}$, but at least its isomorphism class $[\omega(X)]$ is well defined. (Two posets are isomorphic if they are isomorphic as categories.) We call $[\omega(X)]$ the *poset associated* to X.

We say that a poset ω_1 is *stronger* than a poset ω_2 with the same underlying set, if each relation $i \leqslant j$ in ω_2 also holds in ω_1, or equivalently, if ω_2 is a subcategory of ω_1. Likewise the isomorphism class $[\omega_1]$ is *stronger* than $[\omega_2]$ if ω_1 is stronger than some poset isomorphic to ω_2. This defines partial orderings on the set of posets, resp. the set of isomorphism classes of posets, that have underlying set $\{1, \dots, k\}$. Let $\iota_k = \{1 \cong 2 \cong \cdots \cong k\}$ be the *indiscrete* poset with k equivalent elements, and let $\delta_k = \{1, 2, \dots, k\}$ be the *discrete* poset with k unrelated elements. Then ι_k is stronger than any other poset ω on $\{1, 2, \dots, k\}$, which in turn is stronger than δ_k.

Definition 3.1. Given a poset ω with underlying set $\{1, \dots, k\}$ we let

$$F_{[\omega]} D(R^k) \subseteq D(R^k)$$

be the subspectrum whose nth space has q-simplices (the base point $*_q$ and) the lattices $X : [q]^n \to Sub(R^k)$ such that $[\omega(X)]$ is stronger than $[\omega]$. As $[\omega]$ varies, these subspectra filter $D(R^k)$. This is the *poset filtration* on $D(R^k)$, which is indexed on the partially ordered set of isomorphism classes $[\omega]$ of poset structures on $\{1, \dots, k\}$.

We remark that this filtration is somewhat unusual in that it is not indexed on a linearly ordered set, but a partially ordered one.

Let $F_{<[\omega]}D(R^k)$ be the subspectrum of $F_{[\omega]}D(R^k)$ consisting of lattices whose associated poset is strictly stronger than $[\omega]$ (not isomorphic to ω), and let

$$\bar{F}_{[\omega]}D(R^k) = F_{[\omega]}D(R^k)/F_{<[\omega]}D(R^k)$$

be the filtration subquotient for the poset filtration. To describe these quotient spectra we compare with the algebraic K-theory of finite sets.

4. Stable apartments

Let $\mathscr{E}$ be the category of finite sets. It admits the structure of a category with cofibrations and weak equivalences, and has an associated algebraic K-theory spectrum $K(\mathscr{E})$ equivalent to the sphere spectrum S, by the Barratt–Priddy–Quillen theorem. (See [6, Section 4] for more details.) We filter $K(\mathscr{E})$ by subspectra $F_k K(\mathscr{E})$ whose nth space has simplices that are diagrams of sets of cardinality $\leqslant k$. Let $Sub(k)$ be the subcategory of $\mathscr{E}$ of subsets of $\{1, \dots, k\}$ and inclusions among these.

Definition 4.1. Define $A(k)_n$ as the based simplicial set with q-simplices (the base point $*_q$ and) the diagrams

$$X : [q]^n \to Sub(k)$$

satisfying the axioms

(1) $X(i_1, \dots, i_n) = \emptyset$ if some $i_s = 0$,
(2) $X(q, \dots, q) = \{1, \dots, k\}$, and
(3) for each pair of vertices $u < v$ in $[q]^n$ the induced map

$$\operatorname*{colim}_{u \leqslant w < v} X(w) \to X(v)$$

is an injection of finite sets.

The face and degeneracy maps are defined as usual by deletions and repetitions, cf. Definition 2.1. The spaces $A(k)_n$ for $n \geqslant 0$ assemble to a spectrum $A(k)$, called the kth *stable apartment*.

For each lattice $X : [q]^n \to Sub(k)$ we again take note of the special vertices $v_1, \dots, v_k \in [q]^n$ for which the induced map

$$\operatorname*{colim}_{w < v} X(w) \to X(v)$$

is not a bijection. This time these can be canonically numbered, by letting v_i be the minimal vertex in $[q]^n$ such that $i \in X(v_i)$, for each $i \in \{1, \dots, k\}$. Hence to each lattice X of subsets of $\{1, \dots, k\}$ we have a well-defined *associated poset* $\omega(X)$, in which $i \leqslant j$ if and only if $v_i \leqslant v_j$ in $[q]^n$.

Let $F_\omega A(k) \subseteq A(k)$ be the subspectrum whose nth space has q-simplices the lattices X with associated poset $\omega(X)$ stronger than ω. As ω varies, this defines the *poset filtration* on $A(k)$. Let $F_{<\omega} A(k)$ be the subspectrum of lattices with associated poset strictly stronger than ω, and let

$$\bar{F}_\omega A(k) = F_\omega A(k)/F_{<\omega} A(k)$$

be the filtration subquotient.

There is a functor $\mathscr{E} \to \mathscr{P}(R)$ taking a set U to the free R-module $R[U]$ it generates. We identify $R[\{1, \ldots, k\}]$ with R^k by taking i to the ith basis element e_i. There results a functor $Sub(k) \to Sub(R^k)$ and an embedding $A(k) \to D(R^k)$. We think of $A(k)$ as a subspectrum of $D(R^k)$ by way of this embedding. The embedding preserves associated posets, hence respects the poset filtrations, and maps $\bar{F}_\omega A(k)$ into $\bar{F}_{[\omega]} D(R^k)$ for all ω. The symmetric group Σ_k acts on $A(k)$ by permuting the elements of $\{1, \ldots, k\}$, and the embedding is equivariant with respect to the usual homomorphism $\Sigma_k \to GL_k(R)$.

Let ω be a poset structure on $\{1, \ldots, k\}$. Consider a lattice $X: [q]^n \to Sub(k)$ determining a q-simplex in the nth space of $A(k)$, thought of as embedded in $D(R^k)$, that has associated poset $\omega(X) = \omega$. Then the stabilizer for the $GL_k(R)$-action on X is the parabolic subgroup $P_\omega \subseteq GL_k(R)$ leaving invariant the special modules $X(v_i)$ of X for $i = 1, \ldots, k$. These are precisely the submodules $R[U_i] \subseteq R^k$ spanned by the subsets

$$U_i = \{ j \mid j \leqslant i \text{ in } \omega \}$$

for $i = 1, \ldots, k$. In particular this stabilizer is the same for each such X in the stratum $F_\omega A(k)_n \backslash F_{<\omega} A(k)_n$. Also let $(\Sigma_k)_\omega \subseteq \Sigma_k$ be the stabilizer of ω for the Σ_k-action on the set of poset structures on $\{1, \ldots, k\}$.

The following result is [6, 8.6].

Proposition 4.2. $\bar{F}_{[\omega]} D(R^k) \cong GL_k(R)/P_{\omega+} \wedge_{(\Sigma_k)_\omega} \bar{F}_\omega A(k)$.

This leaves us with the task of describing the naively $(\Sigma_k)_\omega$-equivariant homotopy type of $\bar{F}_\omega A(k)$.

5. $(\ell + 1)$-ad homotopy types

A poset is *connected* if, when viewed as a category its geometric realization is path connected. Its *components* are the maximal connected subposets. Let $c(\omega)$ denote the number of components of ω. A poset is *linear* if $i \leqslant j$ or $j \leqslant i$ for each pair i, j in its underlying set. It is *componentwise linear* if each component is linear. The *length* of a linear poset is the maximal number of composable, noninvertible morphisms in the poset. The *length* of a componentwise linear poset is the sum of the lengths of its components. Let $\ell(\omega)$ denote the length of a componentwise linear poset ω.

The following result is from [6, Section 9].

Proposition 5.1. *Let ω be a poset structure on $\{1, \dots, k\}$.*

(1) *If $\omega = \iota_k$ is indiscrete then $F_\omega A(k) \cong S$ is the sphere spectrum with trivial Σ_k-action. If ω is not indiscrete then $F_\omega A(k) \simeq *$ is contractible.*
(2) *If ω is not componentwise linear then $F_{<\omega} A(k) \simeq F_\omega A(k)$* and *$\bar{F}_\omega A(k) \simeq *$.*

Let $F_c A(k) \subseteq A(k)$ be the union of the subspectra $F_\omega A(k)$ over the ω with $\leqslant c$ components. Write $F_{<c} A(k) = F_{c-1} A(k)$ for the union over the ω with $< c$ components, and let $\bar{F}_c A(k) = F_c A(k)/F_{<c} A(k)$ be the filtration subquotient.

The filtration of $A(k)$ by the subspectra $F_c A(k)$ for $c = 1, \dots, k$ will be called the *component filtration.* We will study its analog for the stable building $D(R^k)$ in section 7 below.

If ω has c components, let $F_{\omega,<c} A(k) = F_\omega A(k) \cap F_{<c} A(k)$, and let $\bar{F}_{\omega,c} A(k) = F_\omega A(k)/F_{\omega,<c} A(k)$ be the quotient spectrum. Use similar notation when replacing ω by $< \omega$. Then

$$\bar{F}_\omega A(k) = F_\omega A(k)/F_{<\omega} A(k) \cong \bar{F}_{\omega,c} A(k)/\bar{F}_{<\omega,c} A(k).$$

Now suppose ω is componentwise linear, with $c = c(\omega)$ linear components of length $\ell = \ell(\omega)$. Consider the ℓ componentwise linear posets strictly stronger than ω that are obtained by replacing one of the ℓ noninvertible morphisms in ω by an isomorphism. We denote them by $\omega_1, \dots, \omega_\ell$. Each of these has c components and length $(\ell - 1)$. (There are other posets stronger than ω, and we will consider them below.) Let ε be the 'componentwise indiscrete' poset with the same component sets as ω, obtained by inverting all the morphisms of ω. Then ε is an equivalence relation on $\{1, \dots, k\}$, with c equivalence classes.

Let $\Delta = \Delta\{1, \dots, \ell\}$ be an affine $(\ell - 1)$-simplex with vertices $\{1, \dots, \ell\}$, and let $\partial_s = \Delta\{1, \dots, \hat{s}, \dots, \ell\} \subset \Delta$ be the codimension 1 face where the vertex s is omitted. Let $C\Delta$ be a cone on Δ, and let $\Sigma\Delta = C\Delta/\Delta$ be the unreduced suspension. It is an ℓ-cell, based at the image of Δ. Similarly, there are ℓ contractible subspaces $\Sigma\partial_s \subseteq \Sigma\Delta$. Any intersection of less than ℓ of these is contractible, while the intersection of all $\Sigma\partial_s$ is S^0, i.e., two points. The quotient space of $\Sigma\Delta$ by the union of all the $\Sigma\partial_s$ is homeomorphic to S^ℓ.

Notation 5.2. An $(\ell + 1)$-*ad of spectra* $(X, X_1, \dots, X_\ell)$ is a spectrum X and ℓ subspectra $X_i \subseteq X$ for $i = 1, \dots, \ell$. The homotopy type of the union $X_1 \cup \dots \cup X_\ell$ within X is determined by the homotopy types of the intersections $\bigcap_{i \in U} X_i$ as U ranges through the nonempty subsets of $\{1, \dots, \ell\}$. See [14, Section 0].

Proposition 5.3. *Let ω be componentwise linear, as above, with c components and length ℓ. Let $\Sigma\Delta$ be an ℓ-cell, as above.*

(1) *The ℓ subspectra $\bar{F}_{\omega_s,c} A(k)$* of *$\bar{F}_{\omega,c} A(k)$ with $s = 1, \dots, \ell$ cover $\bar{F}_{<\omega,c} A(k)$.*
(2) *Any intersection of less than ℓ of these subspectra is contractible.*
(3) *The intersection of all ℓ subspectra is $\bar{F}_{\varepsilon,c} A(k) \cong \bar{F}_c A(c)$.*

Hence the $(\ell + 1)$-ad formed by $\bar{F}_{\omega,c} A(k)$ and the ℓ subspectra $\bar{F}_{\omega_s,c} A(k)$ is naively $(\Sigma_k)_\omega$-equivariantly homotopy equivalent to the $(\ell + 1)$-ad formed by $\Sigma\Delta \wedge \bar{F}_c A(c)$ and the ℓ subspectra $\Sigma\partial_s \wedge \bar{F}_c A(c)$.

Proof. (1) The subspectrum $F_{<\omega}A(k)$ of $F_\omega A(k)$ is covered by the ℓ subspectra $F_{\omega_s}A(k)$ together with $c(c-1)$ subspectra $F_{\omega'}A(k)$ where ω' is obtained from ω by adjoining a relation $i \leqslant j$ with i minimal in one component, and j maximal in another component of ω. The latter $c(c-1)$ posets ω' all have $(c-1)$ components, hence are ignored when we pass to the quotient spectrum $\bar{F}_{<\omega,c}A(k)$ of $F_{<\omega}A(k)$ where simplices with associated poset having fewer than c components are collapsed to the base point. See the proof of [6, 9.10] for further discussion.

(2) The intersection of less than ℓ of the spectra $\bar{F}_{\omega_s,c}$ is the quotient of a spectrum $F_{\omega'}A(k)$ by its subspectrum $F_{\omega'}A(k) \cap F_{<c}A(k)$. Here ω' is obtained from ω by inverting some, but not all, of its morphisms. Thus ω' is not indiscrete and $F_{\omega'}A(k)$ is contractible. Similarly, $F_{\omega'}A(k) \cap F_{<c}A(k)$ is a union of $c(c-1)$ contractible spectra such that each multiple intersection is contractible. This is because each such intersection has the form $F_{\omega''}A(k)$ where ω'' is not indiscrete. Hence also $F_{\omega'}A(k) \cap F_{<c}A(k)$ is contractible, and the conclusion follows.

(3) The intersection of all ℓ of the spectra $F_{\omega_s}A(k)$ is $F_\varepsilon A(k)$, where ε is obtained by inverting all morphisms in ω. Collapsing the meet with $F_{<c}A(k)$ to a point identifies the intersection with $\bar{F}_{\varepsilon,c}A(k)$.

If we number the equivalence classes for ε viewed as an equivalence relation as $\{1, \dots, c\}$, then each lattice $X : [q]^n \to Sub(k)$ with associated poset $\omega(X) = \varepsilon$ takes values that are disjoint unions of such equivalence classes. With the given numbering we can identify these with lattices $Y : [q]^n \to Sub(c)$ with discrete associated poset $\omega(Y) = \delta_c$. This provides an isomorphism $\bar{F}_{\varepsilon,c}A(k) \cong \bar{F}_c A(c)$.

For the final statement, only equivariance needs to be checked. A permutation $\pi \in (\Sigma_k)_\omega$ stabilizing ω acts on $\bar{F}_c A(c)$ by way of how it permutes the components of ω. To prove equivariance, construct the $(\ell+1)$-ad homotopy equivalence by adjoining one orbit of $(\ell+1)$-ads at a time, in both $\bar{F}_{\omega,c}A(k)$ and $\Sigma\Delta \wedge \bar{F}_c A(c)$. This starts with the $(\Sigma_k)_\omega$-equivariant isomorphism $\bar{F}_{\varepsilon,c}A(k) \cong \bar{F}_c A(c)$, and ends with a naively equivariant homotopy equivalence of $(\ell+1)$-ads, as claimed. □

Notation 5.4. By Rognes [6, 11.11] the spectrum homology of $\bar{F}_c A(c)$ is free Abelian of rank $(c-1)!$ and concentrated in degree $(2c-2)$. It is denoted by $W_c = H_{2c-2}(\bar{F}_c A(c))$ as a Σ_c-module.

6. Suspended Tits buildings

Let F be a field, and consider a finite-dimensional F-vector space W. The *Tits building* $B(W)$ is the geometric realization of the poset of proper nontrivial F-subspaces of W, and inclusions among these. If W has dimension $k \geqslant 2$ then $B(W)$ has the homotopy type of a wedge of $(k-2)$-spheres by the Solomon–Tits theorem. The group $GL_k(F)$ naturally acts on $B(F^k)$, as well as on its homology

$$St_k(F) = \tilde{H}_{k-2}(B(F^k)).$$

For $k = 1$ we define $St_1(F) = \mathbb{Z}$ with trivial $GL_1(F)$-action. We call $St_k(F)$ the kth *Steinberg module* over F.

Let R be a ring, and consider a finitely generated free R-module V. We define the *Tits building* $B(V)$ as the geometric realization of the poset of proper nontrivial direct summands of V, and split R-module inclusions among these.

Hereafter suppose that F is the quotient field of a PID R. The functor taking an R-module $M \subset R^k = V$ to $F \otimes_R M \subset F^k = W$ induces a map $B(R^k) \to B(F^k)$.

Lemma 6.1. *Let R be a PID with quotient field F. The natural map $B(R^k) \to B(F^k)$ is a simplicial isomorphism.*

Proof. When R is a PID and N is a proper nontrivial subspace of F^k, then $N \cap R^k$ is a proper nontrivial direct summand of R^k. (Proof by induction over the dimension of N.) Then the functors $M \mapsto F \otimes_R M$ and $N \mapsto N \cap R^k$ define inverse isomorphisms between the posets defining $B(R^k)$ and $B(F^k)$. □

Again let V be a finitely generated free R-module. Let $CB(V)$ be the geometric realization of the poset of nontrivial direct summands of V, and split R-module inclusions among these. This poset adjoins the maximal element V to the poset defining $B(V)$, hence $CB(V)$ is a cone on $B(V)$. Let the *suspended Tits building* $\Sigma B(V) = CB(V)/B(V)$ be the quotient space of this cone by its base.

Then $\Sigma B(V)$ is the simplicial set with q-simplices (a base point $*_q$ and) the linear chains

$$0 \neq M_0 \subseteq M_1 \subseteq \ \dots\ \subseteq M_q = V$$

of split inclusions of submodules of V. The face maps d_i, resp. degeneracy maps s_j, are given by deleting M_i, resp. repeating M_j, except d_0 which takes nondegenerate simplices to the base point.

Clearly, $\Sigma B(R^k)$ has the homotopy type of a wedge of $(k-1)$-spheres, and

$$St_k(R) = \tilde{H}_{k-1}(\Sigma B(R^k))$$

is the restriction of the kth Steinberg module to $GL_k(R)$ for $k \geqslant 1$. More generally we let $St(V) = \tilde{H}_{k-1}(\Sigma B(V))$ as a $GL(V)$-module, when V is free of rank k.

7. The component filtration

Definition 7.1. Let $F_c D(R^k)$ be the subspectrum of $D(R^k)$ whose nth space has q-simplices (the base point $*_q$ and) the lattices $X : [q]^n \to Sub(R^k)$ such that $[\omega(X)]$ has at most c components. The sequence of subspectra

$$* \cong F_0 D(R^k) \subset F_1 D(R^k) \subset \cdots \subset F_k D(R^k) = D(R^k)$$

is called the *component filtration* of $D(R^k)$. Let $\bar{F}_c D(R^k) = F_c D(R^k)/F_{c-1} D(R^k)$ be the cth subquotient of this filtration.

The poset filtration on $D(R^k)$ descends to define a filtration on $\bar{F}_c D(R^k)$ indexed by the isomorphism classes of posets with c components. Among these, the componentwise linear posets are initial. For adding relations to a componentwise linear poset with c components either yields another poset in this class, or reduces the number of components.

Let $\tilde{F}_c D(R^k) \subseteq \bar{F}_c D(R^k)$ be the union of the spectra $F_{[\omega]} D(R^k)$ for which ω is componentwise linear with c components, with their intersection with $F_{c-1} D(R^k)$ collapsed to a point. Then the simplices in the nth space of $\tilde{F}_c D(R^k)$ have associated posets that are componentwise linear with c components, together with the base point.

Lemma 7.2. *The inclusion $\tilde{F}_c D(R^k) \subseteq \bar{F}_c D(R^k)$ is a homotopy equivalence.*

Proof. The inclusion is filtered by adjoining simplices which have associated posets with c components that are not all linear, and successively allowing more and more such posets. Each filtration subquotient has the form $\bar{F}_{[\omega]} D(R^k)$ with ω not componentwise linear, which is contractible by Propositions 4.2 and 5.1(2). Hence the inclusion at each step of the filtration is a homotopy equivalence, and the result follows. □

Consider a q-simplex $X : [q]^n \to Sub(R^k)$ in the nth space of $\tilde{F}_c D(R^k)$, and number its special vertices $v_1, \dots, v_k$. Then $\omega = \omega(X)$ has c components, each linear. Let us write $\omega = \lambda_1 \amalg \dots \amalg \lambda_c$, with each λ_i linear. Let k_i be the cardinality of λ_i, and ℓ_i its length. Then $\sum_{i=1}^c k_i = k$ and $\sum_{i=1}^c \ell_i = \ell$ is the length of ω.

Then the special vertices $v_1, \dots, v_k$ of X fall into c linear chains, each unrelated to the others in the product ordering on $[q]^n$. We can reindex them as follows:

$$\begin{array}{l} v_{11} \leqslant v_{12} \leqslant \cdots \leqslant v_{1k_1}, \\ \qquad\vdots \\ v_{c1} \leqslant v_{c2} \leqslant \cdots \leqslant v_{ck_c}. \end{array}$$

The corresponding special modules also fall into c linear chains of direct summands of R^k:

$$\begin{array}{l} 0 \neq M_{11} \subseteq M_{12} \subseteq \cdots \subseteq M_{1k_1} = V_1, \\ \qquad\qquad\vdots \\ 0 \neq M_{c1} \subseteq M_{c2} \subseteq \cdots \subseteq M_{ck_c} = V_c. \end{array}$$

Here $M_{ij} = X(v_{ij})$ for $1 \leqslant i \leqslant c$, $1 \leqslant j \leqslant k_i$, and we write $V_i = M_{ik_i}$ for $1 \leqslant i \leqslant c$. We call the $V_1, \dots, V_c$ the *top modules* of the simplex X.

Lemma 7.3. *The top modules $V_1, \dots, V_c$ form a direct sum decomposition of R^k into nontrivial submodules*

$$R^k = \bigoplus_{i=1}^{c} V_i.$$

Proof. The $V_1, \dots, V_c$ span R^k, because $X(q, \dots, q) = R^k$ is the sum of the special modules $X(v)$ for the special vertices $v \leqslant (q, \dots, q)$, each of which is contained in some $X(v_{ik_i}) = V_i$.

The $V_1, \dots, V_c$ are pairwise independent, since $V_i \cap V_j = X(w)$ where $w \in [q]^n$ is maximal such that $w \leqslant v_{ik_i}$ and $w \leqslant v_{jk_j}$. When $i \neq j$ there are no special vertices $v_s \leqslant w$, since the special vertices lie in c unrelated linear chains. Thus $X(w) = 0$. □

Definition 7.4. Let $R^k = \bigoplus_{i=1}^c V_i$ be a direct sum decomposition of R^k into c nontrivial free submodules. Let

$$E(V_1, \dots, V_c) \subseteq \tilde{F}_c D(R^k)$$

be the subspectrum whose nth space has q-simplices the lattices $X : [q]^n \to Sub(R^k)$ with top modules equal to $V_1, \dots, V_c$. The subgroup of $GL_k(R)$ preserving the unordered set $\{V_1, \dots, V_c\}$ naturally acts on $E(V_1, \dots, V_c)$.

Lemma 7.5. *There is a $GL_k(R)$-equivariant splitting of spectra*

$$\tilde{F}_c D(R^k) \cong \bigvee E(V_1, \dots, V_c),$$

where the wedge sum runs over all unordered direct sum decompositions $R^k = \bigoplus_{i=1}^c V_i$ of R^k into c nontrivial free submodules. The action of $GL_k(R)$ permutes the summands on the right.

Proof. Any face map that alters the top modules of a simplex must reduce the number of connected components of the associated poset, and hence maps that simplex to the base point in $\tilde{F}_c D(R^k)$. Thus the set level decomposition of the simplices of $\tilde{F}_c D(R^k)$ by their top modules persists to the spectrum level. □

Proposition 7.6. *There is a naively equivariant homotopy equivalence*

$$E(V_1, \dots, V_c) \simeq \Sigma B(V_1) \wedge \cdots \wedge \Sigma B(V_q) \wedge \bar{F}_c A(c).$$

Proof. We shall filter the simplices of $E(V_1, \dots, V_c)$ by their associated posets, and the c linear chains of special modules leading up to $V_1, \dots, V_c$. Then we iteratively build $E(V_1, \dots, V_c)$ and $\Sigma B(V_1) \wedge \cdots \wedge \Sigma B(V_c) \wedge \bar{F}_c A(c)$ in parallel, by successively allowing more posets ω, together with all 'compatible' chains of special modules. Let k_i be the rank of V_i, for $i = 1, \dots, c$.

Consider a poset $\omega = \lambda_1 \amalg \cdots \amalg \lambda_c$, with each λ_i linear of cardinality k_i. We label the elements of λ_i by $(i,1), \dots, (i,k_i)$. Let each λ_i have length ℓ_i, and let $\ell = \sum_{i=1}^c \ell_i$.

Also consider c linear chains of split inclusions

$$0 \neq M_{11} \subseteq M_{12} \subseteq \cdots \subseteq M_{1k_1} = V_1,$$

$$\vdots$$

$$0 \neq M_{c1} \subseteq M_{c2} \subseteq \cdots \subseteq M_{ck_c} = V_c$$

such that $M_{i,j-1} = M_{i,j}$ if and only if $(i, j-1) \cong (i,j)$ in λ_i. In particular, we assume that $M_{i,j-1} \neq M_{i,j}$ if $(i, j-1) < (i,j)$. We then say that these $(M_{ij})_{ij}$ are *compatible* with ω.

Iteratively build $E(V_1, \dots, V_c)$ by adjoining the simplices with associated poset equal to the given $\omega = \lambda_1 \amalg \cdots \amalg \lambda_c$, and special modules the c linear chains displayed above. The specific locations in $[q]^n$ of the special vertices may vary, subject to these bounds. This means we are adjoining a copy of $\bar{F}_{\omega,c} A(k)$ along the subspectrum $\bar{F}_{<\omega,c} A(k)$ where the posets are strictly stronger. Thus the iterative step in building $E(V_1, \dots, V_c)$ is to adjoin the first $(\ell+1)$-ad of Proposition 5.3.

In parallel build $\Sigma B(V_1) \wedge \dots \wedge \Sigma B(V_c) \wedge \bar{F}_c A(c)$ by adjoining the subspectrum

$$\Sigma \Delta^{\ell_1 - 1} \wedge \cdots \wedge \Sigma \Delta^{\ell_c - 1} \wedge \bar{F}_c A(c),$$

where each $\Sigma \Delta^{\ell_i - 1}$ is the suspended simplex represented by $M_{i1} \subseteq \cdots \subseteq M_{ik_i} = V_i$ in $\Sigma B(V_i)$.

We naturally identify $\Sigma\Delta^{\ell_1-1} \wedge \cdots \wedge \Sigma\Delta^{\ell_c-1}$ with $\Sigma\Delta$ where Δ is the $(\ell-1)$-simplex spanned by the ℓ vertices of $\Delta^{\ell_1-1}, \ldots, \Delta^{\ell_c-1}$. Thus the iterative step to build $\Sigma B(V_1) \wedge \cdots \wedge \Sigma B(V_c) \wedge \bar{F}_c A(c)$ is to adjoin the second $(\ell+1)$-ad of Proposition 5.3. By that proposition these two $(\ell+1)$-ads are homotopy equivalent.

To complete the proof, proceed by induction over the posets ω with c linear components $\lambda_1, \ldots, \lambda_c$ having underlying sets $(i,1), \ldots, (i,k_i)$ as above, partially ordered by strength. For each such ω, adjoin within $E(V_1, \ldots, V_c)$, resp. $\Sigma B(V_1) \wedge \cdots \wedge \Sigma B(V_c) \wedge \bar{F}_c A(c)$, one copy of the $(\ell+1)$-ad of subspectra in $\bar{F}_{\omega,c} A(k)$, resp. the $(\ell+1)$-ad of subspectra in $\Sigma\Delta \wedge \bar{F}_c A(c)$, for each c-tuple of chains of special modules $(M_{ij})_{ij}$ compatible with ω. By Proposition 5.3 and induction, there results a homotopy equivalence as asserted. Refining the argument to treat one orbit for the action of $GL_k(R)$ at a time, we can also obtain the naively equivariant homotopy equivalence we are claiming. □

Bringing these results together, we have proved:

Proposition 7.7. *There is a naively $GL_k(R)$-equivariant homotopy equivalence*

$$\bar{F}_c D(R^k) \simeq \bigvee \Sigma B(V_1) \wedge \cdots \wedge \Sigma B(V_c) \wedge \bar{F}_c A(c),$$

where the wedge sum runs over the unordered direct sum decompositions $R^k = \bigoplus_{i=1}^{c} V_i$ of R^k into c nontrivial free submodules. The $GL_k(R)$-action permutes the wedge summands on the right.

Recall from Notation 5.4 that $H_*(\bar{F}_c A(c))$ is concentrated in degree $(2c-2)$, where it is free Abelian of rank $(c-1)!$. As a Σ_c-module we denote it W_c.

Theorem 7.8. *Let R be a PID. There is a complex $Z_* = Z_*(R^k)$ of $GL_k(R)$-modules with*

$$Z_{k+c-2} = \bigoplus St(V_1) \otimes \cdots \otimes St(V_c) \otimes W_c,$$

where the sum runs over the unordered direct sum decompositions $R^k = \bigoplus_{i=1}^{c} V_i$ into nontrivial submodules. Here $1 \leqslant c \leqslant k$, so Z_* is concentrated in degrees $k-1 \leqslant * \leqslant 2k-2$. Furthermore,

$$H_n(Z_*) \cong H_n(D(R^k))$$

as $GL_k(R)$-modules.

Proof. The filtration of $D(R^k)$ by the $F_c D(R^k)$ induces a spectral sequence in spectrum homology, with E_1-term $E^1_{c,t} = H_{c+t}(\bar{F}_c D(R^k))$. By Proposition 7.7 and the Künneth theorem we can express this in terms of the homology of the suspended Tits buildings $\Sigma B(V_i)$ and of $\bar{F}_c A(c)$. The homology of $\Sigma B(V_i)$ is concentrated in degree (k_i-1) (by Lemma 6.1, since R is a PID), that of $\bar{F}_c A(c)$ is concentrated in degree $(2c-2)$, and both are torsion free. Hence the spectrum homology of $\bar{F}_c D(R^k)$ is concentrated in degree $\sum_{i=1}^{c}(k_i-1) + (2c-2) = k+c-2$, and so the E^1-term above collapses to the single row $t = k-2$. Suitably reindexed this is the complex Z_*, and its homology computes $H_*(D(R^k))$. □

Theorem 7.9. *Let R be a PID. There is a* component filtration spectral sequence *with*

$$E^1_{s,t} = H_t(GL_k(R); Z_s)$$

that is concentrated in the columns $k-1 \leqslant s \leqslant 2k-2$, and which converges to

$$H_{s+t}(EGL_k(R)_+ \wedge_{GL_k(R)} D(R^k)) \cong H_{s+t}(\bar{F}_k K(R)).$$

Proof. This is the spectral sequence in spectrum homology for the filtration of $EGL_k(R)_+ \wedge_{GL_k(R)} D(R^k)$ by the subspectra $EGL_k(R)_+ \wedge_{GL_k(R)} F_c D(R^k)$ with $s = k + c - 2$. The spectrum homology of the filtration subquotients is readily computed by a collapsing Serre spectral sequence, since $H_*(\bar{F}_c D(R^k))$ is concentrated in degree s. Thus the E^1-term appears as

$$\begin{aligned} E^1_{s,t} &= H_{s+t}(EGL_k(R)_+ \wedge_{GL_k(R)} \bar{F}_c D(R^k)) \\ &\cong H_t(GL_k(R); H_s(\bar{F}_c D(R^k))) \\ &\cong H_t(GL_k(R); Z_s) \end{aligned}$$

for all $s, t \geqslant 0$. □

8. Calculations

Now specialize the above theory to the case when $R = \mathbb{Z}$ is the usual ring of integers. We briefly write St_k for $St_k(\mathbb{Z})$ as a $GL_k(\mathbb{Z})$-module.

Lee and Szczarba showed in [2, 1.3] that $H_0(GL_k(\mathbb{Z}); St_k) = 0$ for all $k \geqslant 2$. This has the following consequence:

Lemma 8.1. *In the component filtration spectral sequence for $R = \mathbb{Z}$ we have*

$$E^1_{s,0} = H_0(GL_k(\mathbb{Z}); Z_s) = 0$$

for all $k-1 \leqslant s < 2k-2$.

Proof. For each $1 \leqslant c < k$ the $GL_k(\mathbb{Z})$-module $Z_s = Z_{k+c-2}$ is a sum of modules co-induced from $St_{k_1} \otimes \cdots \otimes St_{k_c} \otimes W_c$, where at least one $k_i \geqslant 2$. Hence by Lee and Szczarba's result its coinvariants vanish. □

Corollary 8.2. *For each $k \geqslant 2$ the spectrum $\bar{F}_k K(\mathbb{Z})$ is at least $(k-1)$-connected.*

The filtration (1.2) of $K(\mathbb{Z})$ by the subspectra $F_{s+1}K(\mathbb{Z})$ for $s \geqslant 0$ induces the *rank filtration spectral sequence* in spectrum homology, with

$$E^1_{s,t} = H_{s+t}(\bar{F}_{s+1} K(\mathbb{Z})) \tag{8.3}$$

converging to $H_{s+t}(K(\mathbb{Z}))$.

The spectrum homology $H_*(\bar{F}_k K(\mathbb{Z}))$ was computed for $k \leqslant 3$ in [7]. This determines the columns $s = 0, 1$ and 2 of the rank filtration spectral sequence above. (Working modulo finite 2-groups, only Proposition 3.1 and Theorems 3.2(2), 3.4(2) of Rognes [7] are needed. These are proven on pp. 186 and 198–199 of that paper.) By Corollary 8.2 each group $E^1_{s,0} = 0$ for $s \geqslant 1$.

When $s = 3$, we use the component filtration spectral sequence 7.9 for $k = 4$ to compute $E^1_{3,1} \cong H_1(GL_4(\mathbb{Z}); Z_3(\mathbb{Z}^4)) = H_1(GL_4(\mathbb{Z}); St_4)$. By the calculation of Soulé [11], we have $H_1(SL_4(\mathbb{Z}); St_4) \equiv 0$ modulo the Serre class of finite 2-groups. Consequently, $H_1(GL_4(\mathbb{Z}); St_4) \equiv 0$ modulo the same Serre class, and we deduce that $E^1_{3,1} \equiv 0$ modulo finite 2-groups.

The rank filtration spectral sequence for $\mathbb{Z}$ thus appears as follows, modulo finite 2-groups:

5	0	0	0	?	?	?
4	0	0	0	?	?	?
3	0	0	$\mathbb{Z}$	?	?	?
2	0	$\mathbb{Z}/3$	$\mathbb{Z}/3$	?	?	?
1	0	0	0	0	?	?
0	$\mathbb{Z}$	0	0	0	0	0
	0	1	2	3	4	5

(8.4)

We know that $H_*(K(\mathbb{Z}))$ begins $(\mathbb{Z}, 0, 0, \mathbb{Z}/2, \dots)$ by the calculations of Lee and Szczarba [3]. Hence there is a bijective differential

$$d^1_{2,2} : E^1_{2,2} \to E^1_{1,2}$$

modulo finite 2-groups.

Theorem 8.5. *The spectrum homology of* $K(\mathbb{Z})$ *begins*

$$H_*(K(\mathbb{Z})) \cong (\mathbb{Z}, 0, 0, \mathbb{Z}/2, 0, \dots)$$

in degrees $0 \leqslant * \leqslant 4$.

Proof. The 2-primary part of this statement follows from the results of [8, 12, 15]. Modulo finite 2-groups the E^∞-term of the rank filtration spectral sequence has $E^\infty_{00} \cong \mathbb{Z}$ and trivial groups in total degrees $1 \leqslant * \leqslant 4$, by the discussion above. The result follows. □

Theorem 8.6. *The algebraic K-groups of the integers begin*

$$K_*(\mathbb{Z}) \cong (\mathbb{Z}, \mathbb{Z}/2, \mathbb{Z}/2, \mathbb{Z}/48, 0, \dots)$$

in degrees $0 \leqslant * \leqslant 4$.

Proof. The 2-primary result follows from the same references as above. Modulo finite 2-groups the unit map $S \to K(\mathbb{Z})$ is at least 4-connected by the spectrum homology calculation. Hence $\pi_4(S) = 0$ surjects onto $K_4(\mathbb{Z})$, which therefore vanishes. □

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