LOGARITHMIC DE RHAM-WITT COMPLEXES

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1. Witt vectors

Let $a, b \in \mathbb{Z}$ and fix a prime p. Then $a^p \equiv a \mod p$, and more generally $a^{p^n} \equiv a^{p^{n-1}} \mod p^n$. If $a \equiv b \mod p$, then $a^p \equiv b^p \mod p^2$, and more generally, $a^{p^n} \equiv b^{p^n} \mod p^{n+1}$ for each $n \geq 0$. Let $\mathbb{Z}_p = \lim_{n\to\infty} \mathbb{Z}/p^n$ be the ring of p-adic integers. Letting $\omega(a) = \lim_n a^{p^n}$ we obtain a multiplicative (but not additive) section

$$\omega \colon \mathbb{Z}/p \longrightarrow \mathbb{Z}_p$$

called the Teichmüller character.

The ring $\mathbb{Z}_p$ is a complete discrete valuation ring, with maximal ideal $p\mathbb{Z}_p$. The p-adic filtration

$$\cdots \subset p^n\mathbb{Z}_p \subset \cdots \subset p\mathbb{Z}_p \subset \mathbb{Z}_p$$

has subquotients $p^n\mathbb{Z}_p/p^{n+1}\mathbb{Z}_p \cong \mathbb{Z}_p/p\mathbb{Z}_p \cong \mathbb{F}_p$, for each $n \geq 0$. How can we reconstruct $\mathbb{Z}_p$ from these subquotients? Using the section ω we can find set bijections $\mathbb{Z}/p^n \cong \prod_{i=0}^{n-1} \mathbb{F}_p$ for each n, and $\mathbb{Z}_p \cong \prod_{i\geq 0} \mathbb{F}_p$. The theory of Witt vectors tells us how to reconstruct the ring structure on these sets.

Definition 1.1. Let A be a commutative ring. Let

$$W(A) = \prod_{i\geq 0} A$$

be the set of (p-typical) Witt vectors $a = (a_0, a_1, \dots)$. As a set it is evidently functorial in A, with $f \colon A \to A'$ taking a to $f(a) = (f(a_0), f(a_1), \dots)$. The ghost map

$$g \colon W(A) \longrightarrow \prod_{m\geq 0} A$$

maps a to the sequence $g(a) = (g_0, g_1, \dots)$, where

$$\begin{aligned} g_m(a) &= \sum_{i+j=m} p^i a_i^{p^j} \\ &= a_0^{p^m} + p a_1^{p^{m-1}} + \cdots + p^{m-1} a_{m-1}^p + p^n a_m \,. \end{aligned}$$

We call

$$(g_0, g_1, \dots) = (a_0, a_0^p + p a_1, \dots)$$

the ghost coordinates of $(a_0, a_1, \dots)$.

Give the target $\prod_{m\geq 0} A$ of g the product ring structure:

$$\begin{aligned} (g_0, g_1, \dots) + (h_0, h_1, \dots) &= (g_0 + h_0, g_1 + h_1, \dots) \\ (g_0, g_1, \dots) \cdot (h_0, h_1, \dots) &= (g_0 h_0, g_1 h_1, \dots) \,. \end{aligned}$$

Theorem 1.2. *There is a unique functorial commutative ring structure on $W(A)$ such that g is a natural ring homomorphism.*

Date: March 21st 2018.

Example 1.3. $(a_0, a_1, \dots) + (b_0, b_1, \dots) = (s_0(a,b), s_1(a,b), \dots)$ where

$$s_0(a,b) = a_0 + b_0 \qquad \text{and} \qquad s_1(a,b) = a_1 + b_1 - \frac{(a_0+b_0)^p - a_0^p - b_0^p}{p} .$$

and $(a_0, a_1, \dots) \cdot (b_0, b_1, \dots) = (p_0(a,b), p_1(a,b), \dots)$ where

$$p_0(a,b) = a_0 b_0 \qquad \text{and} \qquad p_1(a,b) = a_0^p b_1 + a_1 b_0^p + p a_1 b_1 .$$

Proposition 1.4. *For each $n \ge 0$, s_n and p_n are integer polynomials in $a_0, \dots, a_n, b_0, \dots, b_n$, independent of the ring A.*

Definition 1.5. For each $n \ge 1$, let

$$W_n(A) = \prod_{i=0}^{n-1} A$$

be the set of (p-typical) Witt vectors $a = (a_0, \dots, a_{n-1})$ of length n. Give it the commutative ring structure such that the projection

$$\pi_n \colon W(A) \longrightarrow W_n(A)$$
$$(a_0, a_1, \dots) \longmapsto (a_0, \dots, a_{n-1})$$

is a ring homomorphism.

Let the restriction homomorphism $R \colon W_{n+1}(A) \to W_n(A)$ be given by

$$R(a_0, \dots, a_n) = (a_0, \dots, a_{n-1}) .$$

Then R is a ring homomorphism, and the canonical map

$$\pi \colon W(A) \xrightarrow{\cong} \lim_{n,R} W_n(A)$$

is an isomorphism.

Let the Teichmüller lift $\omega \colon A \to W(A)$ be given by

$$\omega(a_0) = (a_0, 0, \dots)$$

(i.e., the sequence $(a_i)_i$ with $a_i = 0$ for $i \ge 1$) and write $[-]_n \colon A \to W_n(A)$ for $\pi_n \omega$, so that $[a_0]_n$ is the Teichmüller representative of $a_0 \in A$ in $W_n(A)$. The ghost coordinates of $\omega(a_0)$ are $g(\omega(a_0)) = (a_0, a_0^p, \dots)$. These are multiplicative in $g(a_0)$, so ω is a multiplicative homomorphism. The composite

$$[-]_1 \colon A \xrightarrow{\omega} W(A) \xrightarrow{\pi_1} W_1(A)$$

is a multiplicative isomorphism.

Let the Verschiebung map $V \colon W_n(A) \to W_{n+1}(A)$ be given by

$$V(a_0, \dots, a_{n-1}) = (0, a_0, \dots, a_{n-1}) .$$

Then V commutes with R, and passes in the limit to a map $V \colon W(A) \longrightarrow W(A)$, given by $V(a_0, a_1, \dots) = (0, a_0, a_1, \dots)$. The ghost coordinates of $V(a)$ are $g(V(a)) = (0, pg_0, pg_1, \dots)$. These are additive in $g(a) = (g_0, g_1, \dots)$, which implies that V is an additive homomorphism. The V-adic filtration

$$\dots \subset V^2 W(A) \subset V W(A) \subset W(A)$$

is complete and Hausdorff, and there is a short exact sequence

$$0 \to W(A) \xrightarrow{V^n} W(A) \xrightarrow{\pi_n} W_n(A) \to 0 .$$

Hence there is also a short exact sequence

$$0 \to A \xrightarrow{V^n} W_{n+1}(A) \xrightarrow{R} W_n(A) \to 0 .$$

Lemma 1.6.

$$a = \sum_{i \ge 0} V^i \omega(a_i) = \omega(a_0) + V\omega(a_1) + \dots$$

for $a = (a_0, a_1, \dots) \in W(A)$.

Proof. The ghost coordinates of $V^i\omega(a_i)$ are $g_{i+j} = p^i a_i^{p^j}$, for $j \geq 0$. Hence the sum in $\prod_{m\geq 0} A$ of the ghost coordinates of $V^i\omega(a_i)$ is equal to the ghost coordinates of a. □

The Frobenius homomorphism $F\colon W_{n+1}(A) \to W_n(A)$ maps $a = (a_0, \dots, a_n)$ to $F(a)$ with ghost coordinates

$$g(F(a)) = (g_1, g_2, \dots, g_n)\,,$$

where $g(a) = (g_0, \dots, g_n)$ are the ghost coordinates of a. The ghost coordinates of $F(a)$ are additive and multiplicative in those of a, so F is a ring homomorphism. It commutes with R, and passes in the limit to a map $F\colon W(A) \to W(A)$, given by $g(F(a)) = (g_1, g_2, \dots)$.

Example 1.7. $F\colon W_2(A) \to W_1(A) = A$ is given by $F(a_0, a_1) = a_0^p + pa_1$. If A is an $\mathbb{F}_p$-algebra then $F(a_0, a_1, \dots) = (a_0^p, a_1^p, \dots)$, so that $F = W(\varphi)$ where $\varphi\colon A \to A$ is the Frobenius endomorphism of A.

Calculations in ghost coordinates also show that

$$a \cdot V(b) = V(F(a) \cdot b)$$

for $a, b \in W(A)$, and $FV = p$.

Theorem 1.8 (Witt)**.** *If k is a perfect field of characteristic $p > 0$, then $W(k)$ is a complete discrete valuation ring with residue field k and maximal ideal $pW(k)$. In particular, $W(\mathbb{F}_p) \cong \mathbb{Z}_p$.*

2. Fixed points of THH

Let B be a connective commutative (symmetric or orthogonal) ring spectrum, with $A = \pi_0(B)$. One example is given by the Eilenberg–Mac Lane spectrum $B = HA$. Let $\mathbb{T}$ be the circle group, and let $THH(B)$ be the $\mathbb{T}$-equivariant topological Hochschild homology spectrum of B. This is an orthogonal commutative ring $\mathbb{T}$-spectrum. The underlying commutative ring spectrum with $\mathbb{T}$-action is equivalent to the topological realization of

$$B \odot S[1]\colon [q] \mapsto B \otimes S[1]_q\,,$$

where $S[1] = \Delta[1]/\partial\Delta[1]$. Let $C_r \subset \mathbb{T}$ be the cyclic subgroup of order r. We view C_r-fixed points of $\mathbb{T}$-spectra as $\mathbb{T}$-spectra via the r-th root isomorphism $\mathbb{T}/C_r \cong \mathbb{T}$.

Fix a prime p. The $\mathbb{T}$-equivariant model for $THH(B)$ is p-cyclotomic, with an equivalence

$$THH(B)^{\varphi C_p} \simeq THH(B)$$

of commutative ring $\mathbb{T}$-spectra. This leads to restriction maps

$$R\colon THH(B)^{C_{p^n}} \longrightarrow THH(B)^{C_{p^{n-1}}}$$

for all $n \geq 1$. The $\mathbb{T}$-equivariant structure also gives forgetful (Frobenius) maps

$$F\colon THH(B)^{C_{p^n}} \longrightarrow THH(B)^{C_{p^{n-1}}}$$

induced by the projection $\pi\colon (\mathbb{T}/C_{p^{n-1}})_+ \to (\mathbb{T}/C_{p^n})_+$, and Verschiebung maps

$$V\colon THH(B)^{C_{p^{n-1}}} \longrightarrow THH(B)^{C_{p^n}}$$

induced by the transfer map $t\colon (\mathbb{T}/C_{p^n})_+ \to (\mathbb{T}/C_{p^{n-1}})_+$, for all $n \geq 1$.

Here R and F are commutative ring $\mathbb{T}$-spectrum maps, which commute with one another. The Verschiebung map V is only defined up to homotopy. In the stable homotopy category it commutes with R, and is a module map over its target, via F. Hence VF is multiplication by $V(1)$, while FV is multiplication by p:

$$\begin{aligned} RF &= FR \\ RV &\simeq VR \\ x \cdot V(y) &\simeq V(F(x) \cdot y) \\ FV &\simeq p\,. \end{aligned}$$

There is a homotopy cofiber sequence

$$THH(B)_{hC_{p^n}} \xrightarrow{N} THH(B)^{C_{p^n}} \xrightarrow{R} THH(B)^{C_{p^{n-1}}} \xrightarrow{\partial} \Sigma THH(B)_{hC_{p^n}}$$

for each $n \geq 1$.

Proposition 2.1 (Hesselholt-Madsen (1997) Proposition 3.3)**.** *The sequence*

$$0 \to \pi_0 THH(B) \xrightarrow{V^n} \pi_0 THH(B)^{C_{p^n}} \xrightarrow{R} \pi_0 THH(B)^{C_{p^{n-1}}} \to 0$$

is exact. Hence $\pi_0 THH(B)^{C_{p^n}} \cong \prod_{i=0}^n A$ *as a set.*

Sketch proof. The inclusion $\iota\colon THH(B) \to THH(B)_{hC_{p^n}}$ induces an isomorphism

$$\iota\colon \pi_0 THH(B) \cong \pi_0 THH(B)_{hC_{p^n}} \,,$$

and $N \circ \iota \simeq V^n$. If A is p-torsion free, then $F^n V^n = p^n$ acts injectively on $\pi_0 THH(B)$, so V^n is injective. The general case follows by naturality. □

There is a space level map

$$\begin{aligned} [-]_{n+1}\colon \Omega^\infty B &\longrightarrow \Omega^\infty THH(B)^{C_{p^n}} \\ f &\longmapsto f \wedge \cdots \wedge f \end{aligned}$$

for each $n \geq 0$ (with p^n copies of f), such that

$$\Omega^\infty R^n \circ [-]_{n+1}\colon \Omega^\infty B \longrightarrow \Omega^\infty THH(B)$$

is (homotopic to) the inclusion of the 0-skeleton, and

$$\Omega^\infty F^n \circ [-]_{n+1}\colon \Omega^\infty B \longrightarrow \Omega^\infty THH(B)$$

is homotopic to the p^n-th power endomorphism, followed by the inclusion of the 0-skeleton.

Theorem 2.2 (Hesselholt-Madsen (1997) Theorem 3.3)**.** *There is a natural ring isomorphism*

$$I\colon W_{n+1}(A) \xrightarrow{\cong} \pi_0 THH(B)^{C_{p^n}}$$

for each $n \geq 0$*, that commutes with* R*,* F*,* V *(and* $[-]_{n+1}$*!).*

Corollary 2.3. $TR(B;p) = \operatorname{holim}_{n,R} THH(B)^{C_{p^n}}$ *is connective, with*

$$W(A) \xrightarrow{\cong} \pi_0 TR(B;p) \,.$$

$TC(B;p) = \operatorname{hoeq}(\mathrm{id}, F)$ *is* (-2)*-connected, and there is an exact sequence*

$$\pi_0 TC(B;p) \longrightarrow W(A) \xrightarrow{1-F} W(A) \xrightarrow{\partial} \pi_{-1} TC(B;p) \to 0 \,.$$

Sketch proof. The bijection I takes $(a_0, a_1, \ldots, a_n) \in W_{n+1}(A)$ to

$$\sum_{i+j=n} V^i([a_i]_{j+1}) \in \pi_0 THH(B)^{C_{p^n}} \,.$$

Here $V^i([-]_{j+1})$ is the composite

$$A \cong \pi_0 THH(B) \xrightarrow{[-]_{j+1}} \pi_0 THH(B)^{C_{p^j}} \xrightarrow{V^i} \pi_0 THH(B)^{C_{p^n}} \,.$$

Let the ring homomorphism

$$\bar{g} = (R^m F^{n-m})_m \colon \pi_0 THH(B)^{C_{p^n}} \longrightarrow \prod_{m=0}^{n} A$$

be the product of the composite maps

$$\pi_0 THH(B)^{C_{p^n}} \xrightarrow{F^{n-m}} \pi_0 THH(B)^{C_{p^m}} \xrightarrow{R^m} \pi_0 THH(B) \cong A \,.$$

Then $\bar{g} \circ I = g\colon W_{n+1}(A) \to \prod_{m=0}^n A$ is the (truncated) ghost map, and this implies that I is a ring homomorphism. □

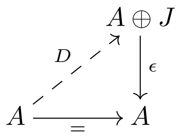
Remark 2.4. (Dress/Siebenecker?) For $B = S$, with $A = \pi_0(B) = \mathbb{Z}$, we have $THH(S)^{C_{p^n}} \cong S^{C_{p^n}} \simeq \bigvee_{i=0}^n BC_{p^i +}$, by the Segal-tom Dieck splitting. So $\pi_0 THH(S)^{C_{p^n}}$ is also isomorphic to the Burnside ring $A(C_{p^n})$. (Describe R, F, V in these terms.)

3. Kähler differentials

We focus on the absolute case, of a commutative ring A. Let J be an A-module. A homomorphism $d\colon A \to J$ is a derivation if the Leibniz rule

$$d(ab) = (da)b + a(db)$$

is satisfied. Let $A \oplus J$ be the split square-zero extension of A by J, with $(a, m) \cdot (b, n) = (ab, an + mb)$. Derivations $d\colon A \to J$ correspond bijectively to ring homomorphisms $D\colon A \to A \oplus J$ over id_A, of the form $D(a) = (a, da)$.



The universal derivation $a \mapsto da$ takes values in A-module of Kähler differentials Ω^1_A, which is generated by symbols db for $b \in A$, subject to the Leibniz rule.

There is an isomorphism

$$HH_1(A) \xrightarrow{\cong} \Omega^1_A$$

taking the class of the Hochschild 1-cycle $a \otimes b \in C_1(A) = A \otimes A$ to $a\, db \in \Omega^1_A$.

The Dennis trace map $\mathrm{tr}\colon K(A) \to HH(A)$ induces a homomorphism $K_1(A) \to HH_1(A)$ taking the class $\{u\} \in K_1(A)$ corresponding to a unit $u \in A^\times$ to the Hochschild class of $u^{-1} \otimes u$, which corresponds to $u^{-1}\, du \in \Omega^1_A$.

The linearization map $THH(A) \to HH(A)$ is 3-connected. Hence there is an isomorphism

$$\pi_1 THH(A) \xrightarrow{\cong} HH_1(A)\,.$$

Inverting these isomorphisms, and extending multiplicatively, we obtain graded A-algebra homomorphisms

$$\Omega^*_A = \Lambda^*_A \Omega^1_A \longrightarrow \pi_* THH(A) \longrightarrow HH_*(A)\,.$$

Here $(\Omega^q_A)_q$ is the sequence of de Rham forms of A. The universal derivation extends to exterior differentials $d\colon \Omega^q_A \to \Omega^{q+1}_A$, making (Ω^*_A, d) a cochain complex. Its cohomology

$$H^q_{dR}(A) = H^q(\Omega^*_A, d)$$

is the de Rham cohomology of A. When A is smooth (over a field of characteristic zero?), this calculates the sheaf cohomology $H^q_{Zar}(X, \mathscr{O}_X)$ for $X = \mathrm{Spec}(A)$ in the Zariski topology.

Theorem 3.1 (Pirashvili/Larsen-Lindenstrauss (2001) Theorem 3.2)**.** *Let A be smooth. There are A-algebra isomorphisms*

$$\Omega^*_A \otimes \pi_* THH(\mathbb{Z}) \cong \pi_* THH(A) \qquad \textit{and} \qquad \Omega^*_A \otimes \bar\pi_* THH(\mathbb{Z}) \cong \bar\pi_* THH(A)\,.$$

Here $\bar\pi_* THH(\mathbb{Z}) \cong E(\lambda_1) \otimes P(\mu_1)$.

Examples of smooth commutative rings include the polynomial rings $\mathbb{Z}[x_1, \dots, x_n]$, the cyclotomic p-integers $\mathbb{Z}[\zeta_p, 1/p]$, and more generally the localization $\mathscr{O}_F[1/\Delta]$ of the ring of integers in any number field F, where Δ is the discriminant of F.

Theorem 3.2 (Larsen-Lindenstrauss (2001) Theorem 3.2)**.** *Let A be a smooth $\mathbb{F}_p$-algebra. There are A-algebra isomorphisms*

$$\Omega^*_{A/\mathbb{F}_p} \otimes \pi_* THH(\mathbb{F}_p) \cong \pi_* THH(A)\,.$$

Here $\pi_* THH(\mathbb{F}_p) \cong P(\mu_0)$, with $|\mu_0| = 2$.

4. de Rham-Witt complexes

(Bloch/Deligne/Illusie (1979), Hesselholt-Madsen/Zink-Langer.)

The Witt- and de Rham-constructions can be merged to define a de Rham-Witt (pro-)complex $(W\Omega^*_A, d)$. This is an inverse system

$$\dots \to (W_{n+1}\Omega^*_A, d) \xrightarrow{R} (W_n\Omega^*_A, d) \to \dots$$

for $n \ge 1$ of cochain complexes

$$\dots \to W_n\Omega^q_A \xrightarrow{d} W_n\Omega^{q+1}_A \to \dots$$

for $q \ge 0$, where each $W_n\Omega^q_A$ is a suitably defined quotient

$$\Omega^q_{W_n(A)} \twoheadrightarrow W_n\Omega^q_A$$

of the de Rham q-forms on $W_n(A)$. In particular, $W_n\Omega^0_A = W_n(A)$ and $(W_1\Omega^*_A, d) = (\Omega^*_A, d)$. There are suitable notions of V- and F-maps acting on the de Rham-Witt complex.

The homomorphisms $W_n(A) \to \pi_0 THH(A)^{C_{p^{n-1}}}$ and $\Omega^q_A \to \pi_q THH(A)$ refine to homomorphisms

$$W_n\Omega^q_A \longrightarrow \pi_q THH(A)^{C_{p^{n-1}}}$$

that are compatible with the R-, V- and F-maps, and take the differential operator d to the operator given by the $\mathbb{T}$-action (for p odd).

The cohomology

$$H^q_{cris}(A/W_n) = H^q(W_n\Omega^q_A, d)$$

defines the crystalline cohomology of A. When A is smooth (and proper?), this calculates the sheaf cohomology $H^q(X, W_n\mathscr{O}_X)$ for $X = \mathrm{Spec}(A)$ in the crystalline topology. Passing to limits over n (using the R-maps) one obtains $H^q_{cris}(A/W)$.

Theorem 4.1. *(Hypotheses? Credits?) Let A be a smooth $\mathbb{F}_p$-algebra. The maps*

$$W_n\Omega^*_A \otimes_{\mathbb{Z}/p^n} \pi_* THH(\mathbb{F}_p)^{C_{p^{n-1}}} \longrightarrow \pi_* THH(A)^{C_{p^{n-1}}}$$

define a pro-isomorphism

$$W\Omega^*_A \otimes_{\mathbb{Z}_p} \pi_* TR(\mathbb{F}_p; p) \longrightarrow \pi_* TR(A; p)\,.$$

Here $\pi_* THH(\mathbb{F}_p)^{C_{p^{n-1}}} \cong \mathbb{Z}/p^n[\mu_0]$ and $\pi_* TR(\mathbb{F}_p; p) \cong \mathbb{Z}_p[\mu_0]$, with $|\mu_0| = 2$.

5. Log rings

(Kato. Fontaine-Illusie. We follow Rognes (2009) Section 5.)

Definition 5.1. A prelog ring (A, M, α) is a commutative ring A, a commutative monoid M and a monoid homomorphism $\alpha\colon M \to (A, \cdot)$. It is a log ring if the restricted homomorphism $\alpha^{-1}(A^\times) \to A^\times$ is an isomorphism.

$$\begin{array}{ccc} \alpha^{-1}(A^\times) & \xrightarrow{\cong} & A^\times \\ \downarrow & & \downarrow \\ M & \xrightarrow{\alpha} & (A, \cdot) \end{array}$$

To each log ring (A, M) we can associate the localization

$$A[M^{-1}] = A \otimes_{\mathbb{Z}[M]} \mathbb{Z}[M^{gp}]\,.$$

Here $\gamma\colon M \to M^{gp}$ is the group completion. There are homomorphisms of log rings

$$(A, A^\times) \to (A, M) \to (A[M^{-1}], A[M^{-1}]^\times)\,,$$

placing (A, M) between A and $A[M^{-1}]$ (with the trivial log structures).

To each A-module J, the split square-zero extension $(A \oplus J, \eta^* M)$ is the log ring with $A \oplus J$ as its underlying ring, and the inverse image $\eta^* M \cong M \times (J, +)$ of M along $\eta\colon A \to A \oplus J$ as the underlying monoid. By definition, a log derivation of (A, M) with values in J is a log ring homomorphism $(D, D^\flat)\colon (A, M) \to (A \oplus J, \eta^* M)$ over $\mathrm{id}_{(A,M)}$:

$$\begin{array}{ccc} & & (A \oplus J, \eta^* M) \\ & \overset{(D,D^\flat)}{\nearrow} & \downarrow \epsilon \\ (A, M) & \xrightarrow{\;=\;} & (A, M) \end{array}$$

Here $D(a) = (a, da)$ for $d\colon A \to J$ a derivation, and $D^\flat(m) = (m, d^\flat(m))$ for $d^\flat\colon M \to J$ a monoid homomorphism.

Proposition 5.2. *The universal log derivation $a \mapsto da$, $m \mapsto d^\flat m = d \log m$ takes values in the A-module of log Kähler differentials $\Omega^1_{(A,M)}$, which is defined by the pushout square*

$$\begin{array}{ccc} A \otimes_{\mathbb{Z}[M]} \Omega^1_{\mathbb{Z}[M]} & \xrightarrow{\psi} & A \otimes M^{gp} \\ \varphi \downarrow & & \downarrow \\ \Omega^1_A & \xrightarrow{j} & \Omega^1_{(A,M)} \end{array}$$

where $\varphi(1 \otimes dm) = d\alpha(m)$ and $\psi(1 \otimes dm) = \alpha(m) \otimes \gamma(m)$.

Proposition 5.3.

$$\Omega^1_{(A,M)} \cong \Omega^1_A \oplus (A \otimes M^{gp})/{\sim}$$

where

$$d\alpha(m) \sim \alpha(m) \otimes \gamma(m)\,.$$

Writing $d \log m$ for $1 \otimes \gamma(m)$, this relation is

$$d\alpha(m) = \alpha(m)\, d \log m\,.$$

6. Hesselholt-Madsen's log THH

If A is a discrete valuation ring, with maximal ideal $(\pi) = \pi A$, fraction field $K = A[\pi^{-1}]$, and residue field $k = A/(\pi)$, the log ring (A, M) with $M = A \cap K^\times$ is generated by the prelog ring $(A, \langle \pi \rangle)$, with $A[\pi^{-1}] = K$. The residue sequence

$$0 \to \Omega^1_A \xrightarrow{j} \Omega^1_{(A,M)} \xrightarrow{\mathrm{res}} k \to 0$$

is exact, where $\mathrm{res}(d \log \pi) = 1$. In this case we write $\Omega^1_{A|K}$ for $\Omega^1_{(A,M)}$.

Hesselholt-Madsen (2003) construct a spectrum $THH(A|K)$, with $\pi_0 THH(A|K) \cong A$ and $\pi_1 THH(A|K) \cong \Omega^1_{A|K}$. They also construct a trace map $\mathrm{tr}\colon K(K) \to THH(A|K)$, giving a vertical map of horizontal homotopy cofiber sequences

$$\begin{array}{ccccccc} K(k) & \longrightarrow & K(A) & \xrightarrow{j} & K(K) & \longrightarrow & \Sigma K(k) \\ \downarrow & & \downarrow & & \downarrow \mathrm{tr} & & \downarrow \\ THH(k) & \longrightarrow & THH(A) & \xrightarrow{j} & THH(A|K) & \longrightarrow & \Sigma THH(k)\,. \end{array}$$

Here the upper row is Quillen's localization sequence. At the level of π_1 and π_0, we obtain the diagram

$$\begin{array}{ccccc} K_1(A) & \xrightarrow{j} & K^\times & \xrightarrow{v} & \mathbb{Z} \\ \downarrow & & \downarrow \mathrm{tr} & & \downarrow \\ \Omega^1_A & \xrightarrow{j} & \Omega^1_{A|K} & \xrightarrow{\mathrm{res}} & k \end{array}$$

where the lower row is the exact residue sequence. Here $\mathrm{tr}(u) = d \log u$ for each $u \in K^\times$.

Proposition 6.1.

$$\bar{\pi}_* THH(\mathbb{Z}_p|\mathbb{Q}_p) \cong E(d\log p) \otimes P(\kappa)$$

with $|d\log p| = 1$ *and* $|\kappa| = 2$.

Proof. This follows from the long exact sequence

$$\cdots \to \bar{\pi}_* THH(\mathbb{F}_p) \longrightarrow \bar{\pi}_* THH(\mathbb{Z}_p) \xrightarrow{j} \bar{\pi}_* THH(\mathbb{Z}_p|\mathbb{Q}_p) \xrightarrow{\partial} \bar{\pi}_{*-1} THH(\mathbb{F}_p) \to \dots$$

where $j(\lambda_1) = 0$, $j(\mu_1) = \kappa^p$, $\partial(d\log p) = 1$ and $\partial(\kappa) = \epsilon_0$. (Check signs!) □

(If also discussing higher chromatic levels, we would write κ_0 for this κ.)

Inverting the isomorphism $\pi_1 THH(A|K) \cong \Omega^1_{A|K}$, and extending multiplicatively, we obtain a graded A-algebra homomorphism

$$\Omega^*_{A|K} = \Lambda^*_A \Omega^1_{A|K} \longrightarrow \pi_* THH(A|K) .$$

Here $(\Omega^q_{A|K}, d)$ is the log de Rham complex of $A|K$. In particular, $\Omega^*_{\mathbb{Z}_p|\mathbb{Q}_p} = E(d\log p)$.

Theorem 6.2 (Hesselholt-Madsen (2003) Theorem B)**.** *There are A-algebra isomorphisms*

$$\Omega^*_{A|K} \otimes_{E(d\log p)} \bar{\pi}_* THH(\mathbb{Z}_p|\mathbb{Q}_p) \cong \Omega^*_{A|K} \otimes P(\kappa) \cong \bar{\pi}_* THH(A|K) ,$$

where $d\kappa = \kappa d\log(-p)$.

Furthermore, $THH(A|K)$ is cyclotomic, and the trace map lifts to a cyclotomic trace map

$$\mathrm{trc}\colon K(K) \longrightarrow TC(A|K;p) .$$

As usual, $TR(A|K;p) = \mathrm{holim}_{n,R}\, THH(A|K)^{C_{p^n}}$ admits a self-map F, and $TC(A|K;p) = \mathrm{hoeq}(\mathrm{id}, F)$.

Hesselholt-Madsen merge the Witt- and log de Rham-constructions to define a logarithmic de Rham-Witt (pro-)complex $(W\Omega^*_{A|K}, d)$. This is an inverse system

$$\cdots \to (W_{n+1}\Omega^*_{A|K}, d) \xrightarrow{R} (W_n\Omega^*_{A|K}, d) \to \dots$$

of cochain complexes

$$\cdots \to W_n\Omega^q_{A|K} \xrightarrow{d} W_n\Omega^{q+1}_{A|K} \to \dots$$

where each $W_n\Omega^q_{A|K}$ is a suitably defined quotient

$$\Omega^q_{W_n(A)|M} \twoheadrightarrow W_n\Omega^q_{A|K}$$

of the log de Rham q-forms on $W_n(A)$, with the prelog structure $M = A \cap K^\times \to A \to W_n(A)$ given by the Teichmüller lift. (Is this log equivalent to $W_n(A) \cap W_n(K)^\times$?)

Theorem 6.3 (Hesselholt-Madsen (2003) Theorem C)**.** *Suppose $\mu_p \subset A^\times$. The maps*

$$W_n\Omega^*_{A|K} \otimes P(\kappa) \longrightarrow \bar{\pi}_* THH(A|K)^{C_{p^{n-1}}}$$

((check the role of κ)) define a pro-isomorphism

$$W\Omega^*_{A|K} \otimes P(\kappa) \longrightarrow \bar{\pi}_* TR(A|K;p) .$$

7. Logarithmic topological Hochschild homology

We follow Rognes (2009) Section 8.

For a general prelog ring (A, M) we have defined logarithmic topological Hochschild homology $THH(A, M)$ as the pushout

$$\begin{array}{ccc} THH(S[M]) = S[B^{cyc}(M)] & \xrightarrow{\rho} & S[B^{rep}(M)] \\ \downarrow{\scriptstyle\alpha} & & \downarrow \\ THH(A) & \xrightarrow{j} & THH(A, M) \end{array}$$

in cyclic commutative rings. Here $B^{cyc}(M) = M \odot S[1]$ is the cyclic bar construction (in commutative monoids), and the replete bar construction $B^{rep}(M)$ is the homotopy pullback

$$\begin{array}{ccc} B^{rep}(M) & \longrightarrow & B^{cy}(M^{gp}) \\ \downarrow & & \downarrow \epsilon \\ M & \xrightarrow{\gamma} & M^{gp} \, . \end{array}$$

In other terms, the augmentation $\epsilon\colon B^{cyc}(M) \to M$ factors as

$$B^{cyc}(M) \xrightarrow{\rho} B^{rep}(M) \xrightarrow{\pi} M \, ,$$

where ρ is an acyclic cofibration and π is a fibration in the group completion model structure on simplicial monoids (Sagave).

For example, with $M = \langle x \rangle = \{x^n \mid n \geq 0\}$ we have

$$B^{cy}(M) \simeq * \sqcup \coprod_{n \geq 1} S^1(n)$$

and

$$B^{cy}(M^{gp}) \simeq \coprod_{n \in \mathbb{Z}} S^1(n)$$

so

$$B^{rep}(M) \simeq \coprod_{n \geq 0} S^1(n)$$

and there is a homotopy cofiber sequence

$$S \longrightarrow S[B^{cy}(M)] \xrightarrow{\rho} S[B^{rep}(M)] \longrightarrow \Sigma S \, .$$

Applying base change to $THH(A)$ we obtain the homotopy cofiber sequence

$$THH(A \otimes_{S[M]} S) \longrightarrow THH(A) \xrightarrow{\rho} THH(A, M) \longrightarrow \Sigma THH(A \otimes_{S[M]} S) \, .$$

In the case $M = \langle \pi \rangle \to (A, \cdot)$ considered by Hesselholt-Madsen, $A \otimes_{S[M]} S = A/(\pi) = k$.

A theory of log structures and log THH for commutative (symmetric or orthogonal) ring spectra B has been developed by Rognes, Sagave and Schlichtkrull, with M taking values in a category of commutative $\mathscr{J}$-space monoids.

8. Logarithmic topological cyclic homology

(Work announced by Rognes at Loen (2009)/in progress by Sagave-Schlichtkrull-Rognes.) In the language of Nikolaus-Scholze (2017), the p-cyclotomic structure on $THH(A)$ is derived from a $\mathbb{T}$-equivariant map

$$\varphi_p\colon THH(A) \longrightarrow THH(A)^{tC_p} \, .$$

For $A = S[M]$ this is induced from the composite

$$B^{cyc}(M) \xrightarrow{\psi_p} B^{cyc}(M)^{hC_p} \xrightarrow{\text{can}} B^{cyc}(M)^{tC_p}$$

where the lift ψ_p is induced from the cyclic map given by

$$M^{q+1} \xrightarrow{\Delta_p} (M^{p(q+1)})^{hC_p}$$

in degree $q \geq 0$. There are compatible maps

$$B^{rep}(M) \xrightarrow{\psi_p} B^{rep}(M)^{hC_p} \xrightarrow{\text{can}} B^{rep}(M)^{tC_p}$$

defined as the map of (homotopy) pullbacks in the diagram

$$\begin{array}{ccccc} B^{cyc}(M^{gp}) & \xrightarrow{\epsilon} & M^{gp} & \xleftarrow{\gamma} & M \\ \downarrow{\scriptstyle \psi_p} & & \downarrow{\scriptstyle \psi_p} & & \downarrow{\scriptstyle \psi_p} \\ B^{cyc}(M^{gp})^{hC_p} & \xrightarrow{\epsilon^{hC_p}} & (M^{gp})^{hC_p} & \xleftarrow{\gamma^{hC_p}} & M^{hC_p} \\ \downarrow{\scriptstyle \mathrm{can}} & & \downarrow{\scriptstyle \mathrm{can}} & & \downarrow{\scriptstyle \mathrm{can}} \\ B^{cyc}(M^{gp})^{tC_p} & \xrightarrow{\epsilon^{tC_p}} & (M^{gp})^{tC_p} & \xleftarrow{\gamma^{tC_p}} & M^{tC_p} \end{array}$$

Here $\psi_p\colon M \to M^{hC_p}$ is the Frobenius lift defined as the composite

$$M \xrightarrow{\Delta_p} (M^p)^{hC_p} \xrightarrow{m^{hC_p}} M^{hC_p} \,,$$

and similarly for $\psi_p\colon M^{gp} \to (M^{gp})^{hC_p}$.

This gives us $\mathbb{T}$-equivariant structure maps

$$\varphi_p\colon THH(S[M], M) \xrightarrow{\psi_p} THH(S[M], M)^{hC_p} \xrightarrow{\mathrm{can}} THH(S[M], M)^{tC_p} \,.$$

by means of the standard maps $S[B^{rep}(M)^{hC_p}] \to S[B^{rep}(M)]^{hC_p}$ and $S[B^{rep}(M)^{tC_p}] \to S[B^{rep}(M)]^{tC_p}$. Applying base change along $THH(S[M]) \to THH(A)$ we likewise obtain the $\mathbb{T}$-map

$$\varphi_p\colon THH(A, M) \xrightarrow{\psi_p} THH(A, M)^{hC_p} \xrightarrow{\mathrm{can}} THH(A, M)^{tC_p} \,.$$

This leads to a definition of the logarithmic topological cyclic homology of (A, M) as the homotopy equalizer

$$TC(A, M; p) \longrightarrow THH(A, M) \underset{G\,\mathrm{can}}{\overset{\varphi_p^{h\mathbb{T}}}{\rightrightarrows}} (THH(A, M)^{tC_p})^{h\mathbb{T}} \,.$$

It remains to compare this definition with that of Hesselholt-Madsen, for $M = A \cap K^\times$, and to determine in what generality the trace maps from $K(A)$ to $TC(A; p) \to THH(A)$ can be extended to trace maps from $K(A[M^{-1}])$ to $TC(A, M; p) \to THH(A, M)$.