ALGEBRAIC K-THEORY OF GROUP RINGS AND TOPOLOGICAL CYCLIC HOMOLOGY

W. LÜCK, H. REICH, J. ROGNES, M. VARISCO

1. Introduction

This is an overview of joint work with Wolfgang Lück, Holger Reich and Marco Varisco.

2. The Borel and Novikov conjectures

Armand Borel formulated the following conjecture, motivated by work of Mostow.

Conjecture 2.1 (Borel (1953)). *Let G be any discrete group. Any two closed manifolds of the homotopy type of BG are homeomorphic.*

This is an analogue of the Poincaré conjecture for aspherical manifolds (i.e., those with contractible universal covering space). The Novikov conjecture is a step towards this conjecture, from the point of view of rational characteristic numbers.

The integral Pontryagin classes $p_i(TM) \in H^{4i}(M;\mathbb{Z})$ of the tangent bundles of smooth manifolds are not topological invariants, but the rational Pontryagin classes are. More explicitly, if $h\colon M' \to M$ is a homeomorphism (of closed smooth or piecewise linear (PL) manifolds), then $h^*p_i(TM) = p_i(TM') \in H^{4i}(M';\mathbb{Q})$. This was proved by Sergei Novikov (1966).

Certain rational polynomials in the Pontryagin classes are known to be homotopy invariants. The first examples are the Hirzebruch L-polynomials $L_1 = p_1/3$, $L_2 = (7p_2 - p_1^2)/35$, $\dots$. By his signature theorem (ca. 1953), for a closed oriented $4i$-dimensional smooth (or PL) manifold M with fundamental class $[M] \in H_{4i}(M;\mathbb{Z})$, the characteristic number $\langle L_i(TM), [M]\rangle = \mathrm{sign}(M)$ equals the signature of the symmetric form $(x, y) \mapsto \langle x \cup y, [M]\rangle$ on $H^{2i}(M;\mathbb{R})$.

The Novikov conjecture predicts that certain other characteristic numbers are also oriented homotopy invariants. Let G be any discrete group and consider a map $u\colon M \to BG$ from a closed oriented n-manifold to its classifying space. To each class $x \in H^{n-4i}(BG;\mathbb{Q})$ the associated higher signature of M is the rational number

$$\mathrm{sign}_x(M, u) = \langle L_i(TM) \cup u^*(x), [M]\rangle \,.$$

Conjecture 2.2 (Novikov (1970)). *If $h\colon M' \to M$ is an orientation-preserving homotopy equivalence, then* $\mathrm{sign}_x(M, u) = \mathrm{sign}_x(M, uh)$.

Using the surgery exact sequence (cf. Wall (1970)), this conjecture can be reformulated in terms of the (symmetric or quadratic) L-theory spectra $\mathbb{L}(-)$ of group rings.

Conjecture 2.3 (Novikov, reformulated). *The L-theory assembly map $a^L\colon BG_+ \wedge \mathbb{L}(\mathbb{Z}) \longrightarrow \mathbb{L}(\mathbb{Z}[G])$ is rationally injective, i.e., the induced homomorphism*

$$a^L_* \otimes \mathbb{Q}\colon H_*(BG; L_*(\mathbb{Z})) \otimes \mathbb{Q} \longrightarrow L_*(\mathbb{Z}[G]) \otimes \mathbb{Q}$$

is injective in each degree.

3. The Hsiang and Farrell–Jones conjectures

Wu-chung Hsiang formulated a similar conjecture for Bass/non-connective algebraic K-theory.

Conjecture 3.1 (Hsiang (1983)). *If G is torsion-free (and BG has the homotopy type of a finite CW complex), then the K-theory assembly map $a^K\colon BG_+ \wedge K(\mathbb{Z}) \longrightarrow K(\mathbb{Z}[G])$ is a rational equivalence.*

Date: November 26th 2014.

This K-theory assembly map can be defined using the inclusion $BG_+ \subset BGL_1(\mathbb{Z}[G])_+ \to \Omega^\infty K(\mathbb{Z}[G])$ and the pairing $K(\mathbb{Z}[G]) \wedge K(\mathbb{Z}) \to K(\mathbb{Z}[G])$. To deal with non-torsion free groups G, we need a more refined assembly map, taking into account some of the subgroups of G.

A *family* $\mathscr{F}$ of subgroups of G is a collection of subgroups, closed under conjugation with elements of G and passage to subgroups. Let $E\mathscr{F}$ denote the universal G-CW space with stabilizers in $\mathscr{F}$. Universality amounts to the condition that $E\mathscr{F}^H$ is contractible for each $H \in \mathscr{F}$. Such a G-CW space exists, and is unique up to G-homotopy equivalence.

The *orbit category* $\mathrm{Or}\, G$ has as objects the homogeneous G-spaces G/H, and as morphisms the G-maps. The rule $G/H \mapsto E\mathscr{F}^H$ defines a contravariant functor $E\mathscr{F}^?$ from $\mathrm{Or}\, G$ to spaces. The rule $G/H \mapsto K(\mathbb{Z}[H])$ can be extended to a covariant functor $K(\mathbb{Z}[?])$ from $\mathrm{Or}\, G$ to spectra. This step involves viewing $K(\mathbb{Z}[H])$ as the algebraic K-theory of the ring with many objects/additive category $\mathbb{Z}[\mathscr{G}^G(G/H)]$ generated by the translation category of the G-set G/H, cf. Davis–Lück (1998).

The smash product

$$E\mathscr{F}_+ \wedge_{\mathrm{Or}\, G} K(\mathbb{Z}[-]) = E\mathscr{F}_+^? \wedge_{\mathrm{Or}\, G} K(\mathbb{Z}[?])$$

is a spectrum defined as a homotopy coend. The G-map $E\mathscr{F} \to *$ induces a natural map

$$a^K \colon E\mathscr{F}_+ \wedge_{\mathrm{Or}\, G} K(\mathbb{Z}[-]) \longrightarrow *_+ \wedge_{\mathrm{Or}\, G} K(\mathbb{Z}[-]) \simeq K(\mathbb{Z}[G]) \,,$$

which we call the K-theory *assembly map* for $\mathscr{F}$. For the minimal family $\mathscr{F} = \{e\}$, $E\{e\} = EG$ and the smash product over the orbit category reduces to the smash product $BG_+ \wedge K(\mathbb{Z})$, and a recovers the previous assembly map. A group is called *virtually cyclic* if it contains a (finite or infinite) cyclic subgroup of finite index.

Conjecture 3.2 (Farrell-Jones (1993)). *Let G be any discrete group, and let $\mathscr{V}\mathscr{C}yc$ be the family of virtually cyclic subgroups of G. Then the K-theory assembly map for $\mathscr{V}\mathscr{C}yc$,*

$$a^K \colon E\mathscr{V}\mathscr{C}yc_+ \wedge_{\mathrm{Or}\, G} K(\mathbb{Z}[-]) \longrightarrow K(\mathbb{Z}[G]) \,,$$

is an equivalence.

4. The Bökstedt–Hsiang–Madsen theorem and our main result

The following theorem was proved using the cyclotomic trace map from connective algebraic K-theory to topological cyclic homology.

Theorem 4.1 (Bökstedt–Hsiang–Madsen (1993)). *Let G be a discrete group such that condition (H') holds.*

(H') *$H_*(BG;\mathbb{Z})$ is of finite type.*

Then the connective K-theory assembly map

$$a^K \colon BG_+ \wedge K(\mathbb{Z}) \longrightarrow K(\mathbb{Z}[G])$$

is rationally injective.

Our aim is to prove a similar theorem for the Farrell–Jones assembly map, which gives a stronger result in the case when G is not torsion-free.

Theorem 4.2 (Lück–Reich–Rognes–Varisco (in preparation)). *Let G be a discrete group such that conditions (H) and (K) hold for each finite cyclic subgroup C of G:*

(H) *$H_*(BZ_GC;\mathbb{Z})$ is of finite type, where Z_GC is the centralizer of C in G;*

(K) *The canonical map*

$$K(\mathbb{Z}[C]) \longrightarrow \prod_p K(\mathbb{Z}_p[C])^\wedge_p$$

is rationally injective in each degree, where p ranges over all primes.

Then the Farrell–Jones assembly map in connective algebraic K-theory

$$a^K \colon E\mathscr{V}\mathscr{C}yc_+ \wedge_{\mathrm{Or}\, G} K(\mathbb{Z}[-]) \longrightarrow K(\mathbb{Z}[G])$$

is rationally injective.

Condition (K) is known to hold when C is the trivial group, which is why there is no explicit condition (K') in the result of Bökstedt–Hsiang–Madsen. Condition (K) holds in degrees $t \leq 1$; in degrees $t \geq 2$ it is expected to hold in all cases, and would follow from the Schneider conjecture (1979), generalizing Leopoldt's conjecture from K_1 to K_t. Condition (H), which encompasses Condition (H'), appears to be an intrinsic limitation of the cyclotomic trace method as applied to this problem.

5. Outline of proof

Since we are only considering rational K-theory, only the finite subgroups of G are really significant. Let $\mathscr{F}in$ be the family of finite subgroups of G.

Proposition 5.1 (Grunewald (2008)). *The family comparison map*

$$E\mathscr{F}in_+ \wedge_{\mathrm{Or}\,G} K(\mathbb{Z}[-]) \longrightarrow E\mathscr{V}\mathscr{C}yc_+ \wedge_{\mathrm{Or}\,G} K(\mathbb{Z}[-])$$

is a rational equivalence.

Furthermore, we may change coefficient rings from the integers $\mathbb{Z}$ to the sphere spectrum $\mathbb{S}$.

Proposition 5.2. *The linearization maps*

$$E\mathscr{F}in_+ \wedge_{\mathrm{Or}\,G} K(\mathbb{S}[-]) \longrightarrow E\mathscr{F}in_+ \wedge_{\mathrm{Or}\,G} K(\mathbb{Z}[-])$$

and

$$K(\mathbb{S}[G]) \longrightarrow K(\mathbb{Z}[G])$$

are rational equivalences.

Hence we may equally well prove that the K-theory assembly map

$$a^K \colon E\mathscr{F}in_+ \wedge_{\mathrm{Or}\,G} K(\mathbb{S}[-]) \longrightarrow K(\mathbb{S}[G])$$

is rationally injective. These reductions fit together in the following commutative diagram.

$$\begin{array}{ccc} E\mathscr{V}\mathscr{C}yc_+ \wedge_{\mathrm{Or}\,G} K(\mathbb{Z}[-]) & \xrightarrow{a^K} & K(\mathbb{Z}[G]) \\ {\scriptstyle \simeq_{\mathbb{Q}}}\uparrow & & \| \\ E\mathscr{F}in_+ \wedge_{\mathrm{Or}\,G} K(\mathbb{Z}[-]) & \xrightarrow{a^K} & K(\mathbb{Z}[G]) \\ {\scriptstyle \simeq_{\mathbb{Q}}}\uparrow & & \uparrow{\scriptstyle \simeq_{\mathbb{Q}}} \\ E\mathscr{F}in_+ \wedge_{\mathrm{Or}\,G} K(\mathbb{S}[-]) & \xrightarrow{a^K} & K(\mathbb{S}[G]) \end{array}$$

At this point we apply the cyclotomic trace map to topological cyclic homology. For each prime p this is a natural map $\mathrm{trc}\colon K(-) \to TC(-;p)$ of functors from symmetric ring spectra to spectra. It restricts and extends to a natural transformation $\mathrm{trc}\colon K(\mathbb{S}[-]) \to TC(\mathbb{S}[-];p)$ of functors from $\mathrm{Or}\,G$ to spectra. Hence there is a commutative square

$$\begin{array}{ccc} E\mathscr{F}in_+ \wedge_{\mathrm{Or}\,G} K(\mathbb{S}[-]) & \xrightarrow{a^K} & K(\mathbb{S}[G]) \\ {\scriptstyle 1\wedge\mathrm{trc}}\downarrow & & \downarrow{\scriptstyle \mathrm{trc}} \\ E\mathscr{F}in_+ \wedge_{\mathrm{Or}\,G} TC(\mathbb{S}[-];p) & \xrightarrow{a^{TC}} & TC(\mathbb{S}[G];p) \end{array}$$

and it would suffice to prove that the left hand vertical map $1 \wedge \mathrm{trc}$ and the lower horizontal map a^{TC} are both rationally injective (in non-negative degrees). To simplify the notation we focus on the case where Hypothesis K is satisfied using only a single prime p. In general, we may use the cyclotomic trace map for each prime p and combine the results.

Remark 5.3. If we had known that the linearization maps $TC(\mathbb{S}[H];p) \to TC(\mathbb{Z}[H];p)$ are rational equivalences (before p-completion, for H finite or equal to G), then it would have sufficed to extend the cyclotomic trace map to a $\mathrm{trc}\colon K(\mathbb{Z}[-]) \to TC(\mathbb{Z}[-];p)$ to a natural transformation of functors from $\mathrm{Or}\,G$ to spectra. This would have been technically simpler, involving categories of modules over rings with many objects instead of over symmetric ring spectra with many objects. These linearization maps are sequential homotopy limits of rational equivalences, but this does not settle the matter.

Proposition 5.4. *Let C be any finite cyclic subgroup of G. If $K(\mathbb{Z}[C]) \to K(\mathbb{Z}_p[C])^\wedge_p$ is rationally injective, then $\mathrm{trc}\colon K(\mathbb{S}[C]) \longrightarrow TC(\mathbb{S}[C];p)$ is rationally injective. If this holds for each finite cyclic subgroup C of G, then the [[left hand vertical]] map*

$$1 \wedge \mathrm{trc}\colon E\mathscr{F}in_+ \wedge_{\mathrm{Or}\,G} K(\mathbb{S}[-]) \longrightarrow E\mathscr{F}in_+ \wedge_{\mathrm{Or}\,G} TC(\mathbb{S}[-];p)$$

is also rationally injective.

The first claim follows because $K(\mathbb{Z}_p[C])^\wedge_p \to TC(\mathbb{Z}_p[C];p)^\wedge_p$ is an equivalence in non-negative degrees, by Hesselholt–Madsen (1997). The second claim follows by using Lück's Chern character for proper equivariant cohomology theories (2002). [[Explain reduction from finite to finite cyclic.]]

Remark 5.5. In the case of the trivial family $\mathscr{F} = \{e\}$, the lower horizontal map

$$a^{TC} \colon BG_+ \wedge TC(\mathbb{S};p) \longrightarrow TC(\mathbb{S}[G];p)$$

is the TC-assembly map considered by Bökstedt–Hsiang–Madsen. It would be nice if we could prove directly that this assembly map is rationally injective, and this is claimed (after p-completion) in Madsen's survey (1994, Theorem 4.5.2), but the simplified proof given there is not quite convincing. Recall that there is a homotopy Cartesian square

$$\begin{array}{ccc} TC(\mathbb{S}[G];p) & \xrightarrow{\alpha} & C(\mathbb{S}[G];p) \\ \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\mathrm{trf}} \\ THH(\mathbb{S}[G]) & \xrightarrow{1-\Delta_p} & THH(\mathbb{S}[G]) \end{array}$$

where the Bökstedt–Hsiang–Madsen functor

$$C(\mathbb{S}[G];p) = \operatorname*{holim}_{n\geq 1} THH(\mathbb{S}[G])_{hC_{p^n}}$$

is the homotopy limit over the transfer maps. The composite $\beta \circ \mathrm{trc} \colon K(\mathbb{S}[G]) \to THH(\mathbb{S}[G])$ is the Waldhausen trace map, in the form given by Bökstedt. There is a natural equivalence

$$THH(\mathbb{S}[G]) \simeq \mathbb{S}[B^{cy}(G)]\,,$$

where $B^{cy}(G)$ is the cyclic bar construction on G, and $\Delta_p \colon B^{cy}(G) \to B^{cy}(G)$ is the p-th power map. Note the decomposition

$$B^{cy}(G) = \coprod_{[g]} B^{cy}_{[g]}(G)$$

where $[g]$ ranges over the conjugacy classes of elements in G, and $B^{cy}_{[g]}(G)$ is the path component that contains the vertex g. The p-th power map takes $B^{cy}_{[g]}(G)$ to $B^{cy}_{[g^p]}(G)$.

One idea is to split the TC-assembly map by splitting each corner in this diagram. The THH-assembly map

$$a^{THH} \colon BG_+ \wedge THH(\mathbb{S}) \longrightarrow THH(\mathbb{S}[G])$$

is induced by the inclusion $BG \cong B^{cy}_e(G) \to B^{cy}(G)$, and is indeed split up to homotopy by the retraction $pr \colon B^{cy}(G)_+ \to B^{cy}_e(G)_+ \cong BG_+$ that sends the other path components to the base point. However, the resulting splitting map

$$pr \colon THH(\mathbb{S}[G]) \longrightarrow BG_+ \wedge THH(\mathbb{S})$$

is not in general compatible with the p-th power map. If $g \neq e$ but $g^p = e$ then the path component $B^{cy}_{[g]}(G)$ maps to the base point under $\Delta_p \circ pr$, but not to the base point under $pr \circ \Delta_p$. Hence this approach does not actually produce a map

$$pr \colon TC(\mathbb{S}[G];p) \longrightarrow BG_+ \wedge TC(\mathbb{S};p)$$

splitting the assembly map up to homotopy, before or after p-completion.

Remark 5.6. To sidestep the difficulty concerning the p-th power map, we return to the original strategy of Bökstedt-Hsiang-Madsen, and pass using the maps α and β from TC onwards to the wedge sum of the functors C and THH. This requires constructing a natural transformation

$$\alpha \vee \beta \colon TC(\mathbb{S}[-];p) \longrightarrow C(\mathbb{S}[-];p) \vee THH(\mathbb{S}[-])$$

of functors from $\mathrm{Or}\,G$ to spectra. The map β arises directly from the definition of TC as a homotopy limit, but the map α is more subtle. The real reason for passing from integral group rings to spherical group rings is to obtain natural splittings of the norm–restriction sequences

$$THH(\mathbb{S}[-])_{hC_{p^n}} \xrightarrow{N} THH(\mathbb{S}[-])^{C_{p^n}} \xrightarrow{R} THH(\mathbb{S}[-])^{C_{p^{n-1}}}\,.$$

These lead to a Segal–tom Dieck splitting

$$TR(\mathbb{S}[-];p) = \operatorname*{holim}_{n} THH(\mathbb{S}[-])^{C_{p^n}} \xrightarrow{\simeq} \prod_{n\geq 0} THH(\mathbb{S}[-])_{hC_{p^n}}\,.$$

The projection away from the factor $n = 0$,

$$TR(\mathbb{S}[-];p) \longrightarrow \prod_{n\ge 1} THH(\mathbb{S}[-])_{hC_{p^n}}\,,$$

is compatible with the Frobenius map F on the left hand side, and the product of the transfer maps $\mathrm{trf}\colon THH(\mathbb{S}[-])_{hC_{p^{n+1}}} \to THH(\mathbb{S}[-])_{hC_{p^n}}$ on the right hand side. Passing to homotopy equalizers with the identity maps, we get the required map

$$\alpha\colon TC(\mathbb{S}[-];p) \longrightarrow \operatorname*{holim}_{n\ge 1} THH(\mathbb{S}[-])_{hC_{p^n}} = C(\mathbb{S}[-];p)\,.$$

Making the Adams transfer equivalence $THH(\mathbb{S}[-])_{hC_{p^n}} \simeq [EC_{p^n +}\wedge THH(\mathbb{S}[-])]^{C_{p^n}}$ and the Segal–tom Dieck splitting natural as a functor from $\mathrm{Or}\,G$ to spectra requires more care than what was available in the literature. The first matter is handled in Reich–Varisco (2014). [[How natural can we make the splittings?]]

The real strategy for proving that a^{TC} is rationally injective in non-negative degrees is therefore to consider yet another commutative diagram

$$\begin{array}{ccc}
E\mathscr{F}in_+ \wedge_{\mathrm{Or}\,G} TC(\mathbb{S}[-];p) & \xrightarrow{\quad a^{TC}\quad} & TC(\mathbb{S}[G];p) \\
\Big\downarrow{\scriptstyle 1\wedge(\alpha\vee\beta)} & & \Big\downarrow{\scriptstyle \alpha\vee\beta} \\
E\mathscr{F}in_+ \wedge_{\mathrm{Or}\,G} (C(\mathbb{S}[-];p) \vee THH(\mathbb{S}[-])) & \xrightarrow{a^{C\vee THH}} & C(\mathbb{S}[G];p) \vee THH(\mathbb{S}[G]))
\end{array}$$

and to prove that the left hand vertical map $1 \wedge (\alpha \vee \beta)$ and the lower horizontal map $a^{C\vee THH}$ are rationally injective (in non-negative degrees).

Proposition 5.7. *The map $\alpha \vee \beta\colon TC(\mathbb{S}[D];p) \longrightarrow C(\mathbb{S}[D];p) \vee THH(\mathbb{S}[D])$ is rationally injective in non-negative degrees, for each finite subgroup D of G. It follows that the [[left hand vertical]] map*

$$1 \wedge (\alpha \vee \beta)\colon E\mathscr{F}in_+ \wedge_{\mathrm{Or}\,G} TC(\mathbb{S}[-];p) \longrightarrow E\mathscr{F}in_+ \wedge_{\mathrm{Or}\,G} (C(\mathbb{S}[-];p) \vee THH(\mathbb{S}[-]))$$

is also rationally injective.

The first claim follows from the homotopy Cartesian square calculating $TC(\mathbb{S}[D];p)$, and the fact that $THH(\mathbb{S}[D])$ is rationally trivial in positive degrees. The second claim follows by using Lück's Chern character, again.

6. Splitting the THH- and C-assembly maps

To split the THH-assembly map

$$a^{THH}\colon E\mathscr{F}in_+ \wedge_{\mathrm{Or}\,G} THH(\mathbb{S}[-]) \longrightarrow THH(\mathbb{S}[G])$$

for the family $\mathscr{F} = \mathscr{F}in$, we introduce the $\mathscr{F}$-part $THH_{\mathscr{F}}(\mathbb{S}[-])$ of $THH(\mathbb{S}[-])$. We let

$$B^{cy}_{\mathscr{F}}(G) = \coprod_{\langle g\rangle\in\mathscr{F}} B^{cy}_{[g]}(G)$$

be the union of the path components in $B^{cy}(G)$ that contain the vertices (g) such that the cyclic group $\langle g\rangle$ is a member of the family $\mathscr{F}$. There is then a natural equivalence

$$THH_{\mathscr{F}}(\mathbb{S}[G]) \simeq \mathbb{S}[B^{cy}_{\mathscr{F}}(G)]\,,$$

and similarly for subgroups H of G. The inclusion $B^{cy}_{\mathscr{F}}(G) \to B^{cy}(G)$, and the projection $B^{cy}(G)_+ \to B^{cy}_{\mathscr{F}}(G)_+$ that sends the other path components to the base point, lead to a natural diagram

$$THH_{\mathscr{F}}(\mathbb{S}[-]) \xrightarrow{in_{\mathscr{F}}} THH(\mathbb{S}[-]) \xrightarrow{pr_{\mathscr{F}}} THH_{\mathscr{F}}(\mathbb{S}[-])$$

that expresses the $\mathscr{F}$-part as a retract of $THH(\mathbb{S}[-])$.

Proposition 6.1. *The left hand vertical map and the lower horizontal map in the commutative square*

$$\begin{array}{ccc}
E\mathscr{F}_+ \wedge_{\mathrm{Or}\,G} THH(\mathbb{S}[-]) & \xrightarrow{\quad a^{THH}\quad} & THH(\mathbb{S}[G]) \\
{\scriptstyle 1\wedge pr_{\mathscr{F}}}\Big\downarrow\simeq & & \Big\downarrow{\scriptstyle pr_{\mathscr{F}}} \\
E\mathscr{F}_+ \wedge_{\mathrm{Or}\,G} THH_{\mathscr{F}}(\mathbb{S}[-]) & \underset{\simeq}{\xrightarrow{a^{THH_{\mathscr{F}}}}} & THH_{\mathscr{F}}(\mathbb{S}[G])
\end{array}$$

are stable equivalences.

This handles the THH-summand of $a^{C \vee THH} \simeq a^C \vee a^{THH}$. To deal with the C-summand, one must first make sense of the $\mathscr{F}$-part $C_{\mathscr{F}}(\mathbb{S}[-];p)$ of the Bökstedt–Hsiang–Madsen functor C. Passing to homotopy limits of homotopy orbits, with respect to transfer maps, we get a commutative diagram

$$\begin{array}{ccc}
E\mathscr{F}_+ \wedge_{\mathrm{Or}\,G} C(\mathbb{S}[-];p) & & \\
\downarrow{\scriptstyle \kappa} & \searrow{\scriptstyle a^C} & \\
\operatorname{holim}_n \big(E\mathscr{F}_+ \wedge_{\mathrm{Or}\,G} THH(\mathbb{S}[-])_{hC_{p^n}}\big) & \longrightarrow & C(\mathbb{S}[G];p) \\
\downarrow{\scriptstyle \simeq} & & \downarrow{\scriptstyle pr_{\mathscr{F}}} \\
\operatorname{holim}_n \big(E\mathscr{F}_+ \wedge_{\mathrm{Or}\,G} THH_{\mathscr{F}}(\mathbb{S}[-])_{hC_{p^n}}\big) & \xrightarrow[\simeq]{} & C_{\mathscr{F}}(\mathbb{S}[G];p)
\end{array}$$

Under the hypothesis of our homological condition (H), Lück–Reich–Varisco (2003) prove that the homotopy orbit–holim exchange map κ is an equivalence, first for $\mathscr{F}$ equal to the family of finite cyclic subgroups of G, and then by a transitivity argument, for the family $\mathscr{F}in$ of all finite subgroups of G. This then concludes the proof that a^C is split injective.

References

[BHM93] M. Bökstedt, W. C. Hsiang, and I. Madsen, *The cyclotomic trace and algebraic K-theory of spaces*, Invent. Math. **111** (1993), no. 3, 465–539.

[DL98] James F. Davis and Wolfgang Lück, *Spaces over a category and assembly maps in isomorphism conjectures in K- and L-theory*, K-Theory **15** (1998), no. 3, 201–252.

[FJ93] F. T. Farrell and L. E. Jones, *Isomorphism conjectures in algebraic K-theory*, J. Amer. Math. Soc. **6** (1993), no. 2, 249–297.

[Gru08] Joachim Grunewald, *The behavior of Nil-groups under localization and the relative assembly map*, Topology **47** (2008), no. 3, 160–202.

[HM97] Lars Hesselholt and Ib Madsen, *On the K-theory of finite algebras over Witt vectors of perfect fields*, Topology **36** (1997), no. 1, 29–101.

[Hsi84] Wu Chung Hsiang, *Geometric applications of algebraic K-theory*, 2 (Warsaw, 1983), PWN, Warsaw, 1984, pp. 99–118.

[Lüc02] Wolfgang Lück, *Chern characters for proper equivariant homology theories and applications to K- and L-theory*, J. Reine Angew. Math. **543** (2002), 193–234.

[LRV03] Wolfgang Lück, Holger Reich, and Marco Varisco, *Commuting homotopy limits and smash products*, K-Theory **30** (2003), no. 2, 137–165. Special issue in honor of Hyman Bass on his seventieth birthday. Part II.

[Mad94] Ib Madsen, *Algebraic K-theory and traces*, Current developments in mathematics, 1995 (Cambridge, MA), Int. Press, Cambridge, MA, 1994, pp. 191–321.

[Nov66] S. P. Novikov, *On manifolds with free abelian fundamental group and their application*, Izv. Akad. Nauk SSSR Ser. Mat. **30** (1966), 207–246 (Russian).

[Nov71] ———, *Analogues hermitiens de la K-théorie*, Actes du Congrès International des Mathématiciens (Nice, 1970), Gauthier-Villars, Paris, 1971, pp. 39–45.

[RV14] Holger Reich and Marco Varisco, *On the Adams isomorphism for equivariant orthogonal spectra* (2014), available at http://arxiv.org/abs/1404.4034.

[Sch79] Peter Schneider, *Über gewisse Galoiscohomologiegruppen*, Math. Z. **168** (1979), no. 2, 181–205 (German).

[Wal70] C. T. C. Wall, *Surgery on compact manifolds*, Academic Press, London, 1970. London Mathematical Society Monographs, No. 1.

Department of Mathematics, University of Oslo, Norway
E-mail address: rognes@math.uio.no
URL: http://folk.uio.no/rognes