Cubical and Cosimplicial Descent

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Outline

Introduction

Cubical Descent

Cosimplicial Descent

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Cubical Descent

Cosimplicial Descent

A small part of Ib Madsen's research and leadership

- Tremendous influence on topology
  - Crucial for topology in Norway
- Guest researchers in Aarhus
  - Bjørn Jahren (1968-1969)
  - Nils Baas
  - Hans Brodersen (1978-1979)
  - J. R. (1991-1992)
  - Bjørn I. Dundas (1994-1996)

A small part of Ib Madsen's research and leadership

- Tremendous influence on topology
  - Crucial for topology in Norway
- PhD students
  - Marcel Bökstedt
    - Iver Ottosen
    - Morten Brun
    - Thomas Kragh
  - Christian Schlichtkrull
    - Mirjam Solberg
  - Kristian Moi

Topological cyclic homology and the cyclotomic trace map

Theorem (Bökstedt–Hsiang–Madsen)

- *There is a natural cyclotomic trace map*

$$K(A) \xrightarrow{trc} TC(A;p) = \operatorname*{holim}_{R,F,n} THH(A)^{C_{p^n}}$$

from algebraic K-theory to topological cyclic homology, lifting the trace map

$$K(A) \xrightarrow{tr} THH(A) \,.$$

- *Calculation of $TC(A;p)$ for $A = S[\Omega X]$.*
- *Assembly map $K(S) \wedge B\Gamma_+ \to K(S[\Gamma])$ is rationally injective for a large class of groups Γ.*

Goodwillie's Kyoto ICM conjecture

Theorem (Dundas-Goodwillie-McCarthy)

Let $A \to B$ be a map of connective S-algebras, such that $\pi_0(A) \to \pi_0(B)$ is a surjection with nilpotent kernel. Then the square

$$\begin{array}{ccc} K(A) & \longrightarrow & K(B) \\ \downarrow{\scriptstyle \mathrm{trc}} & & \downarrow{\scriptstyle \mathrm{trc}} \\ TC(A) & \longrightarrow & TC(B) \end{array}$$

is homotopy Cartesian.

Linearization

Example

The square

$$\begin{array}{ccc} K(S) & \longrightarrow & K(\mathbb{Z}) \\ \downarrow & & \downarrow \\ TC(S) & \xrightarrow{L} & TC(\mathbb{Z}) \end{array}$$

is homotopy Cartesian.

Problem: How to determine L?

Descent

- Joint work with Bjørn I. Dundas.

- Revisit Dundas' proof of $K(A \to B) \simeq TC(A \to B)$.
- How to recover $K(A)$ from K of B and other B-algebras?
- Same for $THH(A)$, $TC(A)$ and the intermediate functors.
- Cubical and cosimplicial versions.

Cubical descent for K, THH and TC

Theorem

Let A and B be connective S-algebras, with 1-connected unit $\eta \colon S \to B$. Then the functors $F = K, THH, \dots, TC$ satisfy cubical descent *at A along $S \to B$ in the sense that*

$$\eta \colon F(A) \xrightarrow{\simeq} \operatorname*{holim}_{T \in P} F(X(T))$$

is an equivalence. Here P is the partially ordered set of nonempty finite $T = \{t_0 < \dots < t_q\} \subset \mathbb{N}$, and $X(T) = A \wedge \bigwedge_{i=0}^{q} B$.

Note: There are S-algebra maps $B \to X(T)$ for all $T \in P$.

Cosimplicial descent for K, THH and TC

Theorem

Let A and B be connective S-algebras, with 1-connected unit $\eta\colon S \to B$. Suppose also that B is commutative. Then the functors $F = K, THH, \dots, TC$ satisfy cosimplicial descent at A along $S \to B$ in the sense that

$$\eta\colon F(A) \xrightarrow{\simeq} \operatorname*{holim}_{[q]\in\Delta} F(Y^q)$$

is an equivalence. Here $[q] = \{0 < \cdots < q\}$ and $Y^q = A \wedge \bigwedge_{i=0}^{q} B$.

Note: There are S-algebra maps $B \to Y^q$ for all $[q] \in \Delta$.

Cosimplicial descent along $S \to H\mathbb{Z}$

Example

The map from $K(S)$ to the homotopy limit of

$$K(H\mathbb{Z}) \substack{\longrightarrow \\ \longleftarrow \\ \longrightarrow} K(H\mathbb{Z} \wedge H\mathbb{Z}) \substack{\longrightarrow \\ \longleftarrow \\ \longrightarrow \\ \longleftarrow \\ \longrightarrow} K(H\mathbb{Z} \wedge H\mathbb{Z} \wedge H\mathbb{Z}) \substack{\longrightarrow \\ \longleftarrow \\ \longrightarrow \\ \longleftarrow \\ \longrightarrow \\ \longleftarrow \\ \longrightarrow} \cdots$$

is an equivalence.

Note: Each $H\mathbb{Z} \wedge \cdots \wedge H\mathbb{Z}$ is equivalent to H of a simplicial ring.

Cosimplicial descent along $S \to MU$

Example

The map from $K(S)$ to the homotopy limit of

$$K(MU) \rightrightarrows K(MU \wedge MU) \rightrightarrows K(MU \wedge MU \wedge MU) \rightrightarrows \dots$$

is an equivalence.

Note: $MU \wedge MU^{\wedge q} \simeq MU \wedge BU_+^q$ for each $q \geq 0$.

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Cubical diagrams, I

- A and B connective S-algebras.
- Unit $\eta\colon S \to B$ a 1-connected map.
- Base change map = 1-cube:

$$A = A \wedge S \xrightarrow{1 \wedge \eta} A \wedge B$$

Cubical diagrams, II

- Square = 2-cube:

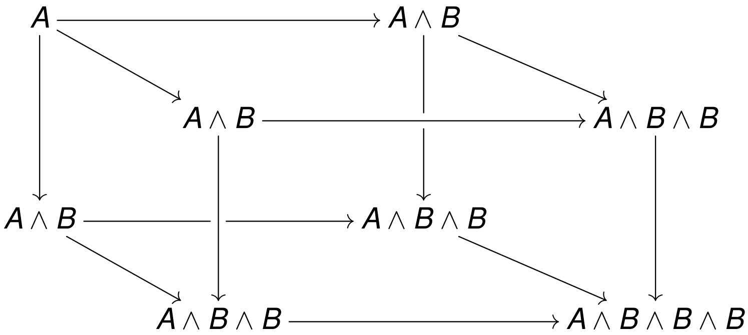
$$\begin{array}{ccc} A & \longrightarrow & A \wedge B \\ \downarrow & & \downarrow \\ A \wedge B & \longrightarrow & A \wedge B \wedge B \end{array}$$

- Canonical map

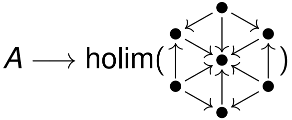
$$A \longrightarrow \operatorname{holim}(A \wedge B \to A \wedge B \wedge B \leftarrow A \wedge B)$$

Cubical diagrams, III

- Cube = 3-cube:



- Canonical map



Cubical diagrams, N

- $P^n_\eta = \mathscr{P}(\{1,\dots,n\})$ partially ordered set of subsets $T \subseteq \{1,\dots,n\}$.
- n-cube in $\mathscr{C}$ = functor $X\colon P^n_\eta \to \mathscr{C}$.

$$P^n_\eta \ni T \longmapsto X(T) \in \mathscr{C}$$

- Canonical map

$$X(\varnothing) \longrightarrow \operatorname*{holim}_{T\in P^n} X(T)$$

where $P^n = P^n_\eta \setminus \{\varnothing\}$.

Definition

Amitsur n-cube $X = X^n(A,B)$:

$$X(T) = A \wedge \bigwedge_{t\in T} B$$

Amitsur ω-cube X

- $P_\eta = \mathscr{P}(\{1, 2, \dots\})$ partially ordered set of finite subsets $T \subseteq \{1, 2, \dots\} = \mathbb{N}$.
- ω-cube in $\mathscr{C}$ = functor $X\colon P_\eta \to \mathscr{C}$.

$$P_\eta \ni T \longmapsto X(T) \in \mathscr{C}$$

- Canonical map

$$X(\varnothing) \longrightarrow \operatorname*{holim}_{T \in P} X(T)$$

where $P = P_\eta \setminus \{\varnothing\}$.

Definition

Amitsur ω-cube $X = X^\omega(A, B)$:

$$X(T) = A \wedge \bigwedge_{t \in T} B$$

Trace maps

Functors from S-algebras to spectra:

- K = algebraic K-theory [Waldhausen, May et al.]
- THH = topological Hochschild homology [Bökstedt]
- TC = topological cyclic homology [Bökstedt-Hsiang-Madsen, Goodwillie et al.]

Natural transformations:

- $\mathrm{tr}\colon K \to THH$ = trace map
- $\mathrm{trc}\colon K \to TC$ = cyclotomic trace map

Higher Blakers-Massey theorem for algebraic K-theory

Theorem (Dundas (1997))

Let A and B be connective S-algebras, with 1-connected unit $\eta\colon S \to B$. The n-cube

$$P^n_\eta \ni T \longmapsto K(X(T)) = K(A \wedge \bigwedge_{t \in T} B)$$

is $(n+1)$-Cartesian, meaning that

$$\eta_n\colon K(A) = K(X(\varnothing)) \longrightarrow \operatorname*{holim}_{T \in P^n} K(X(T))$$

is $(n+1)$-connected.

Cubical descent for algebraic K-theory

Corollary

Let A and B be connective S-algebras, with 1-connected unit $\eta \colon S \to B$. The ω-cube

$$P_\eta \ni T \longmapsto K(X(T)) = K(A \wedge \bigwedge_{t \in T} B)$$

is homotopy Cartesian, meaning that

$$\eta \colon K(A) = K(X(\varnothing)) \longrightarrow \operatorname*{holim}_{T \in P} K(X(T))$$

is an equivalence.

Cubical descent for THH and TC

Theorem

Let A and B be connective S-algebras, with 1-connected unit $\eta\colon S \to B$. The functors $THH, \dots, TC$ satisfy cubical descent at A along $S \to B$ in the sense that

$$\eta\colon THH(A) = THH(X(\varnothing)) \longrightarrow \operatorname*{holim}_{T \in P} THH(X(T))$$

and

$$\eta\colon TC(A) = TC(X(\varnothing)) \longrightarrow \operatorname*{holim}_{T \in P} TC(X(T))$$

are equivalences.

Cubical homotopy limit spectral sequence (SS#1)

- There is a homotopy limit spectral sequence [Bousfield-Kan (1972)]

$$E_2^{s,t} = \lim_P{}^s \pi_t F(X) \Longrightarrow_s \pi_{t-s} \operatorname*{holim}_P F(X)$$

  associated to any P-shaped diagram $T \mapsto F(X(T))$ of spectra.
- In this generality we do not know if the spectral sequence converges to the indicated abutment.
- The right derived functors $\lim_P^s$ appear to be difficult to calculate.

Cubical descent spectral sequence (SS#2)

- Equivalence

$$\operatorname*{holim}_{P} F(X) \simeq \operatorname*{holim}_{n} \operatorname*{holim}_{P^n} F(X)$$

- Tower of fibrations

$$\cdots \to \operatorname*{holim}_{P^{s+1}} F(X) \xrightarrow{p_s} \operatorname*{holim}_{P^s} F(X) \to \ldots$$

- Associated cubical descent spectral sequence

$$E_1^{s,t} = \pi_{t-s} \operatorname{hofib}(p_s) \Longrightarrow_s \pi_{t-s} \operatorname*{holim}_{P} F(X)$$

Strong convergence, I

Theorem

Let A and B be connective S-algebras, with 1-connected unit $\eta\colon S \to B$. For any of the functors $F = K$, THH, $\dots$, TC, and $X = X^\omega(A, B)$ the Amitsur ω-cube, the cubical descent spectral sequence

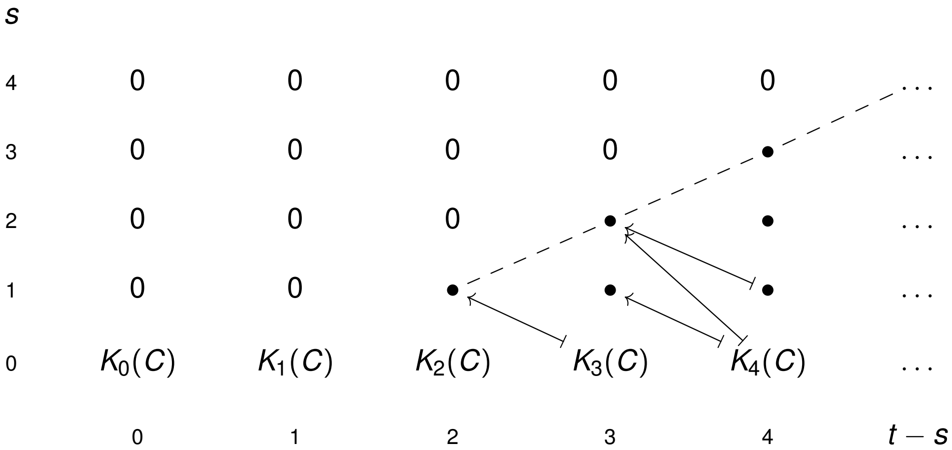
$$E_1^{s,t} = \pi_{t-s}\,\mathrm{hofib}(p_s) \Longrightarrow_s \pi_{t-s} F(A)$$

vanishes above a line of slope $+1$ in the $(t-s, s)$-plane, hence is strongly convergent.

Note: The E_1-term still appears difficult to calculate.

Vanishing line for K-theory cubical descent spectral sequence

$$E_1^{s,t} \Longrightarrow_s K_{t-s}(A) \qquad \text{with } C = A \wedge B$$



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Precosimplicial diagrams, I

- A and B connective S-algebras.
- Unit $\eta\colon S \to B$ a 1-connected map.
- The Amitsur square $X^2(A, B)$ is pulled back from a fork diagram

$$A \longrightarrow A \wedge B \underset{d_1}{\overset{d_0}{\rightrightarrows}} A \wedge B \wedge B$$

- Canonical map

$$A \longrightarrow \operatorname{hoeq}(\, A \wedge B \rightrightarrows A \wedge B \wedge B \,)$$

Precosimplicial diagrams, II

- The Amitsur cube $X^3(A, B)$ is pulled back from a diagram

$$A \longrightarrow A \wedge B \overset{d_0}{\underset{d_1}{\rightrightarrows}} A \wedge B \wedge B \overset{d_0,\, d_1,\, d_2}{\rightrightarrows\!\!\!\rightarrow} A \wedge B \wedge B \wedge B$$

- Canonical map

$$A \longrightarrow \operatorname{holim}\left(A \wedge B \rightrightarrows A \wedge B \wedge B \rightrightarrows\!\!\!\rightarrow A \wedge B \wedge B \wedge B \right)$$

Cosimplicial diagrams

- Δ_η = category of finite totally ordered sets $[q] = \{0 < \cdots < q\}$ for $q \geq -1$, and order-preserving functions $\alpha\colon [p] \to [q]$.
- Coaugmented cosimplicial object = functor $Y\colon \Delta_\eta \to \mathscr{C}$. Canonical map

$$Y^{-1} \longrightarrow \operatorname*{holim}_{[q]\in\Delta} Y^q$$

  where $q \geq 0$ in $\Delta \subset \Delta_\eta$.
- $\Delta^{<n}$ = full subcategory with $0 \leq q < n$.

Precosimplicial diagrams

- M_η = subcategory of Δ_η with morphisms the injective order-preserving functions $\alpha\colon [p] \rightarrowtail [q]$.
- Coaugmented precosimplicial object = functor $Z\colon M_\eta \to \mathscr{C}$. Canonical map

$$Z^{-1} \longrightarrow \operatorname*{holim}_{[q]\in M} Z^q$$

  where $q \geq 0$ in $M \subset M_\eta$.
- $M^{<n}$ = full subcategory with $0 \leq q < n$.

Comparison functors

- Inclusion
$$i\colon M_\eta \subset \Delta_\eta\,.$$
- Pullback over i takes a cosimplicial object to a precosimplicial object by forgetting the codegeneracies.
- Functor
$$e\colon P_\eta \to M_\eta$$
takes $T = \{t_0 < \cdots < t_q\} \subset \mathbb{N}$ to $[q]$ and maps $T \setminus \{t_i\} \subset T$ to $\delta_i\colon [q-1] \rightarrowtail [q]$.
- Pullback over e expands a coaugmented precosimplicial object to an ω-cube.
- Composite functor
$$f = ie\colon P_\eta \to \Delta_\eta\,.$$

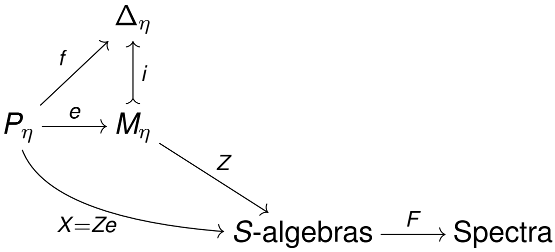
Precosimplicial Amitsur diagram Z

Definition

Coaugmented precosimplicial Amitsur diagram $Z = Z(A, B)$:

$$Z\colon [q] \longmapsto Z^q = A \wedge \bigwedge_{i=0}^{q} B$$

Coface maps use unit $\eta\colon S \to B$.



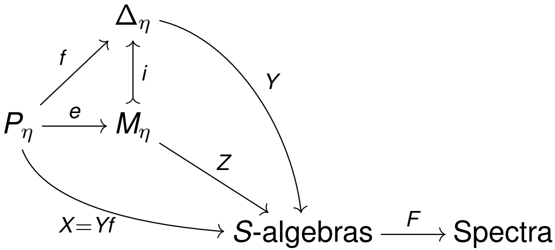
Cosimplicial Amitsur diagram Y

Definition

Suppose that B is commutative. Coaugmented cosimplicial Amitsur diagram $Y = Y(A, B)$:

$$Y\colon [q] \longmapsto Y^q = A \wedge \bigwedge_{i=0}^{q} B$$

Codegeneracy maps use multiplication $\mu\colon B \wedge B \to B$.



Cosimplicial descent for K, THH and TC

- A and B connective S-algebras.
- Unit $\eta \colon S \to B$ is a 1-connected map.
- B is commutative.

Theorem

The functors $F = K, THH, \dots, TC$ satisfy cosimplicial descent at A along $S \to B$ in the sense that

$$\eta \colon F(A) \xrightarrow{\simeq} \operatorname*{holim}_{[q] \in \Delta} F(Y^q)$$

is an equivalence. Here $[q] = \{0 < \cdots < q\}$ and $Y^q = A \wedge \bigwedge_{i=0}^{q} B$.

Left cofinal functors

Theorem (Hopkins (unpublished?), Carlsson (2008))

For each n the restriction $f^n \colon P^n \to \Delta^{<n}$ is left cofinal, meaning that the left fiber $f^n/[q]$ has contractible nerve for each $[q] \in \Delta^{<n}$. Hence $f \colon P \to \Delta$ is left cofinal.

Theorem (Bousfield-Kan (1972))

The canonical map

$$f^* \colon \operatorname*{holim}_{\Delta} F(Y^\bullet) \xrightarrow{\simeq} \operatorname*{holim}_{P} F(X|P)$$

with $X = Yf$ is an equivalence.

Corollary

$F(Y^{-1}) \simeq \operatorname{holim}_\Delta F(Y^\bullet)$ *if and only if* $F(X(\varnothing)) \simeq \operatorname{holim}_P F(X|P)$.

Cosimplicial homotopy limit spectral sequence (SS#3)

- There is a homotopy limit spectral sequence

$$E_2^{s,t} = \lim_{\Delta}{}^s \pi_t F(Y^\bullet) \Longrightarrow_s \pi_{t-s} \operatorname*{holim}_{\Delta} F(Y^\bullet)$$

  associated to any Δ-shaped diagram of spectra, i.e., to any cosimplicial spectrum $[q] \mapsto F(Y^q)$.
- Isomorphic, starting at E_2, to the Bousfield-Kan homotopy spectral sequence described on the next page.

Bousfield-Kan homotopy spectral sequence (SS#4)

- Totalization tower of fibrations

$$\cdots \to \mathrm{Tot}_s\, F(Y^\bullet) \xrightarrow{\tau_s} \mathrm{Tot}_{s-1}\, F(Y^\bullet) \to \ldots$$

- Associated Bousfield-Kan homotopy spectral sequence

$$E_1^{s,t} = \pi_{t-s}\,\mathrm{hofib}(\tau_s) \Longrightarrow_s \pi_{t-s}\,\mathrm{Tot}\, F(Y^\bullet)$$

- Cohomotopy formula for E_2-term

$$E_2^{s,t} = \pi^s \pi_t F(Y^\bullet) \Longrightarrow_s \pi_{t-s}\,\mathrm{Tot}\, F(Y^\bullet)$$

Cosimplicial descent spectral sequence (SS#5)

- Equivalence

$$\operatorname*{holim}_{\Delta} F(Y^\bullet) \simeq \operatorname*{holim}_{n} \operatorname*{holim}_{\Delta^{<n}} F(Y^\bullet)$$

- Tower of fibrations

$$\cdots \to \operatorname*{holim}_{\Delta^{<s+1}} F(Y^\bullet) \xrightarrow{\delta_s} \operatorname*{holim}_{\Delta^{<s}} F(Y^\bullet) \to \ldots$$

- Associated cosimplicial descent spectral sequence

$$E_1^{s,t} = \pi_{t-s}\,\mathrm{hofib}(\delta_s) \Longrightarrow_s \pi_{t-s} \operatorname*{holim}_{\Delta} F(Y^\bullet)$$

Partial totalization and truncated cosimplicial objects

Theorem (Folklore?)

There are natural equivalences

$$\mathrm{Tot}_n F(Y^\bullet) \xrightarrow{\simeq} \operatorname*{holim}_{\Delta^{<n+1}} F(Y^\bullet)$$

for all n, compatible with the Bousfield-Kan equivalence $\mathrm{Tot}\, F(Y^\bullet) \simeq \mathrm{holim}_\Delta F(Y^\bullet)$.

Corollary

The cosimplicial descent spectral sequence is isomorphic, starting at E_1, to the Bousfield-Kan homotopy spectral sequence.

Consensus

Theorem

Let $F(Y^\bullet)$ be a cosimplicial spectrum, with associated ω-cube $F(X|P)$. The

- *cubical descent (SS#2),*
- *cosimplicial descent (SS#5) and*
- *Bousfield-Kan homotopy (SS#4)*

spectral sequences are isomorphic, starting at E_1. They are isomorphic to the

- *cubical homotopy limit (SS#1) and*
- *cosimplicial homotopy limit (SS#3)*

spectral sequences, starting at E_2:

$$E_2^{s,t} = \pi^s \pi_t F(Y^\bullet) \Longrightarrow_s \pi_{t-s} \operatorname*{holim}_{\Delta} F(Y^\bullet) \cong \pi_{t-s} \operatorname*{holim}_{P} F(X|P)$$

Strong convergence, II

- A and B connective S-algebras.
- Unit $\eta\colon S \to B$ is a 1-connected map.
- B is commutative.

Theorem

For any of the functors $F = K, THH, \dots, TC$, and $Y^\bullet = Y^\bullet(A, B)$ the cosimplicial Amitsur diagram with $Y^q = A \wedge \bigwedge_{i=0}^q B$, the cosimplicial descent spectral sequence

$$E_2^{s,t} = \pi^s \pi_t F(Y^\bullet) \Longrightarrow_s \pi_{t-s} F(A)$$

vanishes above a line of slope $+1$ in the $(t-s, s)$-plane, hence is strongly convergent.

Complex bordism

The descent spectral sequence

$$E_2^{s,t} = \pi^s K_t(Y^\bullet) \Longrightarrow_s K_{t-s}(S)$$

with $S \to Y^\bullet$ the Adams–Novikov resolution

$$S \longrightarrow MU \mathrel{\substack{\longrightarrow\\ \longleftarrow\\ \longrightarrow}} MU \wedge MU \mathrel{\substack{\longrightarrow\\ \longleftarrow\\ \longrightarrow\\ \longleftarrow\\ \longrightarrow}} MU \wedge MU \wedge MU \mathrel{\substack{\longrightarrow\\ \longleftarrow\\ \longrightarrow\\ \longleftarrow\\ \longrightarrow\\ \longleftarrow\\ \longrightarrow}} \dots$$

draws attention to $K(MU \wedge BU_+^q)$ for $q \geq 0$.
Question: Is the chromatic filtration seen in the ANSS for $\pi_*(S)$ reflected in this spectral sequence for $K_*(S)$?

Rational collapse

Theorem

The rationalized descent spectral sequence

$$E_2^{s,t} \otimes \mathbb{Q} = \pi^s K_t(Y^\bullet) \otimes \mathbb{Q} \Longrightarrow_s K_{t-s}(S) \otimes \mathbb{Q}$$

collapses at the E_2-term to the edge $s = 0$. Hence $E_2^{s,t}$ is finite for $s > 0$.

Proof.

Use [Goodwillie (1986)] as adapted in [Ausoni-R. (2012)]. □

Homotopy limit property

Theorem (R.-Lunøe-Nielsen (2011))

The comparison maps

$$\Gamma_n \colon THH(MU)^{C_{p^m}} \longrightarrow THH(MU)^{hC_{p^m}}$$
$$\hat{\Gamma}_n \colon THH(MU)^{C_{p^{m-1}}} \longrightarrow THH(MU)^{tC_{p^m}}$$

are p-adic equivalences for all m.

Corollary

The comparison maps

$$\Gamma \colon TF(MU; p) \longrightarrow THH(MU)^{h\mathbb{T}}$$
$$\hat{\Gamma} \colon TF(MU; p) \longrightarrow THH(MU)^{t\mathbb{T}} = TP(MU)$$

are p-adic equivalences.

Thank you!

Best wishes for the future, Ib, and thank you again for all your blossoming and fruitful ideas!