ON THE IMAGE OF MILNOR'S K_4 IN QUILLEN'S K_4 FOR NUMBER FIELDS

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Introduction

Let F be a field, with multiplicative group $F^\times$. Its Milnor K-theory [Mi1] is the graded ring $K_*^M(F)$ defined as the tensor algebra $T(F^\times) = \bigoplus_{n\ge0}(F^\times)^{\otimes n}$ divided by the two-sided homogeneous ideal generated by the terms $x\otimes(1-x)$ for $x\ne 0,1$. The image of $x_1\otimes\cdots\otimes x_n \in (F^\times)^{\otimes n}$ in $K_n^M(F)$ is written $\{x_1,\dots,x_n\}$.

Quillen's higher algebraic K-theory $K_*(F)$ is also a graded ring [Qu], and the isomorphism $F^\times \cong K_1(F)$ extends to a ring homomorphism $T(F^\times)\to K_*(F)$. The product in algebraic K-theory satisfies $\{x\}\cdot\{1-x\}=0$ for $x\ne 0,1$, so this ring homomorphism factors through Milnor K-theory. We obtain a canonical homomorphism

$$\kappa_n \colon K_n^M(F)\to K_n(F)$$

for all $n\ge 0$. This is an isomorphism for $n=0,1$, more or less by definition, and for $n=2$ by H. Matsumuto's theorem [Mi2, §12].

Now suppose F is a number field, with r_1 real embeddings. By the Bass–Tate theorem [BT], $K_n^M(F)\cong(\mathbb{Z}/2)^{r_1}$ for all $n\ge 3$. Multiplication by the symbol $\epsilon=\{-1\}$ induces a homomorphism $\epsilon\cdot\colon K_n^M(F)\to K_{n+1}^M(F)$ which is a surjection for $n=1,2$, and an isomorphism for $n\ge3$. Hence all classes in $K_n^M(F)$ have the form $\epsilon^{n-1}\cdot\{x\}=\{-1,\dots,-1,x\}$ for $x\in F^\times$, when $n\ge1$. The class $\epsilon^n=\{-1,\dots,-1\}$ has nonzero image in $K_n^M(\mathbb{R})$, hence is nonzero in $K_n^M(F)$ when F is formally real, i.e., when $r_1>0$.

It was proved by J.M. Shapiro in [Sh] that κ_3 injects $K_3^M(F)$ into $K_3(F)$. There is a ring map from the stable homotopy groups of spheres $\pi_*^S\to K_*(F)$ taking the Hopf map $\eta\in\pi_1^S$ to $\epsilon\in K_1(F)$. Since $\eta^4=0$ in $\pi_4^S=0$, the symbol $\epsilon^4\in K_4^M(F)$ is in the kernel of κ_4. It follows that the image of $K_n^M(F)$ in $K_n(F)$ is zero for $n\ge5$. This leads us to ask what happens in the remaining case, $n=4$. In many cases it is known that the image is zero. It is the purpose of this paper to give a sometimes nonzero lower bound on the image of $K_4^M(F)\cong(\mathbb{Z}/2)^{r_1}$ in $K_4(F)$.

We prove the following main theorem:

Theorem 0.1. *Let F be a number field with r_1 real embeddings, and $R_F=\mathcal{O}_F[1/2]$ its ring of 2-integers. Let s be the number of distinct primes in $\mathcal{O}_F$ dividing (2). Let $t=\mathrm{rk}_2\,\mathrm{Pic}(R_F)$ and $u=\mathrm{rk}_2\,\mathrm{Pic}_+(R_F)$ be the 2-rank of the Picard group of R_F,* *and the narrow Picard group of R_F, respectively. The 2-rank $\rho = \mathrm{rk}_2 \operatorname{im}(\kappa_4)$ of the image of the canonical map*

$$\kappa_4 \colon K_4^M(F) \cong (\mathbb{Z}/2)^{r_1} \to K_4(F)$$

satisfies $u - t \le \rho \le s + u - 1$. If F is formally real then $\rho < r_1$, otherwise $\rho = 0$.

As a simple example consider $F = \mathbb{Q}(\sqrt{7})$, the real quadratic number field with discriminant 28. It has $r_1 = 2$, $t = 0$ and $u = 1$, so the image of κ_4 is precisely $\mathbb{Z}/2$ in this case.

1. Number fields and cohomology

We review and fix some basic notation.

For an abelian group A and an integer q let ${}_qA = \{a \in A \mid qa = 0\}$ be the q-torsion subgroup. For a prime ℓ let $A\{\ell\} = \bigcup_{\nu \ge 0} {}_{\ell^\nu}A \subseteq A$ be the ℓ-power torsion subgroup. We define the ℓ-rank $\mathrm{rk}_\ell A$ of A as the dimension of the $\mathbb{Z}/\ell$-vector space ${}_\ell A$. When A is finite, this equals the $\mathbb{Z}/\ell$-dimension of A/ℓ. For abelian groups A and B we write $A \rtimes B$ for an extension of B by A, such that there is a short exact sequence $0 \to A \to A \rtimes B \to B \to 0$.

Let F be a number field, with r_1 real embeddings and r_2 pairs of complex embeddings. Let $\mathcal{O}_F$ be its ring of integers, and let $R_F = \mathcal{O}_F[1/2]$ be its ring of 2-integers. Let s be the number of distinct prime ideals in $\mathcal{O}_F$ over the rational prime ideal (2). By Dirichlet's unit theorem, $R_F^\times \cong \mathbb{Z}^{r_1+r_2+s-1} \times \mu(R_F)$, where the group $\mu(R_F)$ of roots of unity in R_F is a finite cyclic group of even order.

Let $\mathrm{Pic}(R_F)$ be the Picard group of R_F, and let $\mathrm{Pic}_+(R_F)$ be the narrow Picard group. These can be defined as follows: Let

$$e \colon F^\times \to \bigoplus_{\mathfrak{P} \nmid (2)} \mathbb{Z}$$

be the divisor map, taking $x = a/b \in F^\times$ with $a, b \in \mathcal{O}_F$ to the sequence $e(x) = (v_{\mathfrak{P}}(a) - v_{\mathfrak{P}}(b))_{\mathfrak{P}}$ where $v_{\mathfrak{P}}$ is the valuation at $\mathfrak{P}$. Then the Picard group is $\mathrm{Pic}(R_F) = \mathrm{cok}(e)$. Let $F_+^\times \subseteq F^\times$ be the subgroup of totally positive units, i.e., the kernel of the sign map

$$\sigma \colon F^\times \to (\mathbb{Z}/2)^{r_1}$$

defined as the sum over the r_1 real embeddings of F of the corresponding maps $F^\times \to \mathbb{R}^\times \to \mathbb{R}^\times/2 \cong \mathbb{Z}/2$. Let $e_+ = e|F_+^\times$ be the restricted divisor map. Then the narrow Picard group is $\mathrm{Pic}_+(R_F) = \mathrm{cok}(e_+)$. Let $t = \mathrm{rk}_2 \mathrm{Pic}(R_F)$ and $u = \mathrm{rk}_2 \mathrm{Pic}_+(R_F)$ be the 2-rank of the Picard group and the narrow Picard group of R_F, respectively.

The Brauer group $\mathrm{Br}(R_F)$ of R_F can be defined in terms of Azumaya algebras or Galois cohomology, and is determined by the Brauer–Hasse–Noether theorem, asserting that there is an exact sequence

$$0 \to \mathrm{Br}(R_F) \to (\mathbb{Z}/2)^{r_1} \oplus (\mathbb{Q}/\mathbb{Z})^s \to \mathbb{Q}/\mathbb{Z} \to 0\,.$$

Thus $\mathrm{Br}(R_F) \cong (\mathbb{Z}/2)^{r_1} \oplus (\mathbb{Q}/\mathbb{Z})^{s-1}$.

Let R be a commutative ring. For M an étale sheaf over $\mathrm{Spec}(R)$, let $H^n_{\text{ét}}(R; M)$ denote the nth étale cohomology group of $\mathrm{Spec}(R)$ with coefficients in the sheaf

M. With $M = \mathbb{G}_m$ (the multiplicative group) there are standard isomorphisms $H^0_{\text{ét}}(R;\mathbb{G}_m) \cong R^\times$, $H^1_{\text{ét}}(R;\mathbb{G}_m) \cong \operatorname{Pic}(R)$ and $H^2_{\text{ét}}(R;\mathbb{G}_m) \cong \operatorname{Br}(R)$ [Ho]. With $M = \mu_q$ (the qth roots of unity) and $1/q \in R$, the short exact Kummer sequence of étale sheaves

$$0 \to \mu_q \to \mathbb{G}_m \xrightarrow{(-)^q} \mathbb{G}_m \to 0$$

induces a long exact sequence in étale cohomology. There result natural isomorphisms $H^0_{\text{ét}}(R;\mu_q) \cong \mu_q(R)$, $H^1_{\text{ét}}(R;\mu_q) \cong R^\times/(R^\times)^q \rtimes {}_q\operatorname{Pic}(R)$ and $H^2_{\text{ét}}(R;\mu_q) \cong \operatorname{Pic}(R)/q \rtimes {}_q\operatorname{Br}(R)$. For integers i we write $\mathbb{Z}/q(i) = \mu_q^{\otimes i}$ for the ith tensor power of μ_q, as an étale sheaf. By a theorem of J. Tate, the natural map

$$\alpha^n \colon H^n_{\text{ét}}(R_F;\mathbb{Z}/q(i)) \to \bigoplus^{r_1} H^n_{\text{ét}}(\mathbb{R};\mathbb{Z}/q(i))$$

is an isomorphism for $n \geq 3$. Here $H^n_{\text{ét}}(\mathbb{R};\mathbb{Z}/q(i)) \cong H^n(\operatorname{Gal}(\mathbb{C}/\mathbb{R});\mathbb{Z}/q(i))$ is easily computed using the periodic resolution for the group cohomology on the right.

Note that with $q = 2$, the étale sheaf $\mathbb{Z}/2(i)$ is independent of i, hence can be briefly written $\mathbb{Z}/2$. The Kummer and Tate formulas above specialize to:

$$\begin{aligned}
&H^0_{\text{ét}}(R_F;\mathbb{Z}/2) \cong \mu_2(R_F) \cong \mathbb{Z}/2\\
&H^1_{\text{ét}}(R_F;\mathbb{Z}/2) \cong R_F^\times/2 \rtimes {}_2\operatorname{Pic}(R_F) \cong (\mathbb{Z}/2)^{r_1+r_2+s+t}\\
&H^2_{\text{ét}}(R_F;\mathbb{Z}/2) \cong \operatorname{Pic}(R_F)/2 \rtimes {}_2\operatorname{Br}(R_F) \cong (\mathbb{Z}/2)^{r_1+s+t-1}\\
&H^n_{\text{ét}}(R_F;\mathbb{Z}/2) \cong H^n_{\text{ét}}(\mathbb{R};\mathbb{Z}/2)^{r_1} \cong (\mathbb{Z}/2)^{r_1}
\end{aligned}$$

with $n \geq 3$ in the last row.

We define $\tilde{H}^1_{\text{ét}}(R_F;\mathbb{Z}/2)$ as the kernel of the natural map $\alpha^1 \colon H^1_{\text{ét}}(R_F;\mathbb{Z}/2) \to \bigoplus^{r_1} H^1_{\text{ét}}(\mathbb{R};\mathbb{Z}/2) \cong (\mathbb{Z}/2)^{r_1}$. Likewise for $\tilde{H}^1_{\text{ét}}(F;\mathbb{Z}/2)$ and $\tilde{H}^2_{\text{ét}}(R_F;\mathbb{Z}/2)$.

Proposition 1.1. *There is a natural exact sequence*

$$\begin{aligned}
0 \to \tilde{H}^1_{\text{ét}}(R_F;\mathbb{Z}/2) \to H^1_{\text{ét}}(R_F;\mathbb{Z}/2) \xrightarrow{\alpha^1} (\mathbb{Z}/2)^{r_1} \to&\\
\to \operatorname{Pic}_+(R_F)/2 \xrightarrow{\pi} \operatorname{Pic}(R_F)/2 \to 0\,.&
\end{aligned}$$

Hence $\operatorname{rk}_2 \operatorname{cok}(\alpha^1) = u - t$ *and* $\operatorname{rk}_2 \tilde{H}^1_{\text{ét}}(R_F;\mathbb{Z}/2) = r_2 + u + s$.

Proof. Consider the following commutative diagram:

$$\begin{array}{ccccccc}
 & & \tilde{H}^1_{\text{ét}}(R_F\mathbb{Z}/2) & \longrightarrow & \tilde{H}^1_{\text{ét}}(F;\mathbb{Z}/2) & \overset{\tilde{\partial}}{\searrow} & \\
 & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & H^1_{\text{ét}}(R_F;\mathbb{Z}/2) & \longrightarrow & H^1_{\text{ét}}(F;\mathbb{Z}/2) & \xrightarrow{\partial} & \bigoplus_{\mathfrak{P}\nmid(2)} H^0_{\text{ét}}(k_{\mathfrak{P}};\mathbb{Z}/2) \\
 & & & \underset{\alpha^1}{\searrow} & \downarrow{\scriptstyle \sigma/2} & & \\
 & & & & (\mathbb{Z}/2)^{r_1} & & \\
 & & & & \downarrow & & \\
 & & & & 0\,. & &
\end{array}$$

The middle row is part of the exact localization sequence in mod 2 étale cohomology, where we write $k_{\mathfrak{P}}$ for the residue field of F at the prime ideal $\mathfrak{P} \subset R_F$. Since $\mathrm{Pic}(F) = 0$ we can identify $H^1_{\text{ét}}(F;\mathbb{Z}/2) \cong F^\times/2$, and the sign map σ descends to a homomorphism $\sigma/2\colon H^1_{\text{ét}}(F;\mathbb{Z}/2) \to (\mathbb{Z}/2)^{r_1}$. It is surjective by the approximation theorem. Its restriction to $H^1_{\text{ét}}(R_F;\mathbb{Z}/2) \subset H^1_{\text{ét}}(F;\mathbb{Z}/2)$ is α^1, and the kernel of $\sigma/2$ is denoted $\tilde{H}^1_{\text{ét}}(F;\mathbb{Z}/2)$. We let $\tilde{\partial}$ be the restriction of the boundary map ∂ to $\tilde{H}^1_{\text{ét}}(F;\mathbb{Z}/2)$.

Then we have an exact sequence

$$0 \to \tilde{H}^1_{\text{ét}}(R;\mathbb{Z}/2) \to H^1_{\text{ét}}(R;\mathbb{Z}/2) \xrightarrow{\alpha^1} (\mathbb{Z}/2)^{r_1} \to \mathrm{cok}(\partial) \to \mathrm{cok}(\tilde{\partial}) \to 0\,.$$

It remains to identify the two cokernels.

First, recognize ∂ as $e/2$, the divisor map reduced mod 2, under the identifications $H^1_{\text{ét}}(F;\mathbb{Z}/2) \cong F^\times/2$ and $H^0_{\text{ét}}(k_{\mathfrak{P}};\mathbb{Z}/2) \cong \mathbb{Z}/2$. Thus $\mathrm{cok}(\partial) \cong \mathrm{cok}(e/2) \cong \mathrm{cok}(e)/2 \cong \mathrm{Pic}(R)/2$.

Second, the surjection $F^\times \to F^\times/2 \cong H^1_{\text{ét}}(F;\mathbb{Z}/2)$ restricts to a surjection $F^\times_+ \to \tilde{H}^1_{\text{ét}}(F;\mathbb{Z}/2)$ of the kernels of σ and $\sigma/2$, respectively. The composite

$$F^\times_+ \to \tilde{H}^1_{\text{ét}}(F;\mathbb{Z}/2) \xrightarrow{\tilde{\partial}} \bigoplus_{\mathfrak{P}\nmid(2)} H^0_{\text{ét}}(k_{\mathfrak{P}};\mathbb{Z}/2)$$

is identified with $(e_+)/2$, the restricted divisor map reduced mod 2. Hence $\mathrm{cok}(\tilde{\partial}) \cong \mathrm{cok}((e_+)/2) \cong \mathrm{cok}(e_+)/2 \cong \mathrm{Pic}_+(R)/2$.

Clearly $\mathrm{Pic}(R)/2 \cong (\mathbb{Z}/2)^t$ and $\mathrm{Pic}_+(R)/2 \cong (\mathbb{Z}/2)^u$ because the groups $\mathrm{Pic}(R)$ and $\mathrm{Pic}_+(R)$ are finite, and the remaining claims follow by exactness. □

We shall also make use of ℓ-adic étale cohomology, with coefficients in the pro-sheaf $\mathbb{Z}_\ell(i) = \{\mathbb{Z}/\ell^\nu\}_\nu$. Since each group $H^n_{\text{ét}}(R_F;\mathbb{Z}/2^\nu(i))$ is finite, there is an isomorphism $H^n_{\text{ét}}(R_F;\mathbb{Z}_2(i)) \cong \lim_\nu H^n_{\text{ét}}(R_F;\mathbb{Z}/2^\nu(i))$.

Lemma 1.2. *Let $i \ge 2$. The finite group $H^2_{\text{ét}}(R_F;\mathbb{Z}_2(i))$ has 2-rank $r_1+s+t-1$ for i even, and $s+t-1$ for i odd.*

Proof. The short exact sequence

$$0 \to H^2_{\text{ét}}(R_F;\mathbb{Z}_2(i))/2 \to H^2_{\text{ét}}(R_F;\mathbb{Z}/2) \to {}_2H^3_{\text{ét}}(\mathbb{R}_F;\mathbb{Z}_2(i)) \to 0$$

and the isomorphism $H^3_{\text{ét}}(R_F;\mathbb{Z}_2(i)) \cong 0$ for i even, $\cong (\mathbb{Z}/2)^{r_1}$ for i odd, ensures that the 2-rank of the finite group $H^2_{\text{ét}}(R_F;\mathbb{Z}_2(i))$ is $r_1+s+t-1$ for i even and $s+t-1$ for i odd. □

2. Milnor K-theory and cohomology

One difficulty in handling the image of Milnor's $K^M_4(F)$ in $K_4(F)$ is that the target group is infinite and contains divisible elements. Let F be a number field, with ring of 2-integers R_F, and keep the notation of the previous section.

Lemma 2.1. *The image of $K^M_4(F)$ in $K_4(F)$ maps injectively to $K_4(F;\mathbb{Z}_2)$ by the coefficient extension homomorphism $K_4(F) \to K_4(F;\mathbb{Z}_2)$.*

Proof. Consider the following commutative diagram:

$$\begin{array}{ccccc}
K_3^M(F) & \longrightarrow & K_3(R_F)\{2\} & \longrightarrow & K_3(F) \\
{\scriptstyle \epsilon\cdot}\downarrow{\scriptstyle \cong} & & {\scriptstyle \epsilon\cdot}\downarrow & & {\scriptstyle \epsilon\cdot}\downarrow \\
K_4^M(F) & \longrightarrow & K_4(R_F)\{2\} & \longrightarrow & K_4(F) \\
 & & \downarrow & & \downarrow \\
 & & K_4(R_F;\mathbb{Z}_2) & \longrightarrow & K_4(F;\mathbb{Z}_2)
\end{array}$$

The canonical map $\kappa_3\colon K_3^M(F) \to K_3(F)$ factors (uniquely) through $K_3(R_F)\{2\}$ because $K_3^M(F) \cong (\mathbb{Z}/2)^{r_1}$ is 2-torsion by the Bass–Tate theorem [BT], and the natural map $K_3(R_F) \to K_3(F)$ is an isomorphism by a theorem of Soulé [So]. Since multiplication by the symbol $\epsilon = \{-1\}$ is an isomorphism from $K_n^M(F)$ to $K_{n+1}^M(F)$ for $n \geq 3$, also the canonical map $\kappa_4\colon K_4^M(F) \to K_4(F)$ factors (uniquely) through $K_4(R_F)\{2\}$. This explains the maps in the upper two rows of the diagram. The maps in the lower part of the diagram are all natural.

By the localization sequence in algebraic K-theory, the natural map $K_4(R_F) \to K_4(F)$ is injective, as is the inclusion $K_4(R_F)\{2\} \subseteq K_4(R_F)$. Since $K_4(R_F)$ is finite, the coefficient extension homomorphism $K_4(R_F)\{2\} \to K_4(R_F;\mathbb{Z}_2)$ is also injective. Finally, by the localization sequence in algebraic K-theory with $\mathbb{Z}_2$-coefficients, the natural map $K_4(R_F;\mathbb{Z}_2) \to K_4(F;\mathbb{Z}_2)$ is injective. This uses that the preceding term in the exact sequence is $\pi_4(\bigvee_{\mathfrak{P}\nmid(2)} K(k_{\mathfrak{P}})_2^\wedge) = 0$. Hence the image of $K_4^M(F)$ in $K_4(F)$ is isomorphic to the image of $K_4^M(F)$ in each of the other groups $K_4(R_F)\{2\}$, $K_4(R_F;\mathbb{Z}_2)$ and $K_4(F;\mathbb{Z}_2)$, respectively. □

The following observation shows why the image of κ_4 cannot be detected in $K_4(F;\mathbb{Z}/2^\nu)$ for $\nu \leq 3$.

Lemma 2.2. *The image of $K_4^M(F)$ in $K_4(F)$ is divisible by* 24.

Proof. Any class in $K_4^M(F)$ has the form $y = \{-1,-1,-1,x\} = \epsilon^3 \cdot \{x\}$ for some $x \in F^\times$. Here $\{-1,-1,-1\} = \epsilon^3$ lifts to $K_3(\mathbb{Q}) \cong K_3(\mathbb{Z}) \cong \mathbb{Z}/48$, where it represents the class of order 2, hence is divisible by 24. Let $\lambda \in K_3(\mathbb{Q})$ be a generator. Then $y = 24 \cdot \lambda \cdot \{x\}$ in $K_4(F)$, as claimed. □

3. Algebraic K-theory and cohomology

We next invoke the extended Bloch–Lichtenbaum spectral sequence of M. Levine, see [BL, 1.3.4] and [Le2, (1.8)]. In its integral form, it is a spectral sequence from motivic cohomology to algebraic K-theory.

Let A be a regular commutative ring of finite type over a regular base ring B of Krull dimension ≤ 1. For example, A can be a number field F, with $B = \mathbb{Q}$, or A can be a ring of ℓ-integers R_F, with $B = \mathbb{Z}[1/\ell]$. Then Levine extends the definition of the motivic cohomology groups $H^n(A;\mathbb{Z}(i))$ given by Bloch, Suslin and Voevodsky in the field case. He then constructs a homological, first quadrant, strongly convergent algebra spectral sequence:

(3.1) $$E^2_{p,q} = H^{q-p}(A;\mathbb{Z}(q)) \Longrightarrow K_{p+q}(A)\,.$$

Likewise, there is a mod m version

$$E^2_{p,q} = H^{q-p}(A;\mathbb{Z}/m(q)) \Longrightarrow K_{p+q}(A;\mathbb{Z}/m) \tag{3.2}$$

converging to algebraic K-theory with mod m coefficients, see [Le2, (1.9)].

We may tensor the integral spectral sequence (3.1) with $\mathbb{Z}_\ell$, to obtain an ℓ-adic spectral sequence. Its E^2-term is $H^{p-q}(A;\mathbb{Z}(q)) \otimes \mathbb{Z}_\ell$, which when each group $H^*(A;\mathbb{Z}/\ell^\nu(q))$ is finite can be identified with $H^{p-q}(A;\mathbb{Z}_\ell(q))$. Likewise its abutment is $K_{p+q}(A)\otimes\mathbb{Z}_\ell$, which when each group $K_*(A;\mathbb{Z}/\ell^\nu)$ is finite can be identified with $K_{p+q}(A;\mathbb{Z}_\ell) = \pi_{p+q}(K(A)^\wedge_\ell)$.

When F is a number field, $A = R_F$ is its ring of ℓ-integers, and $\ell = 2$, Voevodsky's proof of the Milnor conjecture [Vo2] together with [SV], implies that there are natural isomorphisms

$$\begin{aligned} H^n(R_F;\mathbb{Z}/2^\nu(i)) &\cong H^n_{\text{ét}}(R_F;\mathbb{Z}/2^\nu(i)) \\ H^n(R_F;\mathbb{Z}_2(i)) &\cong H^n_{\text{ét}}(R_F;\mathbb{Z}_2(i)) \end{aligned}$$

for all $n \le i$, while $H^n(R_F;\mathbb{Z}/2^\nu(i)) = 0$ and $H^n(R_F;\mathbb{Z}_2(i)) = 0$ for $n > i$. By finiteness of the groups $H^n_{\text{ét}}(R_F;\mathbb{Z}/2^\nu(i))$ and $K_n(R_F;\mathbb{Z}/2^\nu)$ for all n, i, the discussion of the previous paragraph applies, and we have the following 2-adic extended Bloch–Lichtenbaum spectral sequence:

$$E^2_{p,q} = H^{q-p}_{\text{ét}}(R_F;\mathbb{Z}_2(q)) \Longrightarrow K_{p+q}(R_F)\otimes\mathbb{Z}_2 \tag{3.3}$$

with $p, q \ge 0$.

Consider the integral extended Bloch–Lichtenbaum spectral sequence in the case $A = F$. In total degree 1 the edge homomorphism $E^2_{0,1} = H^1(F;\mathbb{Z}(1)) \to K_1(F) \cong F^\times$ is an isomorphism. There is a cup product in motivic cohomology which satisfies the relation $\{x\}\cdot\{1-x\} = 0$ for $x \ne 0, 1$, thus providing a map $K^M_n(F) \to H^n(F;\mathbb{Z}(n))$ for all $n \ge 0$. This is an isomorphism by the Suslin–Nesterenko theorem [NS, 4.9]. Also by multiplicativity of the spectral sequence, we can identify the edge homomorphism $E^2_{0,n} = H^n(F;\mathbb{Z}(n)) \to K_n(F)$ of the extended Bloch–Lichtenbaum spectral sequence with the canonical map $\kappa_n\colon K^M_n(F) \to K_n(F)$. This reduces the question of determining the image of κ_n to that of determining the E^∞-term $E^\infty_{0,n}$.

We now consider the spectral sequences (3.1) for $A = F$, (3.3) for $A = F$ and (3.3) for $A = R_F$. The edge homomorphisms are compatible, yielding a commutative diagram

$$\begin{array}{ccc} H^4(F;\mathbb{Z}(4)) & \longrightarrow & K_4(F) \\ \downarrow & & \downarrow \\ H^4(F;\mathbb{Z}_2(4)) & \longrightarrow & K_4(F;\mathbb{Z}_2) \\ \uparrow & & \uparrow \\ H^4(R_F;\mathbb{Z}_2(4)) & \longrightarrow & K_4(R_F;\mathbb{Z}_2) \end{array}$$

The left hand groups are all isomorphic to $K^M_4(F) \cong \bigoplus^{r_1} H^4(\mathbb{R};\mathbb{Z}_2(4)) \cong (\mathbb{Z}/2)^{r_1}$ by the Suslin–Nesterenko theorem and Voevodsky's proof of the Milnor conjecture.

Thus both left hand vertical maps are isomorphisms. Also the lower right hand vertical map is injective, and the image of $K_4^M(F)$ in $K_4(F)$ is isomorphic to that in $K_4(F;\mathbb{Z}_2)$, by 2.1. Thus the image of κ_4 is isomorphic to the image of the bottom edge homomorphism $H^4(R_F;\mathbb{Z}_2(4)) \to K_4(R_F;\mathbb{Z}_2)$, of the spectral sequence (3.3).

The spectral sequence (3.3) for $A = \mathbb{R}$ has E^2-term

$$E^2_{p,q} = H^{q-p}_{\text{ét}}(\mathbb{R};\mathbb{Z}_2(q)) \cong \begin{cases} \mathbb{Z}/2 & \text{for } p \text{ even,} \\ 0 & \text{for } p \text{ odd.} \end{cases}$$

It abuts to $K_*(\mathbb{R};\mathbb{Z}_2) \cong \pi_*(ko;\mathbb{Z}_2)$, hence there are surjective differentials $d^2_{p,q}$ for all $p \equiv 2 \bmod 4$, while the remaining differentials are all zero.

Similarly, in the spectral sequence (3.2) for $A = \mathbb{R}$ with $\mathbb{Z}/2$-coefficients the E^2-term is $E^2_{p,q} = H^{q-p}_{\text{ét}}(\mathbb{R};\mathbb{Z}/2(q)) \cong \mathbb{Z}/2$ for all $p, q \geq 0$. The abutment is $\pi_*(ko;\mathbb{Z}/2)$, and the pairing of (3.3) with (3.2) implies that there are surjective differentials $d^2_{p,q}$ for all $p \equiv 2, 3 \bmod 4$, while the remaining differentials are all zero.

In total degree 4, the spectral sequence (3.3) yields the exact sequence

$$(3.4) \qquad H^1_{\text{ét}}(R_F;\mathbb{Z}_2(3)) \xrightarrow{d^2_{2,3}} (\mathbb{Z}/2)^{r_1} \to K_4(R_F;\mathbb{Z}_2) \to H^2_{\text{ét}}(R_F;\mathbb{Z}_2(3)) \to 0\,.$$

The image of κ_4 is thus identified with the cokernel of the differential $d^2_{2,3}$. By naturality of (3.3) with respect to the r_1 real embeddings $R_F \to F \to \mathbb{R}$, there is a commutative square:

$$\begin{array}{ccc} H^1_{\text{ét}}(R_F;\mathbb{Z}_2(3)) & \xrightarrow{d^2_{2,3}} & H^4_{\text{ét}}(R_F;\mathbb{Z}_2(4)) \\ \downarrow \alpha^1 & & \cong \downarrow \alpha^4 \\ \bigoplus^{r_1} H^1_{\text{ét}}(\mathbb{R};\mathbb{Z}_2(3)) & \underset{\cong}{\xrightarrow{d^2_{2,3}}} & \bigoplus^{r_1} H^4_{\text{ét}}(\mathbb{R};\mathbb{Z}_2(4)) \end{array}$$

Since the right hand and lower maps are isomorphisms, the cokernel of $d^2_{2,3}$ is identified with the cokernel of the natural homomorphism

$$\alpha^1 \colon H^1_{\text{ét}}(R_F;\mathbb{Z}_2(3)) \to \bigoplus^{r_1} H^1_{\text{ét}}(\mathbb{R};\mathbb{Z}_2(3))\,.$$

This reduces the question about the image of κ_4 to a question in étale cohomology, which however does not seem to be so easy to answer directly.

4. Two-rank formulas

We rewrite the exact sequence (3.4) as follows:

$$(4.1) \qquad 0 \to \operatorname{im}(\kappa_4) \to K_4(R_F) \to H^2_{\text{ét}}(R_F;\mathbb{Z}_2(3)) \to 0\,.$$

We can compute the 2-rank of both the middle and right hand term in this exact sequence, which will provide a lower bound on the 2-rank, thus dimension, of the image group on the left.

Proposition 4.2. *Let F be a formally real number field, i.e., with $r_1 > 0$. For $n \geq 1$, the 2-rank of $K_n(R_F; \mathbb{Z}_2)$ is given by the table:*

n mod 8	0	1	2	3
$\mathrm{rk}_2\, K_n(R_F; \mathbb{Z}_2)$	$s+t-1$	1	$r_1+s+t-1$	r_1

n mod 8	4	5	6	7
$\mathrm{rk}_2\, K_n(R_F; \mathbb{Z}_2)$	$s+u-1$	0	$s+u-1$	1

Proof. Consider the mod 2 extended Bloch–Lichtenbaum spectral sequence (3.2) for R_F. Its E^2-term has

$$E^2_{p,q} = H^{q-p}_{\text{ét}}(R_F; \mathbb{Z}/2) \cong \begin{cases} 0 & \text{for } q < p, \\ \mathbb{Z}/2 & \text{for } q = p, \\ R^\times/2 \rtimes {}_2\operatorname{Pic}(R) & \text{for } q = p+1, \\ \operatorname{Pic}(R)/2 \rtimes {}_2\operatorname{Br}(R) & \text{for } q = p+2, \\ (\mathbb{Z}/2)^{r_1} & \text{for } q \geq p+3. \end{cases}$$

By naturality with respect to the r_1 real embeddings, there are commutative squares:

$$\begin{array}{ccc} H^{q-p}_{\text{ét}}(R_F; \mathbb{Z}/2) & \xrightarrow{d^2_{p,q}} & H^{q-p+3}_{\text{ét}}(R_F; \mathbb{Z}/2) \\ \downarrow \alpha^{q-p} & & \downarrow \alpha^{q-p+3} \\ \bigoplus^{r_1} H^{q-p}_{\text{ét}}(\mathbb{R}; \mathbb{Z}/2) & \xrightarrow{d^2_{p,q}} & \bigoplus^{r_1} H^{q-p+3}_{\text{ét}}(\mathbb{R}; \mathbb{Z}/2) \end{array}$$

For all $q \geq p$ the right hand vertical maps are isomorphisms, so $d^2_{p,q}$ is identified with the natural homomorphism

$$\alpha^{q-p} \colon H^{q-p}_{\text{ét}}(R_F; \mathbb{Z}/2) \to \bigoplus^{r_1} H^{q-p}_{\text{ét}}(\mathbb{R}; \mathbb{Z}/2) \cong (\mathbb{Z}/2)^{r_1}$$

for $p \equiv 2, 3 \bmod 4$, and is zero otherwise.

Let $n = q - p$. The homomorphism $\alpha^0 \colon \mathbb{Z}/2 \to (\mathbb{Z}/2)^{r_1}$ is injective since F is assumed formally real. The homomorphism $\alpha^1 \colon H^1_{\text{ét}}(R; \mathbb{Z}/2) \to (\mathbb{Z}/2)^{r_1}$ was discussed in 1.1. The homomorphism $\alpha^2 \colon H^2_{\text{ét}}(R_F; \mathbb{Z}/2) \to (\mathbb{Z}/2)^{r_1}$ is surjective, with kernel denoted $\tilde{H}^2_{\text{ét}}(R_F; \mathbb{Z}/2)$. ((Proof ?)) The homomorphisms α^n for $n \geq 3$ are all isomorphisms. This leaves an E_3-term concentrated in bidegrees (p, q) with $0 \leq q - p \leq 4$, so located that there is no room for further differentials.

Thus, for $n \geq 1$ the abutment of the spectral sequence is given by the formulas:

$$K_n(R_F; \mathbb{Z}/2) \cong \begin{cases} \tilde{H}^2_{\text{ét}}(R_F; \mathbb{Z}/2) \rtimes \mathbb{Z}/2 & \text{for } n \equiv 0 \bmod 8, \\ H^1_{\text{ét}}(R_F; \mathbb{Z}/2) & \text{for } n \equiv 1 \bmod 8, \\ H^2_{\text{ét}}(R_F; \mathbb{Z}/2) \rtimes \mathbb{Z}/2 & \text{for } n \equiv 2 \bmod 8, \\ (\mathbb{Z}/2)^{r_1-1} \rtimes H^1_{\text{ét}}(R_F; \mathbb{Z}/2) & \text{for } n \equiv 3 \bmod 8, \\ (\mathbb{Z}/2)^{u-t} \rtimes H^2_{\text{ét}}(R_F; \mathbb{Z}/2) & \text{for } n \equiv 4 \bmod 8, \\ (\mathbb{Z}/2)^{r_1-1} \rtimes \tilde{H}^1_{\text{ét}}(R_F; \mathbb{Z}/2) & \text{for } n \equiv 5 \bmod 8, \\ (\mathbb{Z}/2)^{u-t} \rtimes \tilde{H}^2_{\text{ét}}(R_F; \mathbb{Z}/2) & \text{for } n \equiv 6 \bmod 8, \\ \tilde{H}^1_{\text{ét}}(R_F; \mathbb{Z}/2) & \text{for } n \equiv 7 \bmod 8. \end{cases}$$

By Borel's theorem, the rational rank of $K_n(R_F)$ is known, and the asserted 2-rank formulas follow by inspection. □

We can now prove the main theorem, stated in the introduction.

Proof of Theorem 0.1. By (4.1) we have the inequality

$$\mathrm{rk}_2\, K_4(R_F;\mathbb{Z}_2) \le \mathrm{rk}_2\, \mathrm{im}(\kappa_4) + \mathrm{rk}_2\, H^2_{\text{ét}}(R_F;\mathbb{Z}_2(3))\,.$$

(This is an equality when the short exact sequence (4.1) is split, as will be the case when $K_4(R_F)$ has exponent 2.) Substituting the formulas from 1.2 and 4.2 we obtain $(s+u-1) \le \rho + (s+t-1)$ which we rewrite as $\rho \ge u-t$. Clearly $\rho \le \mathrm{rk}_2\, K_4(R_F;\mathbb{Z}_2) = s+u-1$. □

5. Quadratic number fields

The 2-rank of the Picard group and of the narrow Picard group of a quadratic number field was determined using genus theory by P.A. Østvær in his Master's thesis [Ø].

Let $F = \mathbb{Q}(\sqrt{d})$ with $d > 0$ square free.

Then $u - t = 1$ if (a) there exists a prime $p \mid d$ with $p \equiv 3 \bmod 4$, and (b): $d \equiv 5 \bmod 8$, or $p \equiv 1, 7 \bmod 8$ for every odd prime $p \mid d$, or there exists an odd prime $p \mid d$ with $p \equiv 5, 7 \bmod 8$.

The first few examples are $d = 7$, 14, 15, 21, 23, 30, 31, 35, 39, 42, 46, 47, 53, 55, 58, 61, 62, 65, 69 and 70.

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((Reference list incomplete.))