THE MOTIVIC SEGAL CONJECTURE, LECTURE 1

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Abstract. Motivert av Atiyah og Segals kompletteringsteorem for ekvivariant topologisk K-teori formulerte Graeme Segal en tilsvarende formodning om ekvivariant stabil kohomotopi. Jeg vil minne om hva teoremet og formodningen sier. Tanken er at dette skal være det første av en serie foredrag som sikter mot å gi et bevis for en versjon av Segalformodningen i motivisk homotopiteori. Dette var temaet for Thomas Gregersens PhD-avhandling fra 2012.

1. The Atiyah–Segal completion theorem

Let G be a finite group [[or a compact Lie group]]. For each finite G-CW complex X [[or compact Hausdorff G-space]], let the commutative ring $KU^0_G(X)$ be the additive group completion of the commutative semiring [[rig]] of isomorphism classes of G-equivariant complex vector bundles $E \to X$, with the sum and product given by Whitney sum and tensor product, respectively. When $X = *$ such vector bundles are the same as complex G-representations, and $KU^0_G(*) = R(G)$ is the complex representation ring. For instance, when $G = C_p$ is the group of p-th roots of unity, $R(G) \cong \mathbb{Z}[\lambda]/(\lambda^p = 1)$, where λ denotes the class of the standard representation $\mathbb{C}(1)$ of rank one.

In general, $KU^0_G(X)$ is naturally an $R(G)$-module, and can be completed at the augmentation ideal $I(G) = \ker(\dim\colon R(G) \to \mathbb{Z})$ of $R(G)$. When G is a p-group, $I(G)$-adic completion is closely related to p-completion. For instance, when $G = C_p$, $I(G) = (\lambda - 1)$ satisfies $I(G)^p \subset (p)$ and $(p)I(G) \subset I(G)^2$, so $I(G)^\wedge_{I(G)} \cong I(G)^\wedge_p$ and $R(G)^\wedge_{I(G)} \cong \mathbb{Z} \oplus I(G)^\wedge_p$.

If G acts freely on X, there is a natural isomorphism $KU^0(X/G) \cong KU^0_G(X)$, taking the class of a complex vector bundle $F \to X/G$ to the class of its pullback $E = \pi^* F \to X$, where $\pi\colon X \to X/G$ is the canonical projection. This isomorphism reflects the fact that the equivariant K-theory spectrum KU_G is split.

For [[more]] general G-CW complexes X let $KU^0_G(X) = \lim_\alpha KU^0_G(X_\alpha)$, where $\{X_\alpha\}_\alpha$ ranges over the finite subcomplexes of X. [[This limit agrees with the limit over skeleta if X has finite type. The definition is only useful when the derived limits vanish. Completion helps.]] The collapse map $c\colon EG \to *$ induces a homomorphism $\gamma = c^*\colon KU^0_G(X) \to KU^0_G(EG \times X)$. The $R(G)$-module $KU^0_G(EG \times X)$ is already $I(G)$-complete, so $KU^0_G(EG \times X) \cong KU^0_G(EG \times X)^\wedge_{I(G)}$.

Note that $EG \times X$ is G-free. The orbit complex $(EG \times X)/G \cong EG \times_G X$ is the Borel construction, also known as the homotopy orbit space, of the G-space X. By the natural isomorphism above, and a passage to limits, $KU^0_G(EG \times X) \cong KU^0(EG \times_G X)$.

Theorem 1.1 (Atiyah–Segal)**.** *γ induces an isomorphism $KU^0_G(X)^\wedge_{I(G)} \cong KU^0_G(EG \times X)^\wedge_{I(G)}$. In particular, $R(G)^\wedge_{I(G)} \cong KU^0(BG)$. If G is a p-group, then $KU^0(BG)^\wedge_p \cong R(G)^\wedge_p$.*

Let $\mathscr{U}$ be a complete G-universe, i.e., a real inner product space on which G acts by isometries, and which contains countably infinitely many copies of each irreducible

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complex G-representation. [[To be concrete, one may let $\mathscr{U} = \mathbb{R}[G]^\infty$, with $\mathscr{U}^G \cong \mathbb{R}^\infty$.]] For each complex G-representation $V \subset \mathscr{U}$, let $Gr(V \oplus \mathscr{U})$ be the Grassmann G-space of all finite-dimensional complex subspaces U of $V \oplus \mathscr{U}$, with $g \in G$ mapping U to its image $g(U)$, and with $V \oplus 0$ as a G-invariant base point. If $V \subset W$ there is a base point preserving G-map $Gr(V \oplus \mathscr{U}) \to Gr(W \oplus \mathscr{U})$ taking $U \subset V \oplus \mathscr{U}$ to $(W - V) + U \subset W \oplus \mathscr{U}$, where $W - V$ denotes the orthogonal complement of V in W. Let $KU_G(0) = \operatorname{colim}_V Gr(V \oplus \mathscr{U})$.

By equivariant Bott periodicity there is a G-equivalence $KU_G(0) \simeq \Omega^V KU_G(0)$ for each complex G-representation V. Hence there is a G-spectrum KU_G, with V-th space $KU_G(V) \simeq KU_G(0)$ for each complex G-representation $V \subset \mathscr{U}$. It represents the G-equivariant KU-theory functor above, so that $KU^0_G(X) = \pi_0 F(X_+, KU_G)^G$ and $KU^0_G(EG \times X) = \pi_0 F((EG \times X)_+, KU_G)^G$. The strong form of the Atiyah–Segal completion theorem proves that $\gamma = c^* \colon (KU_G)^G \to F(EG_+, KU_G)^G = (KU_G)^{hG}$ (the homotopy fixed point spectrum) becomes an equivalence after $I(G)$-completion. In particular, when G is a p-group it becomes an equivalence after p-completion.

The fixed point spectrum $(KU_G)^G \simeq \bigvee_{[V]} KU$ splits as a wedge sum of copies of KU, one for each irreducible G-representation. In particular, the forgetful map $F \colon (KU_G)^G \to KU$ admits a section (corresponding to the trivial representation), so KU_G is split, and this implies that the homotopy fixed point spectrum $(KU_G)^{hG}$ is equivalent to the function spectrum $F(BG_+, KU)$. Hence there is an Atiyah–Hirzebruch spectral sequence connecting the group cohomology $H^*(BG; \mathbb{Z})$ of G to the completed representation ring $R(G)^\wedge_{I(G)}$. In the case $G = C_p$ the spectral sequence collapses, and $H^*(BC_p; \mathbb{Z}) = \mathbb{Z}[y]/(py)$ is the associated graded of the $I(G)$-adic filtration of $R(G)^\wedge_{I(G)}$, with $I(G)^n / I(G)^{n+1} \cong H^{2n}(BC_p; \mathbb{Z})$ for each $n \geq 0$.

2. Graeme Segal's Burnside ring conjecture

For each finite G-CW complex X, let $\pi^0_G(X_+) = \operatorname{colim}_V [X_+ \wedge S^V, S^V]^G$, where V ranges over the finite dimensional G-representations in $\mathscr{U}$, partially ordered by inclusion. Here $S^V = V \cup \{\infty\}$ is the one-point compactification of V, known as the V-th representation sphere. When $X = *$, so that $X_+ = S^0$, $\pi^0_G(S^0) = \operatorname{colim}_V [S^V, S^V]^G$ is isomorphic to the Burnside ring $A(G)$. The is the additive group completion of the commutative semiring of isomorphism classes of finite G-sets, with sum and product given by disjoint union and Cartesian product, respectively. For instance, when $G = C_p$, $A(C_p) \cong \mathbb{Z}[x]/(x^2 = px)$, where x is the class of the finite G-set G/e. [[Explain ring structure.]]

The identification of $\pi^0_G(S^0)$ with $A(G)$ is a consequence of the Segal–tom Dieck splitting. For example, if $G = C_p$ is of prime order, there is a homotopy fiber sequence

$$\operatorname{Map}(S^V / S^{V^G}, S^V)^G \longrightarrow \operatorname{Map}(S^V, S^V)^G \xrightarrow{R} \operatorname{Map}(S^{V^G}, S^{V^G}),$$

where R takes a G-map $f \colon S^V \to S^V$ to its restriction $f^G \colon S^{V^G} \to S^{V^G}$, and a chain of homotopy equivalences $\operatorname{Map}(S^V / S^{V^G}, S^V)^G \simeq \operatorname{Map}(S^V / S^{V^G}, EG_+ \wedge S^V)^G \simeq \operatorname{Map}(S^V, EG_+ \wedge S^V)^G$. Passing to colimits over $V \subset \mathscr{U}$, we get a homotopy fiber sequence $Q_G(EG_+)^G \to Q_G(S^0)^G \to Q(S^0)$, where $Q_G(X_+) = \operatorname{colim}_V \operatorname{Map}(S^V, X_+ \wedge S^V)$. There is an Adams transfer equivalence $\tau \colon Q(BG_+) \simeq Q_G(EG_+)^G$, leading to a homotopy fiber sequence

$$Q(BG_+) \xrightarrow{N} Q_G(S^0)^G \xrightarrow{R} Q(S^0),$$

which is split by the section $S \colon Q(S^0) \to Q_G(S^0)^G$ induced by the inclusion of representations in $\mathscr{U}^G \cong \mathbb{R}^\infty$ among the representations in $\mathscr{U}$. Hence

$$Q_G(S^0)^G \simeq Q(S^0) \times Q(BG_+).$$

The generators of $\pi_0 Q(BG_+) = \pi_0^S(BG_+) \cong \mathbb{Z}$ and $\pi_0 Q(S^0) = \pi_0^S(S^0) \cong \mathbb{Z}$ correspond to the free G-orbit $x = [G/e]$ and the trivial G-orbit $1 = [G/G]$ in $\pi_0 Q_G(S^0)^G = \pi_G^0(S^0) \cong A(G)$, respectively.

[[Might also use forgetful map $F\colon \mathrm{Map}(S^V, S^V)^G \to \mathrm{Map}(S^V, S^V)$, and detect $A(G)$ inside the ring of "class functions". Alternatively, realize $\pi_*^G(S^0) = \pi_*((S_G)^G)$ as the algebraic K-theory of the category of finite G-sets and G-equivariant bijections.]]

In general, $\pi_G^0(X_+)$ is naturally an $A(G)$-module, and can be completed at the augmentation ideal $J(G) = \ker(\#\colon A(G) \to \mathbb{Z})$ of $A(G)$. When G is a p-group, $J(G)$-adic completion is closely related to p-completion, with $A(G)^\wedge_{J(G)} \cong \mathbb{Z} \oplus J(G)^\wedge_p$.

If G acts freely on X, there is a natural isomorphism $\pi_S^0((X/G)_+) \cong \pi_G^0(X_+)$. This is not as obvious as for topological K-theory, but can be proved as a consequence of the fact that the equivariant sphere spectrum S_G is split.

For [[more]] general G-CW complexes X we provisionally let $\pi_G^0(X_+) = \lim_\alpha \pi_G^0(X_{\alpha+})$, where $\{X_\alpha\}_\alpha$ ranges over the finite subcomplexes of X. [[This limit agrees with the limit over skeleta if X has finite type. The definition is only useful when the derived limits vanish.]] The collapse map $c\colon EG \to *$ induces a homomorphism $\gamma = c^*\colon \pi_G^0(X_+) \to \pi_G^0((EG \times X)_+)$. The $A(G)$-module $\pi_G^0((EG \times X)_+)$ is $J(G)$-complete, so $\pi_G^0((EG \times X)_+) \cong \pi_G^0((EG \times X)_+)^\wedge_{J(G)}$. Since G acts freely on $EG \times X$ we also have a natural isomorphism $\pi_S^0((EG \times_G X)_+) \cong \pi_G^0((EG \times X)_+)$.

Conjecture 2.1 (Segal)**.** *γ induces an isomorphism $\pi_G^0(X_+)^\wedge_{J(G)} \cong \pi_G^0((EG \times X)_+)^\wedge_{J(G)}$. In particular, $A(G)^\wedge_{J(G)} \cong \pi_S^0(BG_+)$. If G is a p-group then $A(G)^\wedge_p \cong \pi^0(BG_+)^\wedge_p$.*

The sphere G-prespectrum $\{V \mapsto S^V\}$ is stably equivalent to the sphere G-spectrum $S_G = \{V \mapsto Q_G(S^V)\}$. It represents G-equivariant stable cohomotopy, so that $\pi_G^0(X_+) = \pi_0 F(X_+, S_G)^G$ and $\pi_G^0((EG \times X)_+) = \pi_0 F((EG \times X)_+, S_G)^G$. To prove the Segal conjecture, it suffices to prove the stronger assertion that $\gamma = c^*\colon (S_G)^G \to F(EG_+, S_G)^G = (S_G)^{hG}$ becomes an equivalence after $J(G)$-adic completion. In particular, for G a p-group it suffices to show that $\gamma\colon (S_G)^G \to (S_G)^{hG}$ becomes an equivalence after p-completion.

By the theorem of Segal and tom Dieck, the fixed point spectrum $(S_G)^G \simeq \bigvee_{(H)} \Sigma^\infty BW_G H_+$ splits as a wedge sum of suspension spectra, indexed over the conjugacy classes of subgroups of G. Here $W_G H = N_G H/H$ is the Weyl group of H in G, where $N_G H$ is the normalizer of H in G. For instance, when $G = C_p$ we have the spectrum level version $(S_G)^G \simeq S \vee \Sigma^\infty BC_{p+}$ of the infinite loop space splitting discussed above. In general the forgetful map $F\colon (S_G)^G \to S$ admits a section (corresponding to the case $H = G$), so S_G is split. It follows that $(S_G)^{hG} = F(EG_+, S_G)^G \simeq F(BG_+, S) = D(BG_+)$ is the functional dual of BG_+, so the (strong) Segal conjecture can be reformulated in non-equivariant terms as asserting that there there is a $J(G)$-adic equivalence $\bigvee_{(H)} \Sigma^\infty BW_G H_+ \simeq D(BG_+)$.

In this case the (second quadrant) Atiyah–Hirzebruch spectral sequence

$$E^2_{s,t} = H^{-s}(BG; \pi_t(S)) \Longrightarrow \pi_{s+t}(S_G)^{hG} \cong \pi_{s+t} D(BG_+)$$

does not collapse, and it is not completely known how it evolves to converge, after $J(G)$-adic completion, to a connective abutment isomorphic to $\pi_*((S_G)^G) \cong \bigoplus_{(H)} \pi_*^S(BW_G H_+)$.

[[Read Atiyah–Segal, Adams, Carlsson, May, ...]]

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