THE MOTIVIC SEGAL CONJECTURE, LECTURE 2

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Abstract. Jeg vil vise hvordan Segalformodningen kan omformuleres, ved hjelp av Tatekonstruksjonen, norm-restriksjonssekvensene, Warwick dualitet og Adams' transferekvivalens, til en form som lettere lar seg bevise v.h.a. den algebraiske Singerkonstruksjonen. Dette er det andre i en serie foredrag som sikter mot å gi et bevis for en versjon av Segalformodningen i motivisk homotopiteori.

1. Stable equivariant homotopy and cohomotopy

Let G be a finite group, and let X and Y be finite based G-CW complexes. Let

$$\{X, Y\}^G = \operatorname*{colim}_V \, [X \wedge S^V, Y \wedge S^V]^G$$

be the abelian group of stable homotopy classes of G-maps $X \to Y$. Here V ranges over the finite dimensional G-representations contained in a fixed complete G-universe $\mathscr{U}$. The stable G-equivariant homotopy category of G-spectra (indexed on $\mathscr{U}$) is constructed so that

$$\{X, Y\}^G = [\Sigma^\infty_G X, \Sigma^\infty_G Y]^G \,,$$

where $\Sigma^\infty_G X$ denotes the suspension G-spectrum on X, and the right hand side is the set of morphisms from $\Sigma^\infty_G X$ to $\Sigma^\infty_G Y$ in the stable G-equivariant homotopy category. In particular, the sphere G-spectrum is $S_G = \Sigma^\infty_G S^0$.

When discussing the Segal conjecture, we considered the *stable G-equivariant homotopy groups*

$$\pi^G_n(Y) = \{S^n, Y\}^G \quad \text{and} \quad \pi^G_{-n}(Y) = \{S^0, Y \wedge S^n\}^G \,,$$

and the *stable G-equivariant cohomotopy groups*

$$\pi^n_G(X) = \{X, S^n\}^G \quad \text{and} \quad \pi^{-n}_G(X) = \{X \wedge S^n, S^0\}^G \,,$$

for $n \ge 0$. In particular,

$$\pi^G_0(S^0) = \pi^0_G(S^0) = \{S^0, S^0\}^G = [S_G, S_G]^G$$

is the Burnside ring.

2. The Wirthmüller equivalence

Let $\iota \colon H \to G$ be the inclusion of a subgroup. The restriction functor ι^* from based G-spaces to based H-spaces has a left adjoint

$$\iota_* X = G_+ \wedge_H X$$

and a right adjoint $\iota_! X = \operatorname{Map}(G_+, X)^H$. After stabilizing, the restriction functor ι^* from G-spectra to H-spectra also has a left and a right adjoint, and the level of stable homotopy categories these are naturally isomorphic.

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Theorem 2.1 (Wirthmüller [?Ada84, Thm. 5.1, 5.2]). *Suppose that X is a finite based H-CW complex and Y is a based finite G-CW complex.*

(a) There is a natural isomorphism

$$\{G_+ \wedge_H X, Y\}^G \cong \{X, \iota^* Y\}^H .$$

(b) There is a natural isomorphism

$$\{\iota^* Y, X\}^H \cong \{Y, G_+ \wedge_H X\}^G .$$

The natural isomorphism between these left and right adjoints can be promoted to a stable equivalence of G-spectra, following Lewis–May–Steinberger [?LMS86, §II.6]. The restriction functor ι^* from G-spectra to H-spectra has a left adjoint $G \ltimes_H (-)$ and a right adjoint $F_H[G, -)$.

Theorem 2.2 ([?LMS86, Thm. II.6.2]). *For H-spectra D, there is a natural equivalence of G-spectra $\omega\colon F_H[G, D) \xrightarrow{\simeq} G \ltimes_H D$.*

We shall say something about the proofs in lecture 3.

3. The Adams transfer equivalence

Let $\rho\colon G \to G/N$ be the projection to a quotient group. The restriction functor ρ^* from based G/N-spaces to based G-spaces has the left adjoint $\rho_* X = X/N$ given by the orbit space, and the right adjoint $\rho_! X = X^N$ given by the fixed point subspace.

Theorem 3.1 (Adams [?Ada84, Thm. 5.3, 5.4]). *Suppose that X is a finite based G-CW complex in which N acts freely away from the base point, and let Y be a finite based G/N-CW complex.*

(a) There is a natural isomorphism

$$\{X/N, Y\}^{G/N} \cong \{X, \rho^* Y\}^G .$$

(b) There is a natural isomorphism

$$\{\rho^* Y, X\}^G \cong \{Y, X/N\}^{G/N} .$$

[[The pullback $\rho^* Y$ is finite based G-CW complex with trivial N-action, and the orbit space X/N is a finite based G/N-CW complex. Comment on the relation between the unstable and stable results.]]

The Adams isomorphism can also be promoted to a stable equivalence of G/N-spectra, following [?LMS86, §II.7]. Let $\mathscr{U}$ be a complete G-universe, and let $\mathscr{U}^N$ be the N-fixed subuniverse. It can also be viewed as a complete G/N-universe.

To each G-spectrum D indexed on $\mathscr{U}^N$, we can associate the orbit prespectrum

$$\{W \mapsto D(W)/N\}$$

for $W \subset \mathscr{U}^N$. The *orbit spectrum* D/N is the G/N-spectrum on $\mathscr{U}^N$ obtained from this prespectrum by spectrification. We can also associate to D the *fixed point spectrum* D^N, with

$$(D^N)(W) = D(W)^N$$

for $W \subset \mathscr{U}^N$. This is already a G/N-spectrum, indexed on $\mathscr{U}^N$.

Let $i\colon \mathscr{U}^N \to \mathscr{U}$ be the inclusion of universes. The restriction functor i^* from G-spectra on $\mathscr{U}$ to G-spectra on $\mathscr{U}^N$ has a left adjoint, i_*. There is a prespectrum

$$\{V \mapsto D(W) \wedge S^{V-W}\}$$

for $V \subset \mathscr{U}$, where $W = V^N$ is the maximal subrepresentation contained in $\mathscr{U}^N$, and $i_* D$ is the G-spectrum on $\mathscr{U}$ obtained from this prespectrum by spectrification.

It is obtained from D by building in deloopings with respect to the representations in $\mathscr{U}$, in addition to those already in $\mathscr{U}^N$. For example, if $N = G$ and $D = \Sigma^\infty X$ is the (naive) suspension spectrum of a based G-space X, indexed on the G-trivial universe $\mathscr{U}^G$, then $i_*D = \Sigma^\infty_G X$ is the genuine suspension spectrum of X, indexed on the complete G-universe $\mathscr{U}$. In particular, $i_*(S) = S_G$.

The fixed point functor $(-)^N$ is right adjoint to the functor $i_*\rho^*$ from G/N-spectra on $\mathscr{U}^N$ to G-spectra on $\mathscr{U}$.

Theorem 3.2 ([?LMS86, Thm. 7.1]). *Let D be an N-free G-spectrum indexed on $\mathscr{U}^N$. There is a natural map*

$$\tau\colon i_*\rho^*(D/N) \longrightarrow i_*D$$

(in the stable homotopy category) of G-spectra on $\mathscr{U}$, and the right adjoint

$$\tilde{\tau}\colon D/N \longrightarrow (i_*D)^N$$

of τ is a stable equivalence of G/N-spectra indexed on $\mathscr{U}^N$.

We shall say something about the proofs in lecture 3.

4. The norm–restriction sequences

Let X be any G-spectrum, for instance the G-equivariant sphere spectrum S_G. The homotopy cofiber sequence $EG_+ \to S^0 \to \tilde{E}G \to \Sigma EG_+$ and the map $X = F(S^0, X) \to F(EG_+, X)$ induce a map of horizontal homotopy cofiber sequences

$$\begin{array}{ccccccc}
[EG_+ \wedge X]^G & \longrightarrow & X^G & \longrightarrow & [\tilde{E}G \wedge X]^G & \longrightarrow & [\Sigma EG_+ \wedge X]^G \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
[EG_+ \wedge F(EG_+, X)]^G & \longrightarrow & F(EG_+, X)^G & \longrightarrow & [\tilde{E}G \wedge F(EG_+, X)]^G & \longrightarrow & [\Sigma EG_+ \wedge F(EG_+, X)]^G
\end{array}$$

Let

$$X_{hG} = EG_+ \wedge_G X \quad , \quad X^{hG} = F(EG_+, X)^G \quad \text{and} \quad X^{tG} = [\tilde{E}G \wedge F(EG_+, X)]^G$$

denote the *homotopy orbit spectrum*, the *homotopy fixed point spectrum* and the *Tate construction*, respectively.

The adjunction counit $i_*i^*X \to X$ is a G-map and a stable equivalence, hence induces a G-equivalence

$$i_*(EG_+ \wedge i^*X) \cong EG_+ \wedge i_*i^*X \xrightarrow{\simeq_G} EG_+ \wedge X\,.$$

Since G acts freely on EG, $D = EG_+ \wedge i^*X$ is a G-free G(-CW) spectrum, and there is a natural Adams transfer equivalence

$$\tau\colon (EG_+ \wedge i^*X)/G \xrightarrow{\simeq} (i_*(EG_+ \wedge i^*X))^G \simeq [EG_+ \wedge X]^G$$

in the stable homotopy category.

The G-map and equivalence $X \to F(EG_+, X)$ induces an equivalence $X_{hG} \simeq F(EG_+, X)_{hG}$, hence the left hand vertical map $[EG_+ \wedge X]^G \to [EG_+ \wedge F(EG_+, X)]^G$ is an equivalence.

[[Explain the geometric fixed point equivalence $[\tilde{E}G \wedge X]^G \simeq \Phi^G(X)$ and the (cyclotomic) equivalence $\Phi^G(S_G) \simeq S$.]]

We obtain the following *norm–restriction* homotopy cofiber sequences:

$$\begin{array}{ccccccc}
X_{hG} & \xrightarrow{N} & X^G & \xrightarrow{R} & \Phi^G(X) & \xrightarrow{\partial} & \Sigma X_{hG} \\
\| & & \downarrow{\scriptstyle \gamma} & & \downarrow{\scriptstyle \hat{\gamma}} & & \| \\
X_{hG} & \xrightarrow{N^h} & X^{hG} & \xrightarrow{R^h} & X^{tG} & \xrightarrow{\partial^h} & \Sigma X_{hG}
\end{array}$$

The Segal conjecture for X, that $\gamma\colon X^G \to X^{hG}$ is an equivalence (after $J(G)$-adic/p-adic completion), is therefore equivalent to the assertion that $\hat{\gamma}\colon \Phi^G(X) \to X^{tG}$ is an equivalence (after $J(G)$-adic/p-adic completion).

In the particular case of $X = S_G$ and $G = C_p$, when $\Phi^{C_p}(S_G) \simeq S$, the diagram appears as follows:

$$\begin{array}{ccccccc}
BG_+ & \xrightarrow{N} & (S_G)^G & \xrightarrow{R} & S & \xrightarrow{\partial} & \Sigma BG_+ \\
\| & & \downarrow{\scriptstyle\gamma} & & \downarrow{\scriptstyle\hat{\gamma}} & & \| \\
BG_+ & \xrightarrow{N^h} & S^{hG} & \xrightarrow{R^h} & S^{tG} & \xrightarrow{\partial^h} & \Sigma BG_+
\end{array}$$

where $BG_+ = \Sigma^\infty BG_+ = S_{hG}$ (and $S^{hG} \simeq F(BG_+, S) = D(BG_+)$). To prove that $\gamma\colon S^G \to S^{hG}$ is a p-adic equivalence, we can instead prove that $\hat{\gamma}\colon S \to S^{tG}$ is a p-adic equivalence.

5. The Tate tower

We can view the Tate construction X^{tG} as the homotopy limit of a tower of spectra

$$X^{tG} \longrightarrow \dots \longrightarrow X^{tG}(k) \longrightarrow \dots \longrightarrow X^{tG}(1) \longrightarrow X^{tG}(0) \simeq \Sigma X_{hG}$$

factoring the boundary map $\partial^h\colon X^{tG} \to \Sigma X_{hG}$ in the lower norm-restriction sequence. Granting this, our aim is to prove that

$$\hat{\gamma}\colon \Phi^G(X) \longrightarrow \operatorname*{holim}_k X^{tG}(k)$$

is a p-adic equivalence.

The *Tate tower* can be defined for each finite group G, but there are certain simplifications in a class of examples that includes the groups C_p of prime order, so we only discuss this case here. More specifically, we assume that that there exists a G-representation V with the property that G acts freely on the unit sphere $S(V)$. These are the *spherical space form groups*. [[They have periodic cohomology, and have been classified, starting with Milnor.]] When $G = C_2$ the minimal example is $V = \mathbb{R}(1)$ with the sign action; when $G = C_p$ with p odd we may take $V = \mathbb{C}(1)$ with the standard action.

In this case we may let $EG = S(\infty V)$ be the unit sphere in $\infty V = \bigoplus^\infty V$, and identify the mapping cone $\tilde{E}G$ of $c\colon EG_+ \to S^0$ with the one-point compactification $S^{\infty V} = \operatorname{colim}_k S^{kV}$. We get an increasing filtration

$$S^0 \subset S^V \subset \dots \subset S^{kV} \subset \dots \subset S^{\infty V} = \tilde{E}G$$

of $\tilde{E}G$, in the category of based G-spaces. If we apply the suspension spectrum functor to G-spectra, then we can extend this filtration to the left as follows:

$$* \longrightarrow \dots \longrightarrow S^{-kV} \longrightarrow \dots \longrightarrow S^{-V} \longrightarrow S^0 \longrightarrow S^V \longrightarrow \dots \longrightarrow S^{kV} \longrightarrow \dots \longrightarrow \tilde{E}G\,.$$

Here S^{kV} is notation for $\Sigma^\infty_G S^{kV}$ when $k \ge 0$, and S^{-kV} is the kV-th desuspension of S_G. This is the spectrification of the G-prespectrum with W-th space equal to S^{W-kV} if $kV \subseteq W$, and $*$ otherwise.

Definition 5.1. Let

$$X^{tG}(k) = [\tilde{E}G/S^{-kV} \wedge F(EG_+, X)]^G$$

for $k \ge 0$. The sequence of G-spectra

$$\tilde{E}G \longrightarrow \dots \longrightarrow \tilde{E}G/S^{-kV} \longrightarrow \dots \longrightarrow \tilde{E}G/S^{-V} \longrightarrow \tilde{E}G/S^0 = \Sigma EG_+$$

induces the Tate tower of spectra

$$X^{tG} \longrightarrow \dots \longrightarrow X^{tG}(k) \longrightarrow \dots \longrightarrow X^{tG}(1) \longrightarrow X^{tG}(0) \simeq \Sigma X_{hG}\,.$$

Lemma 5.2. *(a)* $X^{tG}(0) \simeq \Sigma X_{hG}$*, (b)* $X^{tG}(k) \simeq \Sigma(S^{-kV} \wedge X)_{hG}$*, and (c)* $X^{tG} \simeq \operatorname{holim}_k X^{tG}(k)$*.*

Proof. (a)

$$X^{tG}(0) = [\tilde{E}G/S^0 \wedge F(EG_+, X)]^G = [\Sigma EG_+ \wedge F(EG_+, X)]^G \simeq [\Sigma EG_+ \wedge X]^G ,$$

since G acts freely on ΣEG_+ and $X \to F(EG_+, X)$ is a G-map and an equivalence. By the Adams transfer equivalence, the latter is homotopy equivalent to $\Sigma(EG_+ \wedge_G X) = \Sigma X_{hG}$.

(b) Note that $\tilde{E}G/S^{-kV} \simeq (\tilde{E}G/S^0) \wedge S^{-kV} = \Sigma EG_+ \wedge S^{-kV}$. This follows from

$$(S^{\infty V}/S^0) \wedge S^{-kV} \cong (S^{\infty V} \wedge S^{-kV})/(S^0 \wedge S^{-kV}) \simeq S^{\infty V}/S^{-kV} .$$

Here we use that $S^{\infty V} \cong S^{\infty V + kV} \cong S^{\infty V} \wedge S^{kV}$, and that $S^{kV} \wedge S^{-kV} \simeq S$. Hence

$$X^{tG}(k) = [\tilde{E}G/S^{-kV} \wedge F(EG_+, X)]^G \simeq [\Sigma EG_+ \wedge S^{-kV} \wedge F(EG_+, X)]^G .$$

Like in part (a) we can rewrite this as $[\Sigma EG_+ \wedge S^{-kV} \wedge X]^G \simeq \Sigma(S^{-kV} \wedge X)_{hG}$.

(c) By considering the G-homotopy cofiber sequences $S^{-kV} \to \tilde{E}G \to \tilde{E}G/S^{-kV}$, it suffices to prove that

$$\operatorname*{holim}_k [S^{-kV} \wedge F(EG_+, X)]^G \simeq * .$$

By G-equivariant Spanier–Whitehead duality for the finite G-CW spectrum S^{kV} there is a natural G-equivalence

$$S^{-kV} \wedge F(EG_+, X) \simeq_G F(S^{kV} \wedge EG_+, X) ,$$

so we can rewrite the homotopy limit above as

$$\operatorname*{holim}_k F(S^{kV} \wedge EG_+, X)^G \simeq F(S^{\infty V} \wedge EG_+, X)^G .$$

Here $S^{\infty V} \wedge EG_+ \simeq \tilde{E}G \wedge EG_+$ is the mapping cone of the G-equivalence $c \wedge 1 \colon EG_+ \wedge EG_+ \to S^0 \wedge EG_+$, hence is G-equivariantly contractible. Thus the function spectrum above is G-equivariantly contractible, and the G-fixed point spectrum is also contractible. □

In particular, when $X = S_G$ we have

$$S^{tG}(k) \simeq \Sigma EG_+ \wedge_G S^{-kV} = \Sigma(S^{-kV})_{hG} .$$

For example

$$S^{tC_2}(k) \simeq \Sigma(S^{-k\mathbb{R}(1)})_{hC_2}$$

interpolating between S^{tC_2} and $\Sigma BC_{2+} = \Sigma \mathbb{R}P^\infty_+$, and

$$S^{tC_p}(k) \simeq \Sigma(S^{-k\mathbb{C}(1)})_{hC_p}$$

interpolating between S^{tC_p} and $\Sigma BC_{p+} = \Sigma L^\infty_+$ for p odd, where $L^\infty = S^\infty/C_p$ denotes the infinite dimensional lens space.

6. Thom spaces and Thom spectra

Let V be any G-representation, of dimension d. The *Thom space* of the vector bundle $\xi \colon EG \times_G V \to BG$ is the quotient space

$$Th(\xi) = \frac{D(\xi)}{S(\xi)} \cong \frac{EG \times_G D(V)}{EG \times_G S(V)} \cong EG_+ \wedge_G S^V = (S^V)_{hG} .$$

When $V = 0$ this is BG_+, and when $V = \mathbb{R}^d$ is the trivial representation, it is the d-fold suspension $BG_+ \wedge S^d = \Sigma^d(BG_+)$. In general one often writes

$$BG^V = EG_+ \wedge_G S^V$$

for the Thom space above, and thinks of it as a (twisted) V-fold suspension of BG_+.

If each element $g \in G$ acts on V by an orientation-preserving map, or if we work with cohomology with $\mathbb{F}_2$-coefficients, there is an *orientation class* $U_\xi \in \tilde{H}^d(Th(\xi)) = \tilde{H}^d(BG^V)$, and a *Thom isomorphism*

$$\Phi_\xi = (-) \cup U_\xi \colon H^*(BG) \xrightarrow{\cong} \tilde{H}^{*+d}(BG^V) .$$

We typically use this to identify the cohomology of any Thom space with the cohomology of the base, up to a degree shift.

The inclusion $0 \subset V$ induces a based G-map $S^0 \to S^V$ and a based map $s\colon BG_+ = BG^0 \to BG^V$, which we can identify with the inclusion of the 0-section $BG \to D(\xi)$ in ξ, followed with the collapse map to $Th(\xi) = BG^V$. Let the pullback of the orientation class along this section,

$$e_V = s^*(U_\xi) \in \tilde{H}^d(BG_+) = H^d(BG) ,$$

be the *Euler class* of ξ. [[The corresponding construction for the tangent bundle $TM \to M$ of a closed manifold yields a cohomology class whose value on the fundamental class of the manifold equals the Euler characteristic $\chi(M)$, hence the name.]] The composite homomorphism

$$H^*(BG) \xrightarrow{\Phi_\xi} \tilde{H}^{*+d}(BG^V) \xrightarrow{s^*} H^{*+d}(BG)$$

is given by the cup product with e_V, sending x to $x \cup e_V$.

More generally, the inclusion $kV \subset (k+1)V$ also induces a map of Thom spaces $s\colon BG^{kV} \to BG^{(k+1)V}$, and the composite homomorphism

$$H^*(BG) \xrightarrow{\Phi_{(k+1)\xi}} \tilde{H}^{*+(k+1)d}(BG^{(k+1)V}) \xrightarrow{s^*} \tilde{H}^{*+(k+1)d}(BG^{kV}) \xrightarrow{\Phi_{k\xi}^{-1}} H^{*+d}(BG)$$

is also given by multiplication with the Euler class e_V.

Turning from kV to the virtual representation $-kV$, we only have a virtual vector bundle $-k\xi$, and no Thom space $Th(-k\xi)$. However, we can define a *Thom spectrum*

$$Th(-k\xi) = EG_+ \wedge_G S^{-kV} = (S^{-kV})_{hG}$$

as the homotopy orbits of the G-spectrum S^{-kV}, and will also write BG^{-kV} for this spectrum.

Again there is a Thom isomorphism

$$\Phi_{-k\xi}\colon H^*(BG) \xrightarrow{\cong} H^{*-kd}(BG^{-kV})$$

The map of G-spectra $S^{-(k+1)V} \to S^{-kV}$ induces a spectrum map $s\colon BG^{-(k+1)V} \to BG^{-kV}$, and the composite homomorphism

$$H^*(BG) \xrightarrow{\Phi_{-k\xi}} \tilde{H}^{*-kd}(BG^{-kV}) \xrightarrow{s^*} \tilde{H}^{*-kd}(BG^{-(k+1)V}) \xrightarrow{\Phi_{-(k+1)\xi}^{-1}} H^{*+d}(BG)$$

is still given by multiplication with the Euler class e_V.

With this notation, the Tate tower

$$S^{tG} \longrightarrow \dots \longrightarrow S^{tG}(k+1) \longrightarrow S^{tG}(k) \longrightarrow \dots \longrightarrow \Sigma BG_+$$

is stably equivalent to the tower of Thom spectra

$$S^{tG} \longrightarrow \dots \longrightarrow \Sigma BG^{-(k+1)V} \xrightarrow{s} \Sigma BG^{-kV} \longrightarrow \dots \longrightarrow \Sigma BG^0 .$$

The cohomology of the finite part of this tower defines a sequence

$$\dots \longleftarrow \Sigma H^*(BG^{-(k+1)V}) \xleftarrow{s^*} \Sigma H^*(BG^{-kV}) \longleftarrow \dots \longleftarrow \Sigma H^*(BG^0) .$$

Under the Thom isomorphisms Φ_{-kV}, this is isomorphic to the sequence

$$\dots \longleftarrow \Sigma H^{*+(k+1)d}(BG) \overset{(-)\cup e_V}{\longleftarrow} \Sigma H^{*+kd}(BG) \longleftarrow \dots \longleftarrow \Sigma H^*(BG)$$

where each homomorphism is given by multiplication by the Euler class.

Proposition 6.1. *The colimit of this sequence,*

$$\operatorname*{colim}_k H^*(S^{tG}(k)) \cong \Sigma H^*(BG)[e_V^{-1}]\,,$$

is given by localization of $\Sigma H^*(BG)$ away from the Euler class.

Note the cohomology does not usually convert homotopy colimits to colimits, so this colimit is not the cohomology of S^{tG}. We instead call it the *continuous cohomology* of S^{tG} (with respect to its presentation as the homotopy limit of the Tate tower), and write

$$H^*_c(S^{tG}) = \operatorname*{colim}_k H^*(S^{tG}(k))\,.$$

In fact, the Segal conjecture will tell us that $H^*(S^{tG};\mathbb{F}_p) \cong H^*(S;\mathbb{F}_p)$ is just $\mathbb{F}_p$ concentrated in degree 0, which is often very different from $\Sigma H^*(BG)[e_V^{-1}]$.

7. Limits of stunted real projective and lens spaces

Example 7.1. When $G = C_2$ and $V = \mathbb{R}(1)$, the Thom space $BG^{kV} = (\mathbb{R}P^\infty)^{k\xi} = EG_+ \wedge_G S^{k\mathbb{R}(1)}$ is homeomorphic to the *stunted real projective space*

$$\mathbb{R}P^\infty_k = \mathbb{R}P^\infty/\mathbb{R}P^{k-1}$$

with one cell in each dimension $* \geq k$, in addition to the base point. [[See Atiyah's paper on Thom complexes.]] The map $BG^{kV} \to BG^{(k+1)}V$ corresponds to the map $\mathbb{R}P^\infty_k \to \mathbb{R}P^\infty_{k+1}$ that collapses the (bottom) k-cell to a point.

It is therefore suggestive to introduce the notation

$$\mathbb{R}P^\infty_{-k} = (\mathbb{R}P^\infty)^{-k\xi} = EG_+ \wedge_G S^{-k\mathbb{R}(1)}$$

for the Thom spectrum of $-k\xi$. The Tate tower for C_2 then appears as

$$S^{tC_2} \longrightarrow \dots \longrightarrow \Sigma\mathbb{R}P^\infty_{-k-1} \overset{s}{\longrightarrow} \Sigma\mathbb{R}P^\infty_{-k} \longrightarrow \dots \longrightarrow \Sigma\mathbb{R}P^\infty_+\,.$$

One also introduces the notation

$$\mathbb{R}P^\infty_{-\infty} = \operatorname*{holim}_k \mathbb{R}P^\infty_{-k}$$

so that $S^{tC_2} \simeq \Sigma\mathbb{R}P^\infty_{-\infty}$. The Segal conjecture for C_2 is then the assertion that there is a 2-adic equivalence $\hat\gamma\colon S \longrightarrow \Sigma\mathbb{R}P^\infty_{-\infty}$, or that $\mathbb{R}P^\infty_{-\infty} \simeq_2 S^{-1}$. In this form the conjecture was also studied by Mahowald. [[Reference, priority?]]

In cohomology, $H^*(BC_2;\mathbb{F}_2) = H^*(\mathbb{R}P^\infty;\mathbb{F}_2) = \mathbb{F}_2[x]$ is the polynomial algebra on a generator x in degree $|x| = 1$. The (mod 2) Euler class of the tautological line bundle ξ is $e_V = x$, so the localization $H^*(BC_2;\mathbb{F}_2)[e_V^{-1}] = \mathbb{F}_2[x^{\pm 1}]$ is the ring of Laurent polynomials in x. When the coefficient field $\mathbb{F}_2$ is implicitly understood, we often write $P(x) = \mathbb{F}_2[x]$ and $P(x^{\pm 1}) = \mathbb{F}_2[x^{\pm 1}]$ for this polynomial ring and its localization.

Lemma 7.2. $H^*_c(S^{tC_2};\mathbb{F}_2) = \Sigma P(x^{\pm 1})$.

Example 7.3. When $G = C_p$ and $V = \mathbb{C}(1)$, the Thom space $BG^{kV} = (L^\infty)^{k\xi} = EG_+ \wedge_G S^{k\mathbb{C}(1)}$ is homeomorphic to the *stunted lens space*

$$L^\infty_{2k} = L^\infty/L^{2k-1}$$

with one cell in each dimension $* \geq 2k$, in addition to the base point. Here $L^{2k-1} = S^{2k-1}/C_p$ is the $(2k-1)$-dimensional lens space. The map $BG^{kV} \to BG^{(k+1)}V$ corresponds to the map $L^\infty_{2k} \to L^\infty_{2k+2}$ that collapses the $2k$- and $2k+1$-cells.

We introduce the notation

$$L^\infty_{-2k} = (L^\infty)^{-k\xi} = EG_+ \wedge_G S^{-k\mathbb{C}(1)}$$

for the Thom spectrum of $-k\xi$. The Tate tower for C_p then appears as

$$S^{tC_p} \longrightarrow \dots \longrightarrow \Sigma L^\infty_{-2k-2} \xrightarrow{s} \Sigma L^\infty_{-2k} \longrightarrow \dots \longrightarrow \Sigma L^\infty_+ .$$

Let

$$L^\infty_{-\infty} = \operatorname*{holim}_k L^\infty_{-2k}$$

so that $S^{tC_p} \simeq \Sigma L^\infty_{-\infty}$. The Segal conjecture for C_p asserts that there is a p-adic equivalence $\hat{\gamma}\colon S \longrightarrow \Sigma L^\infty_{-\infty}$.

In cohomology, $H^*(BC_p; \mathbb{F}_p) = H^*(L; \mathbb{F}_2) = \mathbb{F}_p[x, y]/(x^2)$ is the exterior algebra on a generator x in degree $|x| = 1$, tensored with the polynomial algebra on a generator y in degree $|y| = 2$. The (mod p) Euler class of the tautological complex line bundle ξ is $e_V = y$, so the localization $H^*(BC_p; \mathbb{F}_p)[e_V^{-1}] = \mathbb{F}_p[x, y^{\pm 1}]/(x^2)$ is the tensor product of the exterior algebra on x and the Laurent polynomial algebra on y.

Lemma 7.4. *$H^*_c(S^{tC_p}; \mathbb{F}_p) = \Sigma E(x) \otimes P(y^{\pm 1})$.*

8. Limits of Adams spectral sequences

The plan is now to analyze $\hat{\gamma}$ using towers of Adams spectral sequences. For each bounded below spectrum Y with $H_*(Y; \mathbb{F}_p)$ of finite type, there is a strongly convergent Adams spectral sequence

$$E_2^{*,*}(Y) = \operatorname{Ext}^{*,*}_{\mathscr{A}}(H^*(Y; \mathbb{F}_p), \mathbb{F}_p) \Longrightarrow \pi_*(Y^\wedge_p) ,$$

where $\mathscr{A}$ denotes the Steenrod algebra of stable mod p cohomology operations. For $Y = \Phi^G(S_G) \simeq S$ this takes the form

$$E_2^{*,*}(S) = \operatorname{Ext}^{*,*}_{\mathscr{A}}(\mathbb{F}_p, \mathbb{F}_p) \Longrightarrow \pi_*(S^\wedge_p) ,$$

since $H^*(S; \mathbb{F}_p) = \mathbb{F}_p$.

Suppose that G acts freely on $S(V)$, as before. For each $k \geq 0$ the Thom spectrum $S^{tG}(k) \simeq \Sigma BG^{-kV}$ is bounded below and of finite type mod p, so there is a strongly convergent spectral sequence

$$E_2^{*,*}(k) = \operatorname{Ext}^{*,*}_{\mathscr{A}}(\Sigma H^*(BG^{-kV}; \mathbb{F}_p), \mathbb{F}_p) \Longrightarrow \pi_* \Sigma (BG^{-kV})^\wedge_p .$$

These form a tower of spectral sequences, and it can be proved that their algebraic limit

$$\begin{aligned} E_2^{*,*}(\infty) = \lim_k E_2^{*,*}(k) &= \lim_k \operatorname{Ext}^{*,*}_{\mathscr{A}}(\Sigma H^*(BG^{-kV}; \mathbb{F}_p), \mathbb{F}_p) \\ &\cong \operatorname{Ext}^{*,*}_{\mathscr{A}}(\operatorname*{colim}_k \Sigma H^*(BG^{-kV}; \mathbb{F}_p), \mathbb{F}_p) \\ &= \operatorname{Ext}^{*,*}_{\mathscr{A}}(H^*_c(S^{tG}; \mathbb{F}_p), \mathbb{F}_p) \end{aligned}$$

is again a spectral sequence converging strongly to

$$\pi_*(S^{tG})^\wedge_p \cong \pi_* \operatorname*{holim}_k \Sigma (BG^{-kV})^\wedge_p .$$

We now have a map of spectral sequences

$$\hat{\gamma}\colon E_2^{*,*}(S) = \operatorname{Ext}^{*,*}_{\mathscr{A}}(\mathbb{F}_p, \mathbb{F}_p) \longrightarrow \operatorname{Ext}^{*,*}_{\mathscr{A}}(H^*_c(S^{tG}; \mathbb{F}_p), \mathbb{F}_p) = E_2^{*,*}(\infty)$$

converging strongly to

$$\hat{\gamma}\colon \pi_*(S_p^\wedge) \longrightarrow \pi_*(S^{tG})_p^\wedge .$$

The map of E_2-terms is induced by an $\mathscr{A}$-module homomorphism

$$\epsilon\colon H_c^*(S^{tG};\mathbb{F}_p) = \Sigma H^*(BG;\mathbb{F}_p)[e_V^{-1}] \longrightarrow \mathbb{F}_p .$$

When $G = C_2$ this is the homomorphism

$$\epsilon\colon \Sigma H_c^*(S^{tC_2};\mathbb{F}_2) = \Sigma P(x^{\pm 1}) \longrightarrow \mathbb{F}_2$$

that maps Σx^{-1} to 1. When $G = C_p$ it is the homomorphism

$$\epsilon\colon \Sigma H_c^*(S^{tC_p};\mathbb{F}_p) = \Sigma E(x) \otimes P(y^{\pm 1}) \longrightarrow \mathbb{F}_p$$

that maps Σxy^{-1} to 1.

The remaining step of the proof of the Segal conjecture for C_2 and C_p is then provided by the following algebraic theorems, because a map of spectral sequences that is an isomorphism at the E_2-term is an isomorphism at all later terms, and a map of strongly convergent spectral sequences that is an isomorphism at the E_∞-term induces an isomorphism of the abutments.

Theorem 8.1 (W.H. Lin). $\epsilon\colon \Sigma P(x^{\pm 1}) \to \mathbb{F}_2$ *induces an isomorphism*

$$\epsilon^*\colon \operatorname{Ext}_{\mathscr{A}}^{*,*}(\mathbb{F}_2, \mathbb{F}_2) \longrightarrow \operatorname{Ext}_{\mathscr{A}}^{*,*}(\Sigma P(x^{\pm 1}), \mathbb{F}_2) .$$

Theorem 8.2 (J. Gunawardena). $\epsilon\colon \Sigma E(x) \otimes P(y^{\pm 1}) \to \mathbb{F}_p$ *induces an isomorphism*

$$\epsilon^*\colon \operatorname{Ext}_{\mathscr{A}}^{*,*}(\mathbb{F}_p, \mathbb{F}_p) \longrightarrow \operatorname{Ext}_{\mathscr{A}}^{*,*}(\Sigma E(x) \otimes P(y^{\pm 1}), \mathbb{F}_p) .$$

We shall discuss these theorems in a later lecture, using the algebraic Singer construction.

[[Add references to Atiyah, Lin, Gunawardena, Greenlees–May.]]

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