THE MOTIVIC SEGAL CONJECTURE, LECTURE 3

JOHN ROGNES

Abstract. Jeg vil se i mer detalj på hvordan Pontryagin–Thom-konstruksjonen brukes til å bevise Wirthmüller- og Adams transfer-ekvivalensene i stabil ekvivariant homotopiteori.

1. The Wirthmüller equivalence, in more detail

Let $\iota\colon H \to G$ be the inclusion of a subgroup. The restriction functor ι^* from based G-spaces to based H-spaces has a left adjoint

$$\iota_* X = G_+ \wedge_H X$$

and a right adjoint $\iota_! X = \mathrm{Map}(G_+, X)^H$. After stabilizing, the restriction functor ι^* from G-spectra to H-spectra also has a left and a right adjoint, and the level of stable homotopy categories these are naturally isomorphic.

Theorem 1.1 (Wirthmüller [?Ada84, Thm. 5.1, 5.2]). *Suppose that X is a finite based H-CW complex and Y is a based finite G-CW complex.*

(a) There is a natural isomorphism

$$\{G_+ \wedge_H X, Y\}^G \cong \{X, \iota^* Y\}^H .$$

(b) There is a natural isomorphism

$$\{\iota^* Y, X\}^H \cong \{Y, G_+ \wedge_H X\}^G .$$

Proof. (a) This adjunction is determined by the unit H-map

$$\eta_X \colon X \longrightarrow \iota^*(G_+ \wedge_H X)$$

that maps x to $1 \wedge x$, and the counit G-map

$$\epsilon_Y \colon G_+ \wedge_H \iota^* Y \longrightarrow Y$$

that maps $g \wedge y$ to $g \cdot y$. These are natural for stable H-maps in X and stable G-maps in Y, respectively, and the composites

$$G_+ \wedge_H X \longrightarrow G_+ \wedge_H \iota^*(G_+ \wedge_H X) \longrightarrow G_+ \wedge_H X$$

and

$$\iota^* Y \longrightarrow \iota^*(G_+ \wedge_H \iota^* Y) \longrightarrow \iota^* Y$$

are the identities, as required to get an adjunction.

(b) This adjunction is determined by a unit G-map

$$\eta_Y \colon Y \longrightarrow G_+ \wedge_H \iota^* Y$$

and a counit H-map

$$\epsilon_X \colon \iota^*(G_+ \wedge_H X) \longrightarrow X .$$

The unit map is only stably defined. Choose a G-equivariant embedding $e\colon G/H \to V$ of G/H into a G-representation $V \subset \mathscr{U}$. An equivariant tubular neighborhood T of $e(G/H)$ is isomorphic to $G/H \times D(V)$, where $D(V) \subset V$ is the unit disc. Then $\partial T \cong G/H \times S(V)$,

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where $S(V) \subset V$ is the unit sphere. The Pontryagin–Thom construction defines a based G-map

$$t\colon S^V \longrightarrow T/\partial T \cong \frac{G/H \times D(V)}{G/H \times S(V)} \cong G/H_+ \wedge S^V .$$

Here the first map extends the collapse map $T \to T/\partial T$ by sending the complement of T in $V \subset S^V$ to the base point. We define $\eta_Y \in \{Y, G_+ \wedge_H \iota^* Y\}^G$ as the composite

$$Y \wedge S^V \xrightarrow{1\wedge t} Y \wedge G/H_+ \wedge S^V \cong (G_+ \wedge_H \iota^* Y) \wedge S^V .$$

The counit map ϵ_X can be defined without stabilizing, taking (g, x) to $g \cdot x$ if $g \in H$, and to the base point otherwise. These can be checked to be natural for stable G-maps in Y and stable H-maps in X, and the composites

$$\iota^* Y \longrightarrow \iota^*(G_+ \wedge_H \iota^* Y) \longrightarrow \iota^* Y$$

and

$$G_+ \wedge_H X \longrightarrow G_+ \wedge_H \iota^*(G_+ \wedge_H X) \longrightarrow G_+ \wedge_H X$$

are both stably equal to the identity. See [?Ada84, pp. 505–510] for the full details. □

The natural isomorphism between these left and right adjoints can be promoted to a stable equivalence of G-spectra, following Lewis–May–Steinberger [?LMS86, §II.6]. The restriction functor ι^* from G-spectra to H-spectra has a left adjoint $G \ltimes_H (-)$ and a right adjoint $F_H[G, -)$. For a G-spectrum E there is a stable map

$$\bar{t}\colon E \cong \Sigma^{-V} E \wedge S^V \xrightarrow{1\wedge t} \Sigma^{-V} E \wedge (G \ltimes_H \iota^* S^V) \cong G \ltimes_H \iota^*(\Sigma^{-V} E \wedge S^V) \cong G \ltimes_H \iota^* E .$$

Apply this to $E = F_H[G, D)$ for an H-spectrum D, and form the composite

$$\omega\colon F_H[G, D) \xrightarrow{\bar{t}} G \ltimes_H \iota^* F_H[G, D) \xrightarrow{1\ltimes\epsilon} G \ltimes_H D .$$

This is the composite $\iota_! \to \iota_* \iota^* \iota_! \to \iota_*$ constructed from $\bar{t}\colon \mathrm{id} \to \iota_* \iota^*$ and $\epsilon\colon \iota^* \iota_! \to \mathrm{id}$.

Theorem 1.2 ([?LMS86, Thm. II.6.2])**.** *For H-spectra D, the map $\omega\colon F_H[G, D) \longrightarrow G \ltimes_H D$ is a natural equivalence of G-spectra.*

2. The Adams transfer equivalence, in more detail

Let $\rho\colon G \to G/N$ be the projection to a quotient group. The restriction functor ρ^* from based G/N-spaces to based G-spaces has the left adjoint $\rho_* X = X/N$ given by the orbit space, and the right adjoint $\rho_! X = X^N$ given by the fixed point subspace.

Theorem 2.1 (Adams [?Ada84, Thm. 5.3, 5.4])**.** *Suppose that X is a finite based G-CW complex in which N acts freely away from the base point, and let Y be a finite based G/N-CW complex.*

(a) There is a natural isomorphism

$$\{X/N, Y\}^{G/N} \cong \{X, \rho^* Y\}^G .$$

(b) There is a natural isomorphism

$$\{\rho^* Y, X\}^G \cong \{Y, X/N\}^{G/N} .$$

[[The pullback $\rho^* Y$ is finite based G-CW complex with trivial N-action, and the orbit space X/N is a finite based G/N-CW complex.]] [[Comment on the relation between the unstable and stable results.]]

Proof. (a) One first establishes a special case of the motto that "N-free G-CW spectra live in the N-trivial universe $\mathscr{U}^N$", see [?LMS86, p. 65]. More precisely, we need that when X is N-free away from the base point, then the colimit

$$\operatorname*{colim}_{W \subset \mathscr{U}^N} [X \wedge S^{\rho^* W}, \rho^* Y \wedge S^{\rho^* W}]^G$$

defined using N-trivial G-representations $\rho^* W$ only, agrees with

$$\{X, \rho^* Y\}^G = \operatorname*{colim}_V [X \wedge S^V, \rho^* Y \wedge S^V]^G ,$$

as previously defined using all G-representations $V \subset \mathscr{U}$. See [?Ada84, Prop. 5.5] for the proof, which relies on the equivariant form of the Freudenthal suspension theorem.

For each G/N-representation W, we have unstable bijections

$$[X/N \wedge S^W, Y \wedge S^W]^{G/N} \cong [X \wedge S^{\rho^* W}, \rho^* Y \wedge S^{\rho^* W}]^G ,$$

since every G-map from $X \wedge S^{\rho^* W}$ to a space with trivial N-action must factor uniquely through the quotient map

$$q \colon X \wedge S^{\rho^* W} \longrightarrow (X \wedge S^{\rho^* W})/N \cong X/N \wedge S^W .$$

(This uses that N acts trivially on W.) Passing to the colimit over $W \subset \mathscr{U}^N$, and using the cited motto, we get the desired natural isomorphism. [[Relate this to the condition that S_G is split?]]

(b) Adams constructs a stable G-map

$$\tau_X \colon \rho^*(X/N) \longrightarrow X ,$$

which we view as the transfer corresponding to $q \colon X \to X/N$. The natural transformation

$$\{Y, X/N\}^{G/N} \longrightarrow \{\rho^* Y, X\}^G$$

then takes $f \colon Y \to X/N$ to $\tau_X \circ \rho^*(f)$. By an induction argument over the G-cells of X, it suffices to verify that this is an isomorphism in the case $X = G/H_+$, where $H \cap N = e$. In that case $X/N = \bar{G}/\bar{H}_+$, where we set $\bar{G} = G/N$ and $\bar{H} = H/H \cap N$ is the isomorphic image of H in $\bar{G}$. The natural transformation

$$\{Y, \bar{G}/\bar{H}_+\}^{\bar{G}} \longrightarrow \{\rho^* Y, G/H_+\}^G$$

can be identified using the Wirthmüller isomorphism (twice) with the evident isomorphism

$$\{\bar{\iota}^* Y, S^0\}^{\bar{H}} \xrightarrow{\cong} \{\iota^* \rho^* Y, S^0\}^H .$$

It remains to construct the transfer map. Adams does this by choosing a G-map $e \colon X \to V$ into a G-representation $V \subset \mathscr{U}$, with the property that

$$(q, e) \colon X \longrightarrow \rho^*(X/N) \times V$$

is a G-equivariant embedding. An equivariant tubular neighborhood T of $(q, e)(X)$ is isomorphic to $X \times D(V)$, except near the N-fixed base point $*$. By shrinking the radius of the neighborhood to 0 near $*$, we instead get $T \cong X \wedge D(V)_+$, with boundary $\partial T \cong X \wedge S(V)_+$. The Pontryagin–Thom construction defines a based G-map

$$\tau_X \colon X/N \wedge S^V \longrightarrow T/\partial T \cong \frac{X \wedge D(V)_+}{X \wedge S(V)_+} \cong X \wedge S^V ,$$

extending the quotient map $T \to T/\partial T$ by sending the complement of T to the base point. The stable homotopy class of τ_X is Adams' transfer map. See [?Ada84, pp. 511–519] for the various consistency checks that are needed. □

The Adams isomorphism can also be promoted to a stable equivalence of G/N-spectra, following [?LMS86, §II.7]. Let $\mathscr{U}$ be a complete G-universe, and let $\mathscr{U}^N$ be the N-fixed subuniverse. It can also be viewed as a complete G/N-universe.

To each G-spectrum D indexed on $\mathscr{U}^N$, we can associate the orbit prespectrum

$$\{W \mapsto D(W)/N\}$$

for $W \subset \mathscr{U}^N$. The *orbit spectrum* D/N is the G/N-spectrum on $\mathscr{U}^N$ obtained from this prespectrum by spectrification. We can also associate to D the *fixed point spectrum* D^N, with

$$(D^N)(W) = D(W)^N$$

for $W \subset \mathscr{U}^N$. This is already a G/N-spectrum, indexed on $\mathscr{U}^N$. The fixed point functor is right adjoint to the functor $i_*\rho^*$ from G/N-spectra on $\mathscr{U}^N$ to G-spectra on $\mathscr{U}$.

Let $i\colon \mathscr{U}^N \to \mathscr{U}$ be the inclusion of universes. The restriction functor i^* from G-spectra on $\mathscr{U}$ to G-spectra on $\mathscr{U}^N$ has a left adjoint, i_*. There is a prespectrum

$$\{V \mapsto D(W) \wedge S^{V-W}\}$$

for $V \subset \mathscr{U}$, where $W = V^N$ is the maximal subrepresentation contained in $\mathscr{U}^N$, and i_*D is the G-spectrum on $\mathscr{U}$ obtained from this prespectrum by spectrification. It is obtained from D by building in deloopings with respect to the representations in $\mathscr{U}$, in addition to those already in $\mathscr{U}^N$.

For example, if $N = G$ and $D = \Sigma^\infty X$ is the (naive) suspension spectrum of a based G-space X, indexed on the G-trivial universe $\mathscr{U}^G$, then $i_*D = \Sigma^\infty_G X$ is the genuine suspension spectrum of X, indexed on the complete G-universe $\mathscr{U}$. In particular, $i_*(S) = S_G$.

Theorem 2.2 ([?LMS86, Thm. 7.1])**.** *Let D be an N-free G-spectrum indexed on $\mathscr{U}^N$. There is a natural map*

$$\tau\colon i_*\rho^*(D/N) \longrightarrow i_*D$$

(in the stable homotopy category) of G-spectra on $\mathscr{U}$, and the right adjoint

$$\tilde{\tau}\colon D/N \longrightarrow (i_*D)^N$$

of τ is a stable equivalence of G/N-spectra indexed on $\mathscr{U}^N$.

To construct τ, let $\Gamma = G \ltimes N$ be the semidirect product, with respect to the conjugation action of G on N. Let $\Pi = e \ltimes N \subset \Gamma$ be a subgroup isomorphic to N. It is the kernel of a homomorphism $\epsilon\colon \Gamma \to G$, mapping $(g;n)$ to g. Let $\theta\colon \Gamma \to G$ be a different homomorphism, mapping $(g;n)$ to gn. The functor $i_*\theta^*$ maps G-spectra on $\mathscr{U}^N$ to Γ-spectra on $\mathscr{U}$, and takes N-free G(-CW?) spectra to Π-free Γ(-CW?) spectra.

Let $\mathscr{U}'$ be a complete Γ-universe, with $(\mathscr{U}')^\Pi = \mathscr{U}$, and let $j\colon \mathscr{U} \to \mathscr{U}'$ denote the inclusion. There is a transitive Γ-action on N, given by $(g;n)\cdot m = gnmg^{-1}$, with stabilizer $G \ltimes e \cong G$ at e. Hence $N \cong \Gamma/G$ as Γ-sets. Choose a Γ-representation $V \subset \mathscr{U}'$ and a Γ-equivariant embedding $e\colon \Gamma/G \to V$. The Pontryagin–Thom construction gives a based Γ-map $S^V \longrightarrow \Gamma/G_+ \wedge S^V$ and a stable *pretransfer* map

$$t\colon S \longrightarrow \Sigma^\infty N_+$$

of Γ-spectra on $\mathscr{U}'$.

For each N-free G-spectrum D form the Π-free Γ-spectrum $i_*\theta^*D$, and extend the universe further along j to form the Γ-spectrum $j_*i_*\theta^*D$ on $\mathscr{U}'$. Smash this spectrum with t, and rearrange, to obtain the map

$$1 \wedge t\colon j_*(i_*\theta^*D) \longrightarrow j_*(i_*\theta^*D \wedge N_+)\,.$$

By the motto above, this is the image under j_* of a unique map

$$\hat{\tau}\colon i_*\theta^* D \longrightarrow i_*\theta^* D \wedge N_+$$

of Γ-spectra on $\mathscr{U} = (\mathscr{U}')^\Pi$. Pass to the orbit spectra over Π to obtain the desired map

$$\tau\colon i_*\rho^*(D/N) \cong (i_*\theta^* D)/\Pi \xrightarrow{\hat{\tau}/\Pi} (i_*\theta^* D \wedge N_+)/\Pi \cong i_* D$$

of G-spectra on $\mathscr{U}$.

To prove that $\tilde{\tau}\colon D/N \to (i_* D)^N$ is a stable G/N-equivalence, it suffices do deal with the case $D = \Sigma^\infty G/H_+$ where $H \cap N = e$. The case of finite N-free G-CW spectra D will then follow by induction and five lemma arguments. The general case follows by a passage to colimits.

In the special case when $D = \Sigma^\infty G/H_+$, the transfer G-map

$$\tau\colon \Sigma^\infty G/HN_+ \longrightarrow \Sigma^\infty G/H_+$$

can be rewritten as

$$1 \ltimes t\colon G \ltimes_{HN} S \longrightarrow G \ltimes_{HN} \Sigma^\infty HN/H_+$$

where t is the Pontryagin–Thom map associated to the group HN/H. The claim that the adjoint map

$$\tilde{\tau}\colon \Sigma^\infty G/HN_+ \longrightarrow (i_*\Sigma^\infty G/H_+)^N$$

is a stable G/N-equivalence now follows from the Wirthmüller isomorphism, after various compatibility checks.

References

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Department of Mathematics, University of Oslo, Norway

E-mail address: rognes@math.uio.no

URL: http://folk.uio.no/rognes