THE MOTIVIC SEGAL CONJECTURE, LECTURE 4

JOHN ROGNES

Abstract. Jeg vil snakke om de endelige underalgebraene $A(n)$ i Steenrod-algebraen, analysere $A(n)$-modulstrukturen til den kontinuerlige kohomologien $\Sigma P(x^{\pm 1})$ til Tate-konstruksjonen, og skissere Lin-Davis-Mahowald-Adams' bevis av Lins teorem.

1. The Steenrod operations

Let p be a prime, let $H = H\mathbb{F}_p$ be the Eilenberg–Mac Lane ring spectrum representing mod p cohomology, and let $\mathscr{A} = H^*(H)$ be the Steenrod algebra of stable mod p cohomology operations. For $p = 2$ the Steenrod reduced squares

$$Sq^i \colon H^*(X) \to H^{*+i}(X)$$

defined for $i \geq 1$ (Steenrod, 1947) are known to generate $\mathscr{A}$ as an algebra (Serre, 1953). For p odd the Bockstein operator

$$\beta \colon H^*(X) \to H^{*+1}(X)$$

(satisfying $\beta^2 = 0$), together with the Steenrod reduced powers

$$P^i \colon H^*(X) \to H^{*+2i(p-1)}(X)$$

defined for $i \geq 1$ (Steenrod, 1953) are known to generate $\mathscr{A}$ as an algebra (Cartan, 1954, 1955). These operations are all natural in the space X. Setting Sq^0 and P^0 equal to the identities, these operations satisfy the Cartan formula

$$Sq^k(x \cup y) = \sum_{i+j=k} Sq^i(x) \cup Sq^j(y)$$

for $p = 2$, and

$$\beta(x \cup y) = \beta(x) \cup y + (-1)^{|x|} x \cup \beta(y)$$
$$P^k(x \cup y) = \sum_{i+j=k} P^i(x) \cup y + x \cup P^j(y)$$

for p odd, with $x, y \in H^*(X)$. Introducing the formal sum $Sq(x) = \sum_{i \geq 0} Sq^i(x)$, we can write the first and last of these as $Sq(x \cup y) = Sq(x) \cup Sq(y)$ and $P(x \cup y) = P(x) \cup P(y)$. We have $Sq^i(x) = x \cup x = x^2$ for $|x| = i$ and $P^i(x) = x \cup \cdots \cup x = x^p$ for $|x| = 2i$. Finally, $Sq^i(x) = 0$ if $|x| < i$ and $P^i(x) = 0$ if $|x| < 2i$.

For $p = 2$ and $X = \mathbb{R}P^\infty = S^\infty/C_2$ we have $H^*(X) = P(x) = \mathbb{F}_2[x]$ with $|x| = 1$, so $Sq^0(x) = x$, $Sq^1(x) = x^2$ and $Sq^i(x) = 0$ for $i \geq 2$. Hence $Sq(x) = x + x^2 = x(1 + x)$. By the Cartan formula $Sq(x^j) = Sq(x)^j = x^j(1 + x)^j = \sum_{i=0}^{j} \binom{j}{i} x^{i+j}$. Hence

$$Sq^i(x^j) = \binom{j}{i} x^{i+j}$$

for $i, j \geq 0$ determines the action of $\mathscr{A}$ on $H^*(\mathbb{R}P^\infty)$. Only the residue class mod 2 of the binomial coefficient matters.

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For p odd and $X = L^\infty = S^\infty/C_p$ we have $H^*(X) = E(x) \otimes P(y) = \mathbb{F}_p[x, y]/(x^2)$ with $|x| = 1$ and $|y| = 2$, where $\beta(x) = y$ and $\beta(y) = 0$. Then $P^0(x) = x$ and $P^i(x) = 0$ for $i \ge 1$. Furthermore, $P^0(y) = y$, $P^1(y) = y^p$ and $P^i(y) = 0$ for $i \ge 2$, so $P(y) = y + y^p = y(1 + y^{p-1})$. Hence $P(y^j) = P(y)^j = y^j(1 + y^{p-1})^j = \sum_{i=0}^{j} \binom{j}{i} y^{j+i(p-1)}$, so that

$$P^i(y^j) = \binom{j}{i} y^{j+i(p-1)}$$

for $i, j \ge 0$, which determines the action of $\mathscr{A}$ on $H^*(L^\infty)$. Only the residue class mod p of the binomial coefficient matters.

2. The Steenrod algebra

The (non-commutative) algebra structure $\phi\colon \mathscr{A} \otimes \mathscr{A} \to \mathscr{A}$ on $\mathscr{A}$ is given by composition of operations. The Steenrod squares satisfy the Adem relations (Adem, 1952)

$$Sq^a Sq^b = \sum_i \binom{b-i-1}{a-2i} Sq^{a+b-i} Sq^i$$

for $a < 2b$, while the Steenrod powers satisfy

$$P^a P^b = \sum_i (-1)^{a+i} \binom{(p-1)(b-i)-1}{a-pi} P^{a+b-i} P^i$$

for $a < pb$ and

$$P^a \beta P^b = \sum_i (-1)^{a+i} \binom{(p-1)(b-i)}{a-pi} \beta P^{a+b-i} P^i + \sum_i (-1)^{a+i+1} \binom{(p-1)(b-i)-1}{a-pi-1} P^{a+b-i} \beta P^i$$

for $a \le pb$. For instance, $Sq^1 Sq^1 = 0$, $Sq^1 Sq^2 = Sq^3$, $Sq^2 Sq^2 = Sq^3 Sq^1$ and $Sq^3 Sq^2 = 0$. These can be used to simplify composites of the form

$$Sq^I := Sq^{i_1} Sq^{i_2} \dots Sq^{i_r}$$

for $i_1, i_2, \dots, i_r \ge 1$, unless $i_s \ge 2i_{s+1}$ for all $1 \le s < r$. In the latter case we say that $I = (i_1, i_2, \dots, i_r)$ is an admissible sequence of length r, and Sq^I is an admissible monomial. For p odd the Adem relations can simplify composites of the form

$$\beta^\epsilon P^I := \beta^{\epsilon_0} P^{i_1} \beta^{\epsilon_1} \dots P^{i_r} \beta^{\epsilon_r}$$

with $\epsilon_0, \epsilon_1, \dots, \epsilon_r \in \{0, 1\}$ and $i_1, i_2, \dots, i_r \ge 1$, unless $i_s \ge \epsilon_s + p i_{s+1}$ for all $1 \le s < r$. In the latter case we say that $I = (\epsilon_0, i_1, \epsilon_1, \dots, i_r, \epsilon_r)$ is an admissible sequence, and that $\beta^\epsilon P^I$ is an admissible monomial. In fact the admissible monomials of all lengths $r \ge 0$ form a basis for $\mathscr{A}$,

$$\mathscr{A} = \mathbb{F}_2\{Sq^I \mid I \text{ admissible}\}$$

for $p = 2$ and

$$\mathscr{A} = \mathbb{F}_p\{\beta^\epsilon P^I \mid (\epsilon, I) \text{ admissible}\}$$

for p odd, so the Adem relations generate all other relations among the Sq^i for $p = 2$, and among β and the P^i for p odd, in the Steenrod algebra $\mathscr{A}$.

The Adem relations imply that the Steenrod operations Sq^i are decomposable except when $i = 2^j$ for some $j \ge 0$, and that the P^i are decomposable except when $i = p^j$ for some $j \ge 0$. Letting $I(\mathscr{A}) = \ker(\mathscr{A} \to \mathbb{F}_p)$ be the augmentation ideal, the quotient

$$Q(\mathscr{A}) = I(\mathscr{A})/I(\mathscr{A})^2 = \mathrm{cok}(\phi\colon I(\mathscr{A}) \otimes I(\mathscr{A}) \to I(\mathscr{A}))$$

is the vector space of algebra indecomposables. We have

$$Q(\mathscr{A}) = \mathbb{F}_2\{Sq^{2^j} \mid j \ge 0\}$$

for $p = 2$, and

$$Q(\mathscr{A}) = \mathbb{F}_p\{\beta, P^{p^j} \mid j \geq 0\}$$

for p odd.

3. The Hopf algebra structure

The Steenrod algebra admits (Milnor, 1958) the structure of a connected, cocommutative Hopf algebra. The coproduct $\psi\colon \mathscr{A} \to \mathscr{A} \otimes \mathscr{A}$ is defined on the algebra generators by

$$\psi(Sq^k) = \sum_{i+j=k} Sq^i \otimes Sq^j$$

for $p = 2$, and by

$$\psi(\beta) = \beta \otimes 1 + 1 \otimes \beta$$
$$\psi(P^k) = \sum_{i+j=k} P^i \otimes P^j$$

for p odd. The conjugation $\chi\colon \mathscr{A} \to \mathscr{A}$ is the involutive anti-homomorphism recursively determined by the relations

$$\sum_{i+j=k} Sq^i \chi(Sq^j) = 0$$

for $k \geq 1$, $\chi(\beta) = -\beta$, and

$$\sum_{i+j=k} P^i \chi(P^j) = 0$$

for $k \geq 1$. For instance, $\chi(Sq^1) = Sq^1$, $\chi(Sq^2) = Sq^2$ and $\chi(Sq^3) = Sq^2 Sq^1$.

By Milnor–Moore (1965, Section 8), any connected Hopf algebra admits a unique conjugation χ such that $\phi(1 \otimes \chi)\psi = \eta\epsilon = \phi(\chi \otimes 1)\psi$, and such that $\chi(xy) = \chi(y)\chi(x)$. If ϕ is commutative, or ψ is cocommutative, then $\chi^2 = 1$.

Letting $J(\mathscr{A}) = \operatorname{cok}(\mathbb{F}_p \to \mathscr{A})$ be the unit/coaugmentation coideal, the kernel

$$P(\mathscr{A}) = \ker(\psi\colon J(\mathscr{A}) \to J(\mathscr{A}) \otimes J(\mathscr{A}))$$

consists of the coalgebra primitives in $\mathscr{A}$, i.e., the elements $x \in \mathscr{A}$ satisfying $\psi(x) = x \otimes 1 + 1 \otimes x$. For $p = 2$ let $Q_0 = Sq^1$ and recursively define $Q_j = [Sq^{2^j}, Q_{j-1}] = Sq^{2^j} Q_{j-1} + Q_{j-1} Sq^{2^j}$ for $j \geq 1$. For instance, $Q_1 = [Sq^2, Sq^1] = Sq^3 + Sq^2 Sq^1$, and $|Q_j| = 2^j - 1$. For p odd let $Q_0 = \beta$ and define $Q_{j+1} = [P^{p^j}, Q_j]$ for $j \geq 0$. When $p = 2$, these Milnor primitives are precisely the coalgebra primitives of $\mathscr{A}$, so that

$$P(\mathscr{A}) = \mathbb{F}_2\{Q_j \mid j \geq 0\}\,.$$

[[Also discuss p odd.]]

4. Finite sub Hopf algebras

First consider $p = 2$. For $n \geq -1$ let $A(n) \subset \mathscr{A}$ be the subalgebra generated by the Steenrod operations Sq^i for $1 \leq i < 2^{n+1}$. Equivalently, $A(n)$ is the subalgebra generated by the Sq^i for $1 \leq i \leq 2^n$, or by the Sq^{2^j} for $0 \leq j \leq n$. Clearly $A(n-1)$ is a subalgebra of $A(n)$ for each $n \geq 0$. For instance, $A(-1) = \mathbb{F}_2$,

$$A(0) = \mathbb{F}_2\{1, Sq^1\}$$

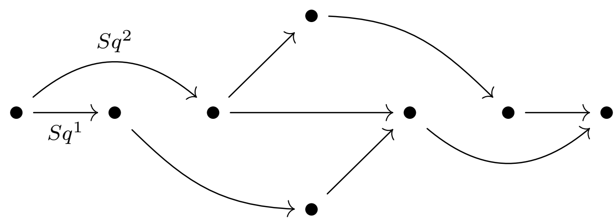
is the exterior algebra on one generator,

$$\bullet \xrightarrow{Sq^1} \bullet$$

and

$$A(1) = \mathbb{F}_2\{1, Sq^1, Sq^2, Sq^1Sq^2, Sq^2Sq^1, Sq^2Sq^2, Sq^2Sq^1Sq^2, Sq^2Sq^2Sq^2\}$$
$$= \mathbb{F}_2\{1, Sq^1, Sq^2, Sq^3, Sq^2Sq^1, Sq^3Sq^1, Sq^5 + Sq^4Sq^1, Sq^5Sq^1\}$$

is a finite algebra of dimension eight.



The next subalgebra, $A(2)$, has dimension 64 and is more difficult to visualize. In general, $A(n)$ is a finite algebra of dimension $2 \cdot 2^2 \cdot \ldots \cdot 2^{n+1} = 2^{\binom{n+2}{2}}$. Since each algebra generator of $\mathscr{A}$ lies in some $A(n)$, we have

$$\mathscr{A} = \bigcup_n A(n) \,.$$

Hence each positive degree element in $\mathscr{A}$ is nilpotent. Clearly the coproduct ψ on $\mathscr{A}$ restricts to a coproduct on $A(n)$, and likewise for the conjugation χ, so each $A(n)$ is a connective, cocommutative sub Hopf algebra of $\mathscr{A}$.

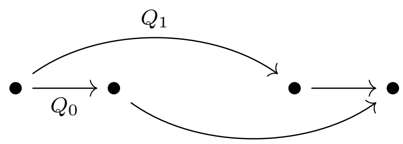
For $n \geq -1$ let $E(n) \subset A(n)$ be the subalgebra generated by the Milnor primitives Q_j for $0 \leq j \leq n$. It turns out that these are commuting exterior generators with no further relations, so that

$$E(n) = E(Q_0, Q_1, \ldots, Q_n)$$

is an exterior algebra on $n+1$ generators. For instance, $E(0) = A(0)$, while

$$E(1) = \mathbb{F}_2\{1, Q_0, Q_1, Q_0Q_1\}$$
$$= \mathbb{F}_2\{1, Sq^1, Sq^3 + Sq^2Sq^1, Sq^3Sq^1\}$$

is of dimension four.



Since each Q_j is coalgebra primitive, this is a primitively generated sub Hopf algebra of $A(n)$. We write

$$E = \bigcup_n E(n) = E(Q_j \mid j \geq 0)$$

for the resulting primitively generated, connected, bicommutative sub Hopf algebra of $\mathscr{A}$.

[[Also discuss p odd.]]

5. Lin and Gunawardena's theorems

In order to make sense of $\mathrm{Ext}^{*,*}_{\mathscr{A}}(H^*_c(S^{tG};\mathbb{F}_p),\mathbb{F}_p)$ for $G = C_p$, $p = 2$ or p odd, we first need to understand the $\mathscr{A}$-module structure of

$$H^*_c(S^{tG};\mathbb{F}_p) = \operatorname*{colim}_k \Sigma H^*(BG^{-kV};\mathbb{F}_p)\,.$$

We previously used to Thom isomorphisms

$$\Phi_{-k\xi}\colon H^{*+kd}(BG;\mathbb{F}_p) \xrightarrow{\cong} H^*(BG^{-kV};\mathbb{F}_p)$$

to get the additive description

$$H^*_c(S^{tG};\mathbb{F}_p) = \Sigma H^*(BG;\mathbb{F}_p)[e_V^{-1}]\,,$$

but the Thom isomorphisms and the homomorphisms given by multiplication by e_V are not in general $\mathscr{A}$-linear, so we need to be more careful to get a description of the left hand side as an $\mathscr{A}$-module. It turns out to be convenient to specify the $\mathscr{A}$-module structure as a sequence of compatible $A(n)$-module structures, for all n, since $\mathscr{A} = \bigcup_n A(n)$.

The Thom isomorphism $\Phi_{-k\xi}$ is $A(n)$-linear if $A(n)$ acts trivially on the orientation class $U_{-k\xi}$, and for a fixed n, this is true whenever k is divisible by a sufficiently high power p^N of p. (When $p = 2$ it suffices that $2^{n+1} \mid k$, because then $Sq(U_{-k\xi}) = Sq(U_\xi)^{-k}$ is an integral power of

$$Sq(U_\xi)^{2^{n+1}} = \sum_{j\ge 0} Sq^j(U_\xi)^{2^{n+1}} = \sum_{j\ge 0} Sq^{j\cdot 2^{n+1}}(U_\xi^{2^{n+1}})\,,$$

so $Sq^i(U_{-k\xi}) = 0$ for $0 < i < 2^{n+1}$. These are the Sq^i that generate $A(n)$.) Likewise, multiplication by e_V^k is $A(n)$-linear when k is divisible by the same sufficiently high power of p. Thus the description of $H^*_c(S^{tG};\mathbb{F}_p)$ as

$$\operatorname*{colim}_{p^N \mid k} \Sigma H^{*+kd}(BG)$$

where k only ranges over the multiples of p^N, and the homomorphisms in the sequence are given by multiplication by $e_V^{p^N}$, is indeed a description given entirely within the category of $A(n)$-modules and $A(n)$-linear homomorphisms.

It follows that $Sq^i(\Sigma x^j \cdot x^k) = Sq^i(\Sigma x^j)\cdot x^k$ whenever $Sq^i \in A(n)$ and $2^{n+1} \mid k$. If we define the binomial coefficient by the formula

$$\binom{j}{i} = \frac{j(j-1)\cdots(j-i+1)}{i(i-1)\dots 1}$$

for $i \ge 0$ and $j \in \mathbb{Z}$, then

$$\binom{j+k}{i} \equiv \binom{j}{i} \mod 2$$

whenever $Sq^i \in A(n)$ and $2^{n+1} \mid k$. Hence the formula

$$Sq^i(\Sigma x^j) = \binom{j}{i}\Sigma x^{i+j}$$

for the $A(n)$-module action on $H^*(\Sigma\mathbb{R}P^\infty;\mathbb{F}_2)$ remains valid in the localization $H^*_c(S^{tC_2};\mathbb{F}_2)$. And, since with these conventions the formula does not depend on n, it remains valid for the entire $\mathscr{A}$-module action.

Proposition 5.1. *$H_c^*(S^{tC_2};\mathbb{F}_2) = \Sigma P(x^{\pm 1})$ with*

$$Sq^i(\Sigma x^j) = \binom{j}{i}\Sigma x^{i+j}$$

for all $i \geq 0$ and $j \in \mathbb{Z}$.

$H_c^*(S^{tC_p};\mathbb{F}_p) = \Sigma E(x) \otimes P(y^{\pm 1})$ with $\beta(x) = y$, $P^i(x) = 0$ for $i > 0$, $\beta(y) = 0$ and

$$P^i(\Sigma y^j) = \binom{j}{i}\Sigma y^{i(p-1)+j}$$

for all $i \geq 0$ and $j \in \mathbb{Z}$.

In particular,

$$Sq^i(\Sigma x^{-1}) = \Sigma x^{i-1}$$
$$P^i(\Sigma xy^{-1}) = (-1)^i \Sigma xy^{(p-1)i-1}$$
$$\beta P^i(\Sigma xy^{-1}) = (-1)^{i+1}\Sigma y^{(p-1)i}$$

[[check the last sign]] for all $i \geq 0$ because

$$\binom{-1}{i} = \frac{(-1)(-2)\cdots(-i)}{i(i-1)\cdots 1} = (-1)^i\,.$$

The homomorphism $\epsilon\colon \Sigma P(x^{\pm 1}) \to \mathbb{F}_2$ given by $\epsilon(\Sigma x^{-1}) = 1$ is $\mathscr{A}$-linear. To see this, one can check that Σx^{-1} is not $\mathscr{A}$-module decomposable, i.e., of the form $Sq^i(\Sigma x^j)$ for any $i \geq 1$. For degree reasons we would have to have $i + j = -1$, but $Sq^i(\Sigma x^{-i-1}) = \binom{-i-1}{i}\Sigma x^{-1}$ and

$$\binom{-i-1}{i} = \frac{(-i-1)(-i-2)\cdots(-i-i)}{i(i-1)\cdots 1} \equiv \binom{2i}{i} \mod 2\,,$$

which is always zero.

Theorem 5.2 (Lin)**.** *$\epsilon\colon \Sigma P(x^{\pm 1}) \to \mathbb{F}_2$ induces isomorphisms*

$$\epsilon_\#\colon \operatorname{Tor}^{\mathscr{A}}_{s,t}(\mathbb{F}_2, \Sigma P(x^{\pm 1})) \xrightarrow{\cong} \operatorname{Tor}^{\mathscr{A}}_{s,t}(\mathbb{F}_2,\mathbb{F}_2)$$

and

$$\epsilon^\#\colon \operatorname{Ext}_{\mathscr{A}}^{s,t}(\mathbb{F}_2,\mathbb{F}_2) \xrightarrow{\cong} \operatorname{Ext}_{\mathscr{A}}^{s,t}(\Sigma P(x^{\pm 1}),\mathbb{F}_2)\,.$$

Theorem 5.3 (Gunawardena)**.** *$\epsilon\colon \Sigma E(x)\otimes P(y^{\pm 1}) \to \mathbb{F}_p$ induces isomorphisms*

$$\epsilon_\#\colon \operatorname{Tor}^{\mathscr{A}}_{s,t}(\mathbb{F}_p, \Sigma E(x)\otimes P(y^{\pm 1})) \xrightarrow{\cong} \operatorname{Tor}^{\mathscr{A}}_{s,t}(\mathbb{F}_p,\mathbb{F}_p)$$

and

$$\epsilon^\#\colon \operatorname{Ext}_{\mathscr{A}}^{s,t}(\mathbb{F}_p,\mathbb{F}_p) \xrightarrow{\cong} \operatorname{Ext}_{\mathscr{A}}^{s,t}(\Sigma E(x)\otimes P(y^{\pm 1}),\mathbb{F}_p)\,.$$

In each case the Ext-isomorphism follows from the Tor-isomorphism and the natural isomorphism

$$\operatorname{Ext}_{\mathscr{A}}^{s,t}(L,\mathbb{F}_p) \cong (\operatorname{Tor}^{\mathscr{A}}_{s,t}(\mathbb{F}_p, L))^*\,,$$

which is valid for each left $\mathscr{A}$-module L [?LDMA80, Lemma 4.3], [?AGM85, Proposition 1.2].

Lin-Davis-Mahowald-Adams and Gunawardena prove the Tor-isomorphisms by first analyzing

$$\operatorname{Tor}^{A(n)}_{s,t}(\mathbb{F}_2, \Sigma P(x^{\pm 1}))$$

and

$$\operatorname{Tor}^{A(n)}_{s,t}(\mathbb{F}_p, \Sigma E(x)\otimes P(y^{\pm 1}))$$

for each $n \geq 0$, and thereafter passing to the colimit over n. Adams-Gunawardena-Miller give a more conceptual proof by first recognizing $\Sigma P(x^{\pm 1}) = R_+(\mathbb{F}_2)$ and $\Sigma E(x) \otimes P(y^{\pm 1}) = R_+(\mathbb{F}_p)$ as special cases of the Singer construction on (left) $\mathscr{A}$-modules M, and then proving more generally that $\epsilon\colon R_+(M) \to M$ is a Tor-equivalence.

6. The dual Steenrod algebra

Following (Milnor, 1958) let $\mathscr{A}_* = H_*(H)$ be the linear dual of the Steenrod algebra. It inherits a product $\phi\colon \mathscr{A}_* \otimes \mathscr{A}_* \to \mathscr{A}_*$ from the ring spectrum structure on H, which is dual to the coproduct ψ on $\mathscr{A}$. It also inherits a coproduct $\psi\colon \mathscr{A}_* \to \mathscr{A}_* \otimes \mathscr{A}_*$, dual to the product ϕ on $\mathscr{A}$, which agrees with the composite

$$H_*(H) \cong \pi_*(H \wedge S \wedge H) \xrightarrow{\eta_*} \pi_*(H \wedge H \wedge H) \cong H_*(H) \otimes H_*(H)$$

induced by the unit map $\eta\colon S \to H$ and the Künneth isomorphism on the right hand side. The conjugation $\chi\colon \mathscr{A}_* \to \mathscr{A}_*$ is dual to the conjugation $\chi\colon \mathscr{A} \to \mathscr{A}$, and agrees with the homomorphism

$$H_*(H) = \pi_*(H \wedge H) \xrightarrow{\gamma_*} \pi_*(H \wedge H) = H_*(H)$$

induced by the twist equivalence $\gamma\colon H \wedge H \to H \wedge H$. In other words, $\mathscr{A}_*$ is the connected, commutative Hopf algebra dual to the Steenrod algebra.

The n-th space in the spectrum H is the Eilenberg–Mac Lane space $K(\mathbb{F}_p, n)$, and there is a canonical homomorphism

$$x_*\colon \tilde{H}_{*+1}(K(\mathbb{F}_p, 1)) \longrightarrow \operatorname*{colim}_n \tilde{H}_{*+n}(K(\mathbb{F}_p, n)) = H_*(H)\,.$$

It is induced by the map $\Sigma^\infty K(\mathbb{F}_p, 1) \to \Sigma H$ representing the generator $x \in H^1(K(\mathbb{F}_p, 1))$. Here $K(\mathbb{F}_2, 1) \simeq \mathbb{R}P^\infty$ and $K(\mathbb{F}_p, 1) \simeq L^\infty$ for p odd, so

$$H_*(\mathbb{R}P^\infty) = \mathbb{F}_2\{\alpha_i \mid i \geq 0\}$$

with α_i dual to x^i in $H^*(\mathbb{R}P^\infty) = P(x)$, and

$$H_*(L^\infty) = E(\alpha_1) \otimes \mathbb{F}_p\{\alpha_{2i} \mid i \geq 0\}$$

with α_1 dual to x and α_{2i} dual to y^i in $H^*(L^\infty) = E(x) \otimes P(y)$.

The loop space structure on $K(\mathbb{F}_p, 1) \simeq \Omega K(\mathbb{F}_p, 2)$ makes $H_*(K(\mathbb{F}_p, 1))$ an algebra, and x_* takes all decomposables (for this so-called Pontryagin product) to zero. However, the algebra indecomposables all turn out to map non-trivially to $\mathscr{A}_*$. These are dual to the coalgebra primitives in $H^*(K(\mathbb{F}_p, 1))$, which are the classes x^{2^j} for $j \geq 0$ when $p = 2$, and the classes x and y^{p^j} for $j \geq 0$ when p is odd. Thus the algebra indecomposables are α_{2^j} for $j \geq 0$ when $p = 2$, and α_1 and α_{2p^j} for $j \geq 0$ when p is odd.

For $p = 2$ let $\xi_j = x_*(\alpha_{2^j}) \in \mathscr{A}_*$ for each $j \geq 0$, with $|\xi_j| = 2^j - 1$ and $\xi_0 = 1$. For p odd let $\tau_j = x_*(\alpha_{2p^j}) \in \mathscr{A}_*$ for each $j \geq 0$, with $|\tau_j| = 2p^j - 1$. [[The class $x_*(\alpha_1) = 1$.]] Furthermore, let $\xi_j = \beta_*(\tau_j) = y_*(\alpha_{2p^j}) \in \mathscr{A}_*$, with $|\xi_j| = 2p^2 - 2$, where $\beta_*\colon H_*(H) \to H_{*-1}(H)$ is induced by the Bockstein map $\beta\colon H \to \Sigma H$, and $y_* = \beta_* x_*$ is induced by the map $\beta \circ x\colon K(\mathbb{F}_p, 1) \to \Sigma^2 H$ representing $y = \beta(x) \in H^2(K(\mathbb{F}_p, 1))$. Again $\xi_0 = 1$.

Theorem 6.1 (Milnor)**.** *There are algebra isomorphisms*

$$\mathscr{A}_* = P(\xi_k \mid k \geq 1)$$

for $p = 2$, and

$$\mathscr{A}_* = E(\tau_k \mid k \geq 0) \otimes P(\xi_k \mid k \geq 1)$$

for p odd. The coproduct on $\mathscr{A}_*$ satisfies

$$\psi(\xi_k) = \sum_{i+j=k} \xi_i^{2^j} \otimes \xi_j$$

for $p = 2$, and

$$\psi(\tau_k) = \tau_k \otimes 1 + \sum_{i+j=k} \xi_i^{p^j} \otimes \tau_j$$
$$\psi(\xi_k) = \sum_{i+j=k} \xi_i^{p^j} \otimes \xi_j$$

for p odd.

[[Deduce these formulas from the coaction on $H_*(K(\mathbb{F}_p, 1))$?]]

For instance,

$$\psi(\xi_1) = \xi_1 \otimes 1 + 1 \otimes \xi_1$$
$$\psi(\xi_2) = \xi_2 \otimes 1 + \xi_1^p \otimes \xi_1 + 1 \otimes \xi_2$$

for $p = 2$ and for p odd, and

$$\psi(\tau_0) = \tau_0 \otimes 1 + 1 \otimes \tau_0$$
$$\psi(\tau_1) = \tau_1 \otimes 1 + \xi_1^p \otimes \tau_0 + 1 \otimes \tau_1$$

for p odd. These formulas for the coproduct in $\mathscr{A}_*$ are often more convenient for calculations than the Adem relations for the product in $\mathscr{A}$.

We write $\bar{\xi}_j = \chi(\xi_j)$ and $\bar{\tau}_j = \chi(\tau_j)$ for the images of the Milnor generators under the conjugation. These are recursively determined by

$$\sum_{i+j=k} \xi_i^{p^j} \chi(\xi_j) = 0$$

for any p and $k \geq 1$, and

$$\tau_k + \sum_{i+j=k} \xi_i^{p^j} \chi(\tau_j) = 0$$

for p odd and $k \geq 0$. Hence $\bar{\xi}_1 = -\xi_1$, $\bar{\xi}_2 = -\xi_2 + \xi_1^{p+1}$, $\bar{\tau}_0 = -\tau_0$ and $\bar{\tau}_1 = -\tau_1 + \tau_0 \xi_1^p$. We shall write ξ_1 in place of $\bar{\xi}_1$ for $p = 2$.

Corollary 6.2. *We have algebra isomorphisms*

$$\mathscr{A}_* = P(\bar{\xi}_k \mid k \geq 1)$$

for $p = 2$ and

$$\mathscr{A}_* = E(\bar{\tau}_k \mid k \geq 1) \otimes P(\bar{\xi}_k \mid k \geq 1)$$

for p odd, and the coproducts satisfy

$$\psi(\bar{\xi}_k) = \sum_{i+j=k} \bar{\xi}_i \otimes \bar{\xi}_j^{2^i}$$

for $p = 2$ and

$$\psi(\bar{\tau}_k) = 1 \otimes \bar{\tau}_k + \sum_{i+j=k} \bar{\tau}_i \otimes \bar{\xi}_j^{p^i}$$
$$\psi(\bar{\xi}_k) = \sum_{i+j=k} \bar{\xi}_i \otimes \bar{\xi}_j^{p^i}$$

for p odd.

Proof. These formulas follow from the fact that $\chi\colon A \to A$ is an anti-homomorphism, meaning that $\chi(xy) = \chi(y)\chi(x)$ for all $x, y \in A$. In terms of homomorphisms, this asserts that $\chi\phi = \phi(\chi \otimes \chi)\gamma\colon A \otimes A \to A$. Dually, $\psi\chi = \gamma(\chi \otimes \chi)\psi\colon A_* \to A_* \otimes A_*$, which gives the claimed formulas. □

7. Comparison of bases

In order to be able to translate facts about $\mathscr{A}_*$ back to $\mathscr{A}$, we need to understand the perfect pairing $\mathscr{A} \otimes \mathscr{A}_* \to \mathbb{F}_p$ in terms of the given bases. This is the special case $X = H$ of the Kronecker pairing $\langle\ ,\ \rangle\colon H^*(X) \otimes H_*(X) \to \mathbb{F}_p$. It turns out that Sq^i is dual to ξ_1^i in the basis for $\mathscr{A}$ that is dual to the monomial basis

$$\{\xi_1^{e_1}\xi_2^{e_2}\cdots\xi_r^{e_r} \mid e_1, \dots, e_r \ge 0\}$$

for $\mathscr{A}_*$. This basis for $\mathscr{A}$ is known as the Milnor basis, and is different from the basis $\{Sq^I\}$ consisting of the admissible monomials.

Lemma 7.1. *For $p = 2$ let $j \ge 1$ and $i = 2^j - 1$. Then*

$$\langle Sq^i, \xi_j\rangle = \begin{cases} 1 & \text{if } j = 1, \\ 0 & \text{otherwise.} \end{cases}$$

More generally, let $e_1, e_2, \dots, e_r \ge 0$ and $i = e_1 + (2^2-1)e_2 + \cdots + (2^r-1)e_r$. Then

$$\langle Sq^i, \xi_1^{e_1}\xi_2^{e_2}\cdots\xi_r^{e_r}\rangle = \begin{cases} 1 & \text{if } e_2 = \cdots = e_r = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. $\langle Sq^i, \xi_j\rangle = \langle Sq^i, x_*(\alpha_{2^j})\rangle = \langle x^*(Sq^i), \alpha_{2^j}\rangle = \langle Sq^i(x), \alpha_{2^j}\rangle$. Here $Sq^1(x) = x^2$ and $Sq^i(x) = 0$ for $i \ge 2$, so $\langle Sq^1, \xi_1\rangle = \langle x^2, \alpha_2\rangle = 1$, while $\langle Sq^i, \xi_j\rangle = \langle 0, \alpha_{2^j}\rangle = 0$ for $j \ge 2$. It follows that

$$\begin{aligned}\langle Sq^i, \xi_j^e\rangle &= \langle Sq^i, \Phi(\xi_j \otimes \cdots \otimes \xi_j)\rangle = \langle \Psi(Sq^i), \xi_j \otimes \cdots \otimes \xi_j\rangle \\ &= \sum_{i_1+\cdots+i_e=i} \langle Sq^{i_1} \otimes \cdots \otimes Sq^{i_e}, \xi_j \otimes \cdots \otimes \xi_j\rangle = \sum_{i_1+\cdots+i_e=i} \langle Sq^{i_1}, \xi_j\rangle \cdots \langle Sq^{i_e}, \xi_j\rangle\,,\end{aligned}$$

where Φ and Ψ denote the e-fold products and coproducts, respectively. Here $\langle Sq^{i_1}, \xi_j\rangle \cdots \langle Sq^{i_e}, \xi_j\rangle$ equals 1 if $i_1 = \cdots = i_e = 1$ and $j = 1$, and is 0 otherwise. Hence $\langle Sq^i, \xi_j^e\rangle$ is 1 if $i = e$ and $j = 1$, or if $e = 0$, and is 0 otherwise. In the same fashion,

$$\begin{aligned}\langle Sq^i, \xi_1^{e_1}\cdots\xi_r^{e_r}\rangle &= \langle Sq^i, \Phi(\xi_1^{e_1} \otimes \cdots \otimes \xi_r^{e_r})\rangle = \langle \Psi(Sq^i), \xi_1^{e_1} \otimes \cdots \otimes \xi_r^{e_r}\rangle \\ &= \sum_{i_1+\cdots+i_r=i} \langle Sq^{i_1} \otimes \cdots \otimes Sq^{i_r}, \xi_1^{e_1} \otimes \cdots \otimes \xi_r^{e_r}\rangle = \sum_{i_1+\cdots+i_r=i} \langle Sq^{i_1}, \xi_1^{e_1}\rangle \cdots \langle Sq^{i_r}, \xi_r^{e_r}\rangle\,,\end{aligned}$$

where Φ and Ψ now denote r-fold products and coproducts, respectively. Here $\langle Sq^{i_1}, \xi_1^{e_1}\rangle \cdots \langle Sq^{i_r}, \xi_r^{e_r}\rangle$ equals 1 if $i_1 = e_1$ and $e_2 = \cdots = e_r = 0$, and is 0 otherwise. Hence $\langle Sq^i, \xi_1^{e_1}\cdots\xi_r^{e_r}\rangle$ equals 1 if $i = e_1$ and $e_2 = \cdots = e_r = 0$, and is 0 otherwise. □

8. Finite quotient Hopf algebras

The finite sub Hopf algebras $A(n) \subset \mathscr{A}$ have dual finite quotient Hopf algebras $A(n)_* = \mathscr{A}_*/I(n)_*$.

Definition 8.1. For $p = 2$ and $n \geq -1$ let

$$I(n)_* = (\xi_1^{2^{n+1}}, \xi_2^{2^n}, \dots, \xi_n^{2^2}, \xi_{n+1}^2, \xi_k \mid k \geq n+2) \subset \mathscr{A}_*$$

be the ideal generated by the listed elements $\xi_k^{2^e}$ with $k \geq 1$, $e \geq 0$ and $k + e \geq n + 2$, and let

$$A(n)_* = \mathscr{A}_*/I(n)_* = P_{2^{n+1}}(\xi_1) \otimes P_{2^n}(\xi_2) \otimes \cdots \otimes E(\xi_{n+1})$$

be the quotient algebra. [[Also discuss p odd.]]

Here P_h denotes the truncated polynomial algebra of height h, with $P_2 = E$. Note that $A(n)_*$ is a finite algebra of dimension $2^{n+1} \cdot 2^n \cdot \ldots \cdot 2 = 2^{\binom{n+2}{2}}$. For instance, $A(-1)_* = \mathbb{F}_2$, $A(0)_* = E(\xi_1)$ and $A(1)_* = P_4(\xi_1) \otimes E(\xi_2)$. Since $I(n)_* \subset I(n-1)_*$ we get an infinite sequence of surjective algebra homomorphisms

$$\mathscr{A}_* \to \cdots \to A(n)_* \to A(n-1)_* \to \cdots \to A(0)_* \to \mathbb{F}_2 .$$

Lemma 8.2. *$I(n)_*$ is a Hopf ideal in $\mathscr{A}_*$.*

This means that the coproduct $\psi \colon \mathscr{A}_* \to \mathscr{A}_* \otimes \mathscr{A}_*$ maps $I(n)_*$ into the kernel $I(n)_* \otimes \mathscr{A}_* + \mathscr{A}_* \otimes I(n)_*$ of $\mathscr{A}_* \otimes \mathscr{A}_* \to A(n)_* \otimes A(n)_*$, hence induces a coproduct

$$\psi \colon A(n)_* \longrightarrow A(n)_* \otimes A(n)_*$$

making the diagram of horizontal extensions

$$\begin{array}{ccccc} I(n)_* & \longrightarrow & \mathscr{A}_* & \longrightarrow & A(n)_* \\ \downarrow & & \downarrow \psi & & \downarrow \psi \\ I(n)_* \otimes \mathscr{A}_* + \mathscr{A}_* \otimes I(n)_* & \longrightarrow & \mathscr{A}_* \otimes \mathscr{A}_* & \longrightarrow & A(n)_* \otimes A(n)_* \end{array}$$

commute.

Proof. It suffices to observe that

$$\psi(\xi_k^{2^e}) = \psi(\xi_k)^{2^e} = \sum_{i+j=k} (\xi_i^{2^j} \otimes \xi_j)^{2^e} = \sum_{i+j=k} \xi_i^{2^{j+e}} \otimes \xi_j^{2^e}$$

maps to zero in $A(n)_* \otimes A(n)_*$ whenever $k \geq 1$, $e \geq 0$ and $k + e \geq n + 2$. In fact $\xi_i^{2^{j+e}} \in I(n)_*$ under these conditions, unless $i = 0$, in which case $j = k$ and $\xi_j^{2^e} \in I(n)_*$. □

It follows that $A(n)_*$ is itself a connected, commutative Hopf algebra, and the infinite sequence above consists of Hopf algebra homomorphisms. This also implies that $I(n)_*$ is closed under the conjugation, hence can equally well be generated by the conjugate classes $\bar{\xi}_k^{2^e}$ for $k \geq 1$, $e \geq 0$ and $k + e \geq k + 2$.

Proposition 8.3. *The quotient Hopf algebra $A(n)_*$ of $\mathscr{A}_*$ is dual to the sub Hopf algebra $A(n)$ of $\mathscr{A}$.*

Proof. The dual of $A(n)_*$ consists of the classes in $\mathscr{A}$ that annihilate $I(n)_*$ under the Kronecker pairing with $\mathscr{A}_*$. For $1 \leq i < 2^{n+1}$ we have

$$\langle Sq^i, \xi_s^{2^e} \rangle = 0$$

whenever $s \geq 1$, $e \geq 0$ and $s + e \geq n + 2$ (which for $s = 1$ implies $e \geq n + 1$), so these Sq^i annihilate the ideal generators of $I(n)_*$. A general class in $I(n)_*$ is a sum of terms $\alpha \cdot \xi_s^{2^e}$, with $\alpha \in \mathscr{A}_*$, and

$$\langle Sq^k, \alpha \cdot \xi_s^{2^e} \rangle = \sum_{i+j=k} \langle Sq^i, \alpha \rangle \langle Sq^j, \xi_s^{2^e} \rangle .$$

For $1 \le k < 2^{n+1}$ we have $0 \le j < 2^{n+1}$, so the right hand factor vanishes unless $j = 0$ and $i = k$, leaving $\langle Sq^k, \alpha\rangle\langle Sq^0, \xi_s^{2^e}\rangle$, which is zero for degree reasons. Hence the algebra generators of $A(n)$ all lie in the dual of $A(n)_*$, proving that $A(n)$ is contained in that dual.

It remains to prove that the images of the Sq^{2^j} for $0 \le j \le n$ suffice to span the algebra indecomposables of the dual of $A(n)_*$. By duality, this is equivalent to asking that the coalgebra primitives $P(A(n)_*)$ map injectively to the span of $\xi_1^{2^j}$ for $0 \le j \le n$.

[[Refer to Milnor (1958) for the opposite inclusion?]] □

[[The quotient Hopf algebra $E(n)_* = \mathscr{A}_*/(\xi_1^2, \dots, \xi_{n+1}^2, \xi_k \mid k \ge n+2) = E(\xi_1, \dots, \xi_{n+1})$ of $\mathscr{A}_*$ is dual to the sub Hopf algebra $E(n)$ of $\mathscr{A}$.]]

Let k be a field. We sometimes use the notation $A//B = A \otimes_B k$ for the tensor product over an augmented subalgebra B of a k-algebra A. It is the coequalizer in left A-modules

$$A \otimes B \rightrightarrows A \longrightarrow A//B$$

of the two homomorphisms $A \otimes B \to A$ taking $a \otimes b$ to ab and $a\epsilon(b)$, respectively, where $\epsilon\colon B \to k$ is the augmentation. As such, it equals the quotient $A/AI(B)$, where $I(B) = \ker(\epsilon)$ is the augmentation ideal of B, and $AI(B)$ is the left ideal in A generated by $I(B)$. If B is normal in A, in the sense that $AI(B) = I(B)A$, then this quotient is a quotient algebra of A, but this does not hold in general.

If A is a Hopf algebra and B a sub Hopf algebra, then for $a \in A$, $b \in I(B)$ we have $\psi(ab) = \psi(a)\psi(b)$ and $\psi(b) \in B \otimes I(B) + I(B) \otimes B$, so $\psi(ab) \in A \otimes AI(B) + AI(B) \otimes A$. Hence $AI(B)$ is a Hopf ideal in A, and $A//B$ inherits a coalgebra structure from A, making the quotient map $A \to A//B$ a unital coalgebra homomorphism.

[[More generally so for A a left or right B-module coalgebra.]]

The following theorem is due to Milnor–Moore (1965, Theorem 4.4). It implies that a Hopf algebra is free as a (left or right) module over any sub Hopf algebra, as long as they are both connected.

Theorem 8.4. *If B is a connected Hopf algebra and A is a connected left B-module coalgebra such that $i\colon B \to A$ is injective, then*

$$A \cong B \otimes (k \otimes_B A)$$

as left B-modules and right $k \otimes_B A$-comodules. If A is instead a connected right B-module coalgebra, then

$$A \cong (A \otimes_B k) \otimes B$$

as right B-modules and left $A \otimes_B k$-comodules.

Dually, we can consider the cotensor product $A_* \,\square_{B_*}\, k$ over a unital quotient coalgebra B_* of a k-coalgebra A_*. It is the equalizer in left A_*-comodules

$$A_* \,\square_{B_*}\, k \longrightarrow A_* \rightrightarrows A_* \otimes B_*$$

of the two homomorphisms $A_* \to A_* \otimes B_*$ given by the coproduct on A_* followed by the projection to B_* in the right hand factor, and A_* tensored with the unit of B_*, respectively. If $A_* \to B_*$ is dual to $B \subset A$, the cotensor product is dual to the tensor product $A \otimes_B k = A/AI(B)$, hence is the left A_*-subcomodule of A_* that annihilates $AI(B)$ under the pairing with A. It is not in general a subcoalgebra of A_*. If $A_* \to B_*$ is a surjection of Hopf algebras, then the two homomorphisms are algebra homomorphisms, so the equalizer is a subalgebra of A_*, making the inclusion $A_* \square_{B_*} k \to A_*$ an (augmented) algebra homomorphism.

[[More generally so for A_* a left or right B_*-comodule algebra.]]

Here is the dual Milnor–Moore theorem (1965, Theorem 4.7).

Theorem 8.5. *If B_* is a connected Hopf algebra and A_* is a connected left B_*-comodule algebra such that $j\colon A_* \to B_*$ is surjective, then*

$$A_* \cong B_* \otimes (k \,\square_{B_*}\, A_*)$$

as left B_*-comodules and right $k \,\square_{B_*}\, A_*$-modules. If A_* is instead a connected right B_*-comodule algebra, then

$$A_* \cong (A_* \,\square_{B_*}\, k) \otimes B_*$$

as right B_*-comodules and left $A_* \,\square_{B_*}\, k$-modules.

Proposition 8.6. *The subalgebra $E(\xi_1^{2^n}, \bar\xi_2^{2^{n-1}}, \dots, \bar\xi_{n+1}) = A(n)_* \,\square_{A(n-1)_*}\, \mathbb{F}_2$ of $A(n)_*$ is dual to the quotient coalgebra $A(n)//A(n-1) = A(n) \otimes_{A(n-1)} \mathbb{F}_2$ of $A(n)$.*

Proof. We know that $A(n)$ is a free right $A(n-1)$-module by the Milnor–Moore theorem, so $A(n) \otimes_{A(n-1)} \mathbb{F}_2$ has dimension $2^{\binom{n+2}{2}}/2^{\binom{n+1}{2}} = 2^{n+1}$, hence by duality $A(n)_* \,\square_{A(n-1)_*}\, \mathbb{F}_2$ also has this dimension. It is the equalizer of two algebra homomorphisms

$$A(n)_* \rightrightarrows A(n)_* \otimes A(n-1)_* \,,$$

hence is a subalgebra of $A(n)_*$. It contains the $n+1$ elements $\xi_1^{2^n}, \bar\xi_2^{2^{n-1}}, \dots, \bar\xi_{n+1}$, because

$$\psi(\bar\xi_k^{2^e}) = \sum_{i+j=k} \bar\xi_i^{2^e} \otimes \bar\xi_j^{2^{i+e}} \,,$$

and $\bar\xi_j^{2^{i+e}}$ maps to zero in $A(n-1)_*$ for $i+j=k \ge 1$, $e \ge 0$ and $k+e \ge n+1$, unless $j=0$. Thus the image of $\bar\xi_k^{2^e}$ in $A(n)_* \otimes A(n-1)_*$ is $\bar\xi_k^{2^e} \otimes 1$ under both homomorphisms. These $n+1$ elements generate the exterior algebra $E(\xi_1^{2^n}, \bar\xi_2^{2^{n-1}}, \dots, \bar\xi_{n+1})$ inside $A(n)_*$, of dimension 2^{n+1}. Hence, by a dimension count, this is the whole of $A(n)_* \,\square_{A(n-1)_*}\, \mathbb{F}_2$. □

9. Some bicomodule algebras and bimodule coalgebras

Definition 9.1. For $n \ge 0$ let

$$J(n)_* = (\xi_2^{2^n}, \dots, \xi_n^{2^2}, \xi_{n+1}^2, \xi_k \mid k \ge n+2) \subset \mathscr{A}_*$$

be the ideal generated by the elements $\xi_k^{2^e}$ with $k \ge 2$, $e \ge 0$ and $k+e \ge n+2$, and let

$$C(n)_* = \mathscr{A}_*/J(n)_* = P(\xi_1) \otimes P_{2^n}(\xi_2) \otimes \cdots \otimes E(\xi_{n+1})$$

be the quotient algebra. Let $C(n) \subset \mathscr{A}$ be the dual sub coalgebra.

For instance, $C(0)_* = P(\xi_1)$ is dual to $C(0) = \mathbb{F}_2\{Sq^i \mid i \ge 0\}$, and $C(1)_* = P(\xi_1) \otimes E(\xi_2)$.

The ideal $J(n)_*$ is not a Hopf ideal, so $C(n)_*$ is not a Hopf algebra. However, it admits interesting left and right coactions.

Lemma 9.2. *$J(n)_*$ is an $A(n)_*$-$A(n-1)_*$ bicomodule ideal in $\mathscr{A}_*$.*

This means that the left $A(n)_*$-coaction $\mathscr{A}_* \to \mathscr{A}_* \otimes \mathscr{A}_* \to A(n)_* \otimes \mathscr{A}_*$ maps $J(n)_*$ into the kernel $A(n)_* \otimes J(n)_*$ of $A(n)_* \otimes \mathscr{A}_* \to A(n)_* \otimes C(n)_*$, and that the right $A(n-1)_*$-coaction $\mathscr{A}_* \to \mathscr{A}_* \otimes \mathscr{A}_* \to \mathscr{A}_* \otimes A(n-1)_*$ maps $J(n)_*$ into the kernel $J(n)_* \otimes A(n-1)_*$ of $\mathscr{A}_* \otimes A(n-1)_* \to C(n)_* \otimes A(n-1)_*$.

Proof. The left and right coactions take $\xi_k^{2^e}$ with $k \geq 2$, $e \geq 0$ and $k + e \geq n + 2$ to the images of

$$\psi(\xi_k^{2^e}) = \sum_{i+j=k} \xi_i^{2^{j+e}} \otimes \xi_j^{2^e}$$

in $A(n)_* \otimes \mathscr{A}_*$ and $\mathscr{A}_* \otimes A(n{-}1)_*$, respectively. Regarding the left $A(n)_*$-coaction, $\xi_i^{2^{j+e}}$ with $i + j = k$ maps to zero in $A(n)_*$ unless $i = 0$, so $\psi(\xi_k^{2^e})$ maps to $1 \otimes \xi_k^{2^e}$, which lies in the ideal $A(n)_* \otimes J(n)_*$. Regarding the right $A(n{-}1)_*$-coaction, $\xi_i^{2^{j+e}}$ with $i + j = k$ lies in $J(n)_*$ unless $i = 0$ or $i = 1$, so $\psi(\xi_k^{2^e})$ maps to

$$1 \otimes \xi_k^{2^e} + \xi_1^{2^{k-1+e}} \otimes \xi_{k-1}^{2^e} \,,$$

and $\xi_k^{2^e}$ and $\xi_{k-1}^{2^e}$ both map to zero in $A(n{-}1)_*$. Both coactions are algebra maps, so this implies the claim. $\square$

It follows that these coactions induce a left $A(n)_*$-coaction $C(n)_* \to A(n)_* \otimes C(n)_*$ and a right $A(n{-}1)_*$-coaction $C(n)_* \to C(n)_* \otimes A(n{-}1)_*$, and that these two coactions commute. In particular the diagram

$$\begin{array}{ccccc}
\mathscr{A}_* & \xrightarrow{\Psi} & \mathscr{A}_* \otimes \mathscr{A}_* \otimes \mathscr{A}_* & \longrightarrow & A(n)_* \otimes \mathscr{A}_* \otimes A(n{-}1)_* \\
\downarrow & & & & \downarrow \\
C(n)_* & & \longrightarrow & & A(n)_* \otimes C(n)_* \otimes A(n{-}1)_*
\end{array}$$

commutes.

Lemma 9.3. *The surjection $\mathscr{A}_* \to C(n)_*$ is a homomorphism of $A(n)_*$-$A(n{-}1)_*$ bicomodule algebras. Dually, the inclusion $C(n) \subset \mathscr{A}$ is a homomorphism of $A(n)$-$A(n{-}1)$ bimodule coalgebras.*

Definition 9.4. We define

$$B(n)_* = C(n)_*[\xi_1^{-1}] = \operatorname*{colim}_k \Sigma^{-k} C(n)_* \,,$$

and let $B(n) = \lim_k \Sigma^{-k} C(n)$ be the dual of $B(n)_*$.

Lemma 9.5. *Multiplication by $\xi_1^{2^{n+1}} \colon \Sigma^{2^{n+1}} C(n)_* \to C(n)_*$ is an $A(n)_*$-$A(n{-}1)_*$ bicomodule homomorphism.*

Proof. The left $A(n)_*$-coaction takes $\xi_1^{2^{n+1}}$ to $1 \otimes \xi_1^{2^{n+1}}$, while the right $A(n{-}1)_*$-coaction takes $\xi_1^{2^{n+1}}$ to $\xi_1^{2^{n+1}} \otimes 1$, hence multiplication by $\xi_1^{2^{n+1}}$ commutes with both coactions. $\square$

Lemma 9.6. *The short exact sequence*

$$0 \to \Sigma^{2^{n+1}} C(n)_* \xrightarrow{\xi_1^{2^{n+1}} \cdot} C(n)_* \longrightarrow A(n)_* \to 0$$

and the injection

$$C(n)_* \longrightarrow B(n)_* = \operatorname*{colim}_j \Sigma^{-j \cdot 2^{n+1}} C(n)_*$$

both consist of $A(n)$-$A(n{-}1)$ bicomodules and $A(n)$-$A(n{-}1)$ bicomodule homomorphisms. Dually, the short exact sequence

$$0 \to A(n) \longrightarrow C(n) \longrightarrow \Sigma^{2^{n+1}} C(n) \to 0$$

and the surjection

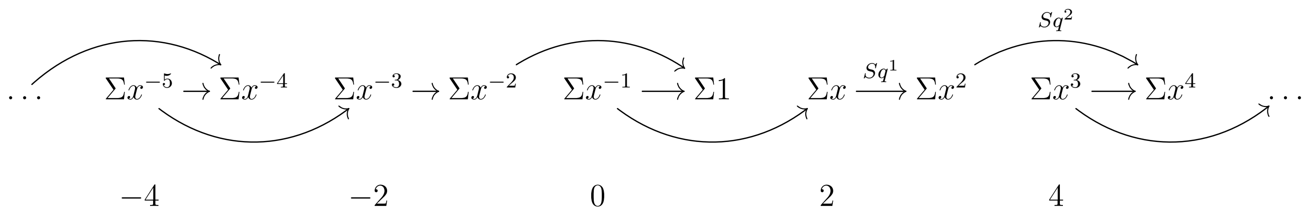
$$B(n) = \lim_j \Sigma^{-j \cdot 2^{n+1}} C(n) \longrightarrow C(n)$$

both consist of $A(n)$-$A(n{-}1)$ bimodules and $A(n)$-$A(n{-}1)$ bimodule homomorphisms.

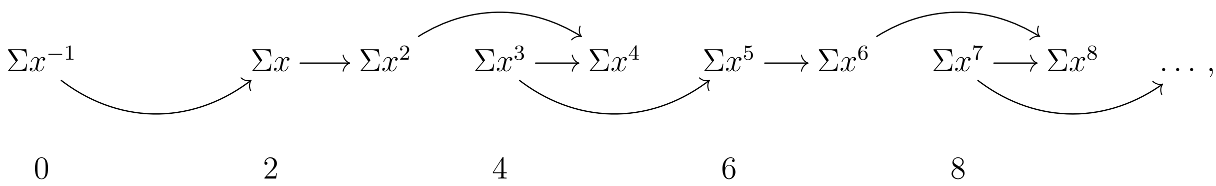
Proof. Note that $I(n)_*$ is the ideal generated by $J(n)_*$ and $\xi_1^{2^n}$, and multiplication by $\xi_1^{2^n}$ acts injectively on $C(n)_*$. $\square$

10. The $A(n)$-module structure of $\Sigma P(x^{\pm 1})$ [[and $\Sigma E(x) \otimes P(y^{\pm 1})$]]

Consider the $A(1)$-module structure on $\Sigma P(x^{\pm 1})$, where $A(1) \subset \mathscr{A}$ is generated by Sq^1 and Sq^2.



It is 4-periodic, and is obtained by non-trivial extensions from infinitely many copies of the cyclic $A(1)$-module $A(1)//A(0) = \mathbb{F}_2\{1, Sq^2, Sq^1Sq^2, Sq^2Sq^1Sq^2\}$, suspended by all integer multiples of four. The quotient by the submodule $A(1)F_{<0}$ generated by classes in negative degree appears as follows



and there is a short exact sequence of $A(1)$-modules

$$0 \to A(1)//A(0) \longrightarrow \frac{\Sigma P(x^{\pm 1})}{A(1)F_{<0}} \longrightarrow \Sigma^4 \frac{\Sigma P(x^{\pm 1})}{A(1)F_{<0}} \to 0 \,.$$

These patterns generalize to all larger n.

Definition 10.1. Let $F_{<k} = \mathbb{F}_2\{\Sigma x^i \mid i+1 < k\} \subset \Sigma P(x^{\pm 1})$ be the subspace of classes in degree less than k, and let $A(n)F_{<k} \subset \Sigma P(x^{\pm 1})$ be the left $A(n)$-submodule generated by $F_{<k}$.

We get a tower of surjections

$$\dots \longrightarrow \frac{\Sigma P(x^{\pm 1})}{A(n)F_{<k}} \longrightarrow \frac{\Sigma P(x^{\pm 1})}{A(n)F_{<k+1}} \longrightarrow \dots \,,$$

in the category of $A(n)$-modules. Since $A(n)$ is bounded above (and below) we get an isomorphism

$$\Sigma P(x^{\pm 1}) \cong \lim_k \frac{\Sigma P(x^{\pm 1})}{A(n)F_{<k}} \,,$$

where in each degree the limit is achieved at a finite stage.

Multiplication by $x^{2^{n+1}}$ acts $A(n)$-linearly on $\Sigma P(x^{\pm 1})$, hence induces isomorphisms

$$\Sigma^{2^{n+1}} \frac{\Sigma P(x^{\pm 1})}{A(n)F_{<k}} \cong \frac{\Sigma P(x^{\pm 1})}{A(n)F_{<k+2^{n+1}}} \,.$$

Restricting $k = -j \cdot 2^{n+1}$ to integer multiples of 2^{n+1}, we can write

$$\Sigma P(x^{\pm 1}) \cong \lim_j \frac{\Sigma P(x^{\pm 1})}{A(n)F_{<-j\cdot 2^{n+1}}} \cong \lim_j \Sigma^{-j\cdot 2^{n+1}} \frac{\Sigma P(x^{\pm 1})}{A(n)F_{<0}} \,.$$

Proposition 10.2. *There is an $A(n)$-module [[coalgebra?]] isomorphism*

$$C(n) \otimes_{A(n-1)} \mathbb{F}_2 \xrightarrow{\cong} \frac{\Sigma P(x^{\pm 1})}{A(n)F_{<0}}$$

mapping the class of $c \otimes 1$ to the class of $c(\Sigma x^{-1})$.

Proof. The displayed map is well defined because

$$Sq^{2^j}(\Sigma x^{-1}) = Sq^{2^n}(\Sigma x^{2^j-1-2^n}) \in A(n)F_{<0}$$

for $0 < j < n$, and these Sq^{2^j} generate $A(n-1)$.

By the Milnor–Moore theorem $C(n)$ is free as a right $A(n-1)$-module, and $c(\Sigma x^{-1})$ lies in $A(n)F_{<2^{n+1}}$ when $c \in A(n)$, so there is a map of short exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A(n) \otimes_{A(n-1)} \mathbb{F}_2 & \longrightarrow & C(n) \otimes_{A(n-1)} \mathbb{F}_2 & \longrightarrow & \Sigma^{2^{n+1}} C(n) \otimes_{A(n-1)} \mathbb{F}_2 & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & A(n)F_{<2^{n+1}}/A(n)F_{<0} & \longrightarrow & \Sigma P(x^{\pm 1})/A(n)F_{<0} & \longrightarrow & \Sigma P(x^{\pm 1})/A(n)F_{<2^{n+1}} & \longrightarrow & 0 \end{array}$$

of left $A(n)$-modules, where the right hand vertical map is isomorphic to the 2^{n+1}-th suspension of the middle vertical map. The left hand vertical map is surjective because

$$Sq^i(\Sigma x^{-1}) = \Sigma x^{i-1}$$

for $0 \le i < 2^{n+1}$ generate the target as an $A(n)$-module, and these Sq^i lie in $A(n)$. That target contains precisely one generator in each congruence class of degrees modulo 2^{n+1}, so it has the same dimension as $A(n) \otimes_{A(n-1)} \mathbb{F}_2$. Hence the left hand vertical map is an isomorphism, and it follows by induction that the middle vertical map is also an isomorphism. □

Theorem 10.3. *There is an $A(n)$-module isomorphism*

$$B(n) \otimes_{A(n-1)} \mathbb{F}_2 \xrightarrow{\cong} \Sigma P(x^{\pm 1}) .$$

Proof. The canonical homomorphism

$$\begin{aligned} B(n) \otimes_{A(n-1)} \mathbb{F}_2 \cong \big(\lim_j \Sigma^{-j\cdot 2^{n+1}} C(n)\big) \otimes_{A(n-1)} \mathbb{F}_2 &\longrightarrow \lim_j \big(\Sigma^{-j\cdot 2^{n+1}} C(n) \otimes_{A(n-1)} \mathbb{F}_2\big) \\ &\cong \lim_j \big(\Sigma^{-j\cdot 2^{n+1}} \frac{\Sigma P(x^{\pm 1})}{A(n)F_{<0}}\big) \cong \lim_j \big(\frac{\Sigma P(x^{\pm 1})}{A(n)F_{<-j\cdot 2^{n+1}}}\big) \cong \Sigma P(x^{\pm 1}) \end{aligned}$$

is an isomorphism, because in each degree both limits are achieved for all sufficiently large j, so $\Sigma^{-j\cdot 2^{n+1}} C(n) \otimes_{A(n-1)} \mathbb{F}_2$ maps isomorphically and compatibly to both sides, in a range of degrees that grows to cover all degrees as j increases to ∞. □

To proceed from here, Davis and Mahowald [?LDMA80, Lemma 1.3] obtained the following splitting, after base change along $A(n) \subset \mathscr{A}$.

Lemma 10.4. *There is an $\mathscr{A}$-module isomorphism*

$$\mathscr{A} \otimes_{A(n)} \frac{\Sigma P(x^{\pm 1})}{A(n)F_{<0}} \cong \bigoplus_{j \ge 0} \Sigma^{j\cdot 2^{n+1}} (\mathscr{A} \otimes_{A(n-1)} \mathbb{F}_2) .$$

Proof. The composition

$$\mu \colon \mathscr{A} \otimes_{A(n)} C(n) \subset \mathscr{A} \otimes_{A(n)} \mathscr{A} \to \mathscr{A}$$

induces a splitting $\mu \otimes_{A(n-1)} 1$ of the short exact sequence

$$0 \to \mathscr{A} \otimes_{A(n-1)} \mathbb{F}_2 \longrightarrow \mathscr{A} \otimes_{A(n)} C(n) \otimes_{A(n-1)} \mathbb{F}_2 \longrightarrow \mathscr{A} \otimes_{A(n)} \Sigma^{2^{n+1}} C(n) \otimes_{A(n-1)} \mathbb{F}_2 \to 0$$

of left $\mathscr{A}$-modules. Hence there is an $\mathscr{A}$-module splitting of the short exact sequence

$$0 \to \mathscr{A} \otimes_{A(n-1)} \mathbb{F}_2 \longrightarrow \mathscr{A} \otimes_{A(n)} \Sigma P(x)/A(n)F_{<0} \longrightarrow \mathscr{A} \otimes_{A(n)} \Sigma^{2^{n+1}} (\Sigma P(x)/A(n)F_{<0}) \to 0\,.$$

Iterating, these combine to define the asserted isomorphism. □

Corollary 10.5.

$$\operatorname{Tor}_{s,t}^{A(n)}(\mathbb{F}_2, \frac{\Sigma P(x^{\pm 1})}{A(n)F_{<0}}) \cong \bigoplus_{j \geq 0} \operatorname{Tor}_{s,t}^{A(n-1)}(\mathbb{F}_2, \Sigma^{j \cdot 2^{n+1}} \mathbb{F}_2)\,.$$

Proof. Apply $\operatorname{Tor}_{s,t}^{\mathscr{A}}(\mathbb{F}_2, -)$ and change-of-rings. □

Corollary 10.6.

$$\operatorname{Tor}_{s,t}^{A(n)}(\mathbb{F}_2, \Sigma P(x^{\pm 1})) \cong \bigoplus_{j \in \mathbb{Z}} \operatorname{Tor}_{s,t}^{A(n-1)}(\mathbb{F}_2, \Sigma^{j \cdot 2^{n+1}} \mathbb{F}_2)\,.$$

Proof. Pass to the (achieved) limit over the desuspensions $\Sigma P(x^{\pm 1})/A(n)F_{<k}$ for (negative) multiples k of 2^{n+1}. □

Corollary 10.7.

$$\operatorname{Tor}_{s,t}^{\mathscr{A}}(\mathbb{F}_2, \Sigma P(x^{\pm 1})) \cong \operatorname{Tor}_{s,t}^{\mathscr{A}_*}(\mathbb{F}_2, \mathbb{F}_2)\,.$$

Proof. Pass to the colimit over n, checking that only the summand with $j = 0$ survives. □

This approach requires some careful control of the splitting maps and their behavior under passage from $A(n)$ to $A(n+1)$. We shall instead give the details in the more conceptual argument of [?AGM85].

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Department of Mathematics, University of Oslo, Norway
E-mail address: rognes@math.uio.no
URL: http://folk.uio.no/rognes