THE MOTIVIC SEGAL CONJECTURE, LECTURE 5

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Abstract. ((TBW))

1. Some bicomodule algebras

Definition 1.1. In the dual Steenrod algebra $\mathscr{A}_* = P(\xi_k \mid k \geq 1)$ consider the ideals

$$I(n)_* = (\xi_k^{2^e} \mid k \geq 1, e \geq 0, k + e \geq n + 2)$$

for $n \geq -1$, and

$$J(n)_* = (\xi_k^{2^e} \mid k \geq 2, e \geq 0, k + e \geq n + 2)$$

for $n \geq 0$. Let

$$A(n)_* = \mathscr{A}_*/I(n)_* = P_{2^{n+1}}(\xi_1) \otimes P_{2^n}(\xi_2) \otimes \cdots \otimes P_{2^2}(\xi_n) \otimes E(\xi_{n+1})$$

and

$$C(n)_* = \mathscr{A}_*/J(n)_* = P(\xi_1) \otimes P_{2^n}(\xi_2) \otimes \cdots \otimes P_{2^2}(\xi_n) \otimes E(\xi_{n+1})$$

be the associated quotient algebras, and let

$$B(n)_* = C(n)_*[1/\xi_1] = P(\xi_1^{\pm 1}) \otimes P_{2^n}(\xi_2) \otimes \cdots \otimes P_{2^2}(\xi_n) \otimes E(\xi_{n+1})$$

be the localization obtained by inverting ξ_1.

We have inclusions

$$\begin{array}{ccccc}
J(n+1)_* & \rightarrowtail & J(n)_* & & \\
\downarrow & & \downarrow & & \\
I(n+1)_* & \rightarrowtail & I(n)_* & \rightarrowtail & I(n-1)_*
\end{array}$$

for $n \geq 0$, and associated algebra homomorphisms

$$\begin{array}{ccccccc}
 & & B(n+1)_* & \twoheadrightarrow & B(n)_* & & \\
 & & \uparrow & & \uparrow & & \\
\mathscr{A}_* & \twoheadrightarrow & C(n+1)_* & \twoheadrightarrow & C(n)_* & & \\
 & & \downarrow & & \downarrow & & \\
 & & A(n+1)_* & \twoheadrightarrow & A(n)_* & \twoheadrightarrow & A(n-1)_* \,.
\end{array}$$

In particular, $C(0)_* = P(\xi_1)$ and $B(0)_* = P(\xi_1^{\pm 1})$. The kernels of the algebra homomorphisms $\gamma_* \colon C(n)_* \twoheadrightarrow C(0)_*$ and $\beta_* \colon B(n)_* \twoheadrightarrow B(0)_*$ are the ideals generated by $(\xi_2, \dots, \xi_{n+1})$, in each case.

Lemma 1.2. *$I(n)_*$ is a Hopf ideal, for each $n \geq -1$, so that there is a unique Hopf algebra structure on $A(n)_*$ making $\mathscr{A}_* \twoheadrightarrow A(n)_*$ a Hopf algebra homomorphism. It follows that $A(n+1)_* \twoheadrightarrow A(n)_*$ is also a Hopf algebra homomorphism.*

Date: October 24th 2014.

Proof. [[Calculate.]] □

We use the Hopf algebra homomorphisms $\mathscr{A}_* \to A(n+1)_* \to A(n)_* \to A(n-1)_*$ to view $\mathscr{A}_*$, $A(n+1)_*$ and $A(n)_*$ as $A(n)_*$-$A(n-1)_*$-bicomodule algebras. The homomorphisms $\mathscr{A}_* \to A(n+1)_* \to A(n)_*$ are then $A(n)_*$-$A(n-1)_*$-bicomodule algebra homomorphisms.

Lemma 1.3. *$J(n)_*$ is an $A(n)_*$-$A(n-1)_*$-bicomodule ideal, for each $n \geq 0$, so that there is a unique $A(n)_*$-$A(n-1)_*$-bicomodule algebra structure on $C(n)_*$ making $\mathscr{A}_* \twoheadrightarrow C(n)_*$ an $A(n)_*$-$A(n-1)_*$-bicomodule algebra homomorphism.*

Proof. [[Calculate.]] □

If follows that $C(n+1)_* \twoheadrightarrow C(n)_*$ and $C(n)_* \twoheadrightarrow A(n)_*$ are also $A(n)_*$-$A(n-1)_*$-bicomodule algebra homomorphisms, when $C(n+1)_*$ is given the $A(n)_*$-$A(n-1)_*$-bicomodule algebra structure induced from the $A(n+1)_*$-$A(n)_*$-bicomodule algebra structure that was just defined.

Lemma 1.4. *There is a short exact sequence*

$$0 \to \Sigma^{2^{n+1}} C(n)_* \xrightarrow{\xi_*} C(n)_* \to A(n)_* \to 0$$

of $A(n)_*$-$A(n-1)_*$-bicomodules, where ξ_* is given by multiplication by $\xi_1^{2^{n+1}}$. Hence

$$B(n)_* = \operatorname*{colim}_j \left(\Sigma^{-j\cdot 2^{n+1}} C(n)_*\right)$$

has a unique $A(n)_*$-$A(n-1)_*$-bicomodule algebra structure making the inclusion $C(n)_* \rightarrowtail B(n)_*$ an $A(n)_*$-$A(n-1)_*$-bicomodule algebra homomorphism.

Proof. [[Calculate.]] □

It follows that $B(n+1)_* \twoheadrightarrow B(n)_*$ and $C(n)_* \rightarrowtail B(n)_*$ are also $A(n)_*$-$A(n-1)_*$-bicomodule algebra homomorphisms, when $B(n+1)_*$ has the $A(n)_*$-$A(n-1)_*$-bicomodule algebra structure induced from the $A(n+1)_*$-$A(n)_*$-bicomodule algebra structure that was just defined.

Proposition 1.5. *There is an isomorphism*

$$C(n)_* \cong A(n)_* \otimes (\mathbb{F}_2 \,\square_{A(n)_*}\, C(n)_*)$$

of left $A(n)_*$-comodules and right $\mathbb{F}_2 \,\square_{A(n)_*}\, C(n)_*$-modules. There is also an isomorphism

$$C(n)_* \cong (C(n)_* \,\square_{A(n-1)_*}\, \mathbb{F}_2) \otimes A(n-1)_*$$

of right $A(n-1)_*$-comodules and left $C(n)_* \,\square_{A(n-1)_*}\, \mathbb{F}_2$-modules.

Proof. $A(n)_*$, $A(n-1)_*$ and $C(n)_*$ are connected, and $C(n)_* \twoheadrightarrow A(n)_*$ and $C(n)_* \twoheadrightarrow A(n-1)_*$ are surjective, so this follows from [MM65, Theorem 4.7]. □

It follows that $C(n)_*$ is [[cofree, hence]] injective as a left $A(n)_*$-comodule and as a right $A(n-1)_*$-comodule, so the short exact sequence in Lemma 1.4 admits a splitting in either one of these categories.

Lemma 1.6. $\mathbb{F}_2 \,\square_{A(n)_*}\, C(n)_* = P(\xi_1^{2^{n+1}})$ *and*

$$C(n)_* \,\square_{A(n-1)_*}\, \mathbb{F}_2 = P(\xi_1^{2^n}) \otimes E(\bar{\xi}_2^{2^{n-1}}, \dots, \bar{\xi}_{n+1})$$

as subalgebras of $C(n)_*$.

Proof. We prove this inductively, starting from $\mathbb{F}_2 \,\square_{A(n)_*} A(n)_* = \mathbb{F}_2$ and

$$A(n)_* \,\square_{A(n-1)_*} \mathbb{F}_2 = E(\xi_1^{2^n}, \bar{\xi}_2^{2^{n-1}}, \dots, \bar{\xi}_{n+1}) \,.$$

Applying $\mathbb{F}_2 \,\square_{A(n)_*} (-)$ to the (split) short exact sequence in Lemma 1.4, we get an exact sequence

$$0 \to \Sigma^{2^{n+1}} X_* \xrightarrow{\xi_*} X_* \longrightarrow \mathbb{F}_2 \to 0$$

where $X_* = \mathbb{F}_2 \,\square_{A(n)_*} C(n)_*$. We have already seen that $\xi_1^{2^{n+1}} \in C(n)_*$ lifts to X_*. It follows by an induction on degrees that the inclusion $P(\xi_1^{2^{n+1}}) \subseteq X_*$ is an equality of subalgebras of $C(n)_*$.

On the other hand, applying $(-) \,\square_{A(n-1)_*} \mathbb{F}_2$ to the same (split) short exact sequence, we get a exact sequence

$$0 \to \Sigma^{2^{n+1}} Y_* \xrightarrow{\xi_*} Y_* \longrightarrow E(\xi_1^{2^n}, \bar{\xi}_2^{2^{n-1}}, \dots, \bar{\xi}_{n+1}) \to 0$$

where $Y_* = C(n)_* \,\square_{A(n-1)_*} \mathbb{F}_2$. We can check directly that $\xi_1^{2^n}, \bar{\xi}_2^{2^{n-1}}, \dots, \bar{\xi}_{n+1} \in C(n)_*$ lift to Y_*. It follows by another induction on degrees that the inclusion

$$P(\xi_1^{2^n}) \otimes E(\bar{\xi}_2^{2^{n-1}}, \dots, \bar{\xi}_{n+1}) \subseteq Y_*$$

is in fact an equality of subalgebras of $C(n)_*$. □

Definition 1.7. Let $\gamma'_* \colon C(n)_* \twoheadrightarrow P(\xi_1^{2^{n+1}})$ be the composite of $\gamma_* \colon C(n)_* \twoheadrightarrow C(0)_*$ and the (additive, homogeneous) retraction $P(\xi_1) \twoheadrightarrow P(\xi_1^{2^{n+1}})$, and let $\beta'_* \colon B(n)_* \twoheadrightarrow P(\xi_1^{\pm 2^{n+1}})$ be the composite of $\beta_* \colon B(n)_* \twoheadrightarrow B(0)_*$ and the retraction $P(\xi_1^{\pm 1}) \twoheadrightarrow P(\xi_1^{\pm 2^{n+1}})$.

Lemma 1.8. *The horizontal composites in the commutative diagram*

$$\begin{array}{ccccc} B(n)_* & \longrightarrow & A(n)_* \otimes B(n)_* & \xrightarrow{1\otimes\beta'_*} & A(n)_* \otimes P(\xi_1^{\pm 2^{n+1}}) \\ \uparrow & & \uparrow & & \uparrow \\ C(n)_* & \longrightarrow & A(n)_* \otimes C(n)_* & \xrightarrow{1\otimes\gamma'_*} & A(n)_* \otimes P(\xi_1^{2^{n+1}}) \end{array}$$

are isomorphisms of left $A(n)_*$-comodules.

Proof. The lower horizontal composite

$$\alpha'_* \colon C(n)_* \longrightarrow A(n)_* \otimes P(\xi_1^{2^{n+1}})$$

induces an isomorphism

$$1 \,\square\, \alpha'_* \colon \mathbb{F}_2 \,\square_{A(n)_*} C(n)_* \longrightarrow P(\xi_1^{2^{n+1}}) \,,$$

because we can identify it with the composite of $\mathbb{F}_2 \square_{A(n)_*} C(n)_* \subset C(n)_*$ and $\gamma'_* \colon C(n)_* \twoheadrightarrow P(\xi_1^{2^{n+1}})$. Since $A(n)_*$ is connected and $C(n)_*$ is left $A(n)_*$-comodule injective, it follows that α'_* is also an isomorphism. [[More details?]] The upper horizontal composite

$$B(n)_* \longrightarrow A(n)_* \otimes P(\xi_1^{\pm 2^{n+1}})$$

is obtained from α'_* by inverting multiplication by $\xi_1^{2^{n+1}}$, hence is also an isomorphism. □

Lemma 1.9. *The composite of $C(n)_* \square_{A(n-1)_*} \mathbb{F}_2 \subset C(n)_*$ and $\gamma_* \colon C(n)_* \twoheadrightarrow C(0)_*$ maps $\bar{\xi}_k^{2^e}$ to $\xi_1^{2^e(2^k-1)} = \xi_1^{2^{n+1}-2^e}$ for $1 \le k \le n+1$ and $k + e = n+1$. Hence the composite is injective, with image $M(n)_* \subset P(\xi_1)$ the subalgebra generated by the classes $\xi_1^{2^{n+1}-2^e}$ for $0 \le e \le n$. This subalgebra contains all classes ξ_1^m with $m > (n-1)2^{n+1} + 1$.*

Proof. It follows from the recursive definition of $\bar\xi_k = \chi(\xi_k)$ that

$$\bar\xi_k \equiv \xi_1^{2^k-1} \mod (\xi_2, \dots, \xi_k)\,.$$

Hence a monomial generator

$$\xi_1^{j\cdot 2^{n+1}} \cdot \xi_{n+1-e_1}^{2^{e_1}} \cdots \xi_{n+1-e_r}^{2^{e_r}}$$

for $C(n)_* \,\Box_{A(n-1)_*} \mathbb{F}_2$, where $j \ge 0$, $0 \le r \le n+1$ and $0 \le e_1 < \dots < e_r \le n$, is mapped to ξ_1^m with

$$m = j\cdot 2^{n+1} + (2^{n+1} - 2^{e_1}) + \dots + (2^{n+1} - 2^{e_r}) = (j+r)2^{n+1} - (2^{e_1} + \dots + 2^{e_r})\,.$$

It follows that m uniquely determines j, r and $e_1, \dots, e_r$, so the composite homomorphism is injective. It is elementary to check that the integers m that occur are those that can be written as sums of non-negative multiples of $2^{n+1} - 2^e$ for $0 \le e \le n$. It is also elementary to check that these sums include all integers greater than $(n-1)2^{n+1} + 1$. □

Definition 1.10. Let $\gamma''_* \colon C(n)_* \twoheadrightarrow M(n)_*$ be the composite of $\gamma_* \colon C(n)_* \twoheadrightarrow C(0)_*$ and the retraction $P(\xi_1) \twoheadrightarrow M(n)_*$.

Lemma 1.11. *The composite*

$$C(n)_* \longrightarrow C(n)_* \otimes A(n-1)_* \xrightarrow{\gamma''_*\otimes 1} M(n)_* \otimes A(n-1)_*$$

is an isomorphism of right $A(n-1)_*$-comodules.

Proof. Let α''_* denote the composite right $A(n-1)_*$-comodule homomorphism. The induced homomorphism

$$\alpha''_* \,\Box\, 1 \colon C(n)_* \,\Box_{A(n-1)_*} \mathbb{F}_2 \longrightarrow M(n)_*$$

is an isomorphism, because we can identify it with the composite of $C(n)_* \,\Box_{A(n-1)_*} \mathbb{F}_2 \subset C(n)_*$ and $\gamma''_* \colon C(n)_* \twoheadrightarrow M(n)_*$. Since $A(n-1)_*$ is connected and $C(n)_*$ is right $A(n-1)_*$-comodule injective, it follows that α''_* is also an isomorphism. [[More details?]] □

Multiplication with $\xi_1^{2^{n+1}}$ on $C(n)_*$ is not directly compatible with multiplication with $\xi_1^{2^{n+1}}$ on $M(n)_*$, because the retraction $P(\xi_1) \twoheadrightarrow M(n)_*$ does not commute with such multiplication. This is why the following result does not immediately follow by localization from the previous lemma.

Lemma 1.12. *The composite*

$$B(n)_* \longrightarrow B(n)_* \otimes A(n-1)_* \xrightarrow{\beta_*\otimes 1} B(0)_* \otimes A(n-1)_*$$

is an isomorphism of right $A(n-1)_*$-comodules.

Proof. Consider the commutative diagram of right $A(n-1)_*$-comodules

$$\begin{array}{ccccc}
\Sigma^{-j\cdot 2^{n+1}} C(n)_* & \longrightarrow & \Sigma^{-j\cdot 2^{n+1}} C(0)_* \otimes A(n-1)_* & \twoheadrightarrow & \Sigma^{-j\cdot 2^{n+1}} M(n)_* \otimes A(n-1)_* \\
\downarrow & & \downarrow & & \\
B(n)_* & \longrightarrow & B(0)_* \otimes A(n-1)_* & &
\end{array}$$

where the left hand vertical map is a canonical map from the definition of $B(n)_*$ as a colimit, and the upper horizontal composite is the isomorphism α''_* of Lemma 1.11. For each fixed degree, there exists a sufficiently large j such that the two vertical maps and the upper right hand horizontal map are all isomorphism in that degree. This uses that $A(n)_*$, $A(n-1)_*$ and the kernel of $C(0)_* \twoheadrightarrow M(n)_*$ are bounded above and below. Hence the two remaining horizontal maps are also isomorphisms in the fixed degree. Since the lower horizontal map does not depend on j, it is an isomorphism in every degree. $\square$

2. The dual bimodule coalgebras

We now dualize the definitions and results of the previous section.

Definition 2.1. Let $\mathscr{A} = (\mathscr{A}_*)^*$ be the Steenrod algebra, let

$$A(n) = (A(n)_*)^*$$

be the Hopf algebra dual to $A(n)_*$, let

$$C(n) = (C(n)_*)^*$$

be the $A(n)$-$A(n-1)$-bimodule coalgebra dual to $C(n)_*$, and let

$$B(n) = (B(n)_*)^* = \lim_j \left(\Sigma^{-j\cdot 2^{n+1}} C(n)\right)$$

be the $A(n)$-$A(n-1)$-bimodule coalgebra dual to $B(n)_*$.

Then $A(n) \rightarrowtail A(n+1) \rightarrowtail \mathscr{A}$ are Hopf algebra homomorphisms, there is a short exact sequence

$$0 \to A(n) \to C(n) \xrightarrow{\xi} \Sigma^{2^{n+1}} C(n) \to 0$$

of $A(n)$-$A(n-1)$-bimodules, where ξ is dual to ξ_*, and the diagram

$$\begin{array}{ccccc}
 & & B(n+1) & \longleftarrow\!\!\prec & B(n) \\
 & & \downarrow\!\!\!\downarrow & & \downarrow\!\!\!\downarrow \\
\mathscr{A} & \longleftarrow\!\!\prec & C(n+1) & \longleftarrow\!\!\prec & C(n) \\
 & & \uparrow & & \uparrow \\
 & & A(n+1) & \longleftarrow\!\!\prec & A(n)
\end{array}$$

consists of $A(n)$-$A(n-1)$-bimodule coalgebra homomorphisms. There is an isomorphism

$$C(n) \cong A(n) \otimes (\mathbb{F}_2 \otimes_{A(n)} C(n))$$

of left $A(n)$-modules and right $\mathbb{F}_2 \otimes_{A(n)} C(n)$-comodules, and an isomorphism

$$C(n) \cong (C(n) \otimes_{A(n-1)} \mathbb{F}_2) \otimes A(n-1)$$

of right $A(n-1)$-modules and left $C(n) \otimes_{A(n-1)} \mathbb{F}_2$-comodules. In particular, $C(n)$ is free as a left $A(n)$-module and also as a right $A(n-1)$-module (but not as an $A(n)$-$A(n-1)$-bimodule).

Definition 2.2. Let $C(0) = \mathbb{F}_2\{Sq^k \mid k \geq 0\}$ and $B(0) = \mathbb{F}_2\{Sq^k \mid k \in \mathbb{Z}\}$, with Sq^k dual to ξ_1^k. The surjection $B(0) \twoheadrightarrow C(0)$ maps Sq^k to 0 for $k < 0$. Let $\beta \colon B(0) \rightarrowtail B(n)$ be dual to β_*, and let $\beta' \colon \mathbb{F}_2\{Sq^{j\cdot 2^{n+1}} \mid j \in \mathbb{Z}\} \rightarrowtail B(n)$ be the restriction of β to the subspace spanned by the Sq^k with $2^{n+1}|k$.

Proposition 2.3. *(a) The composite*

$$A(n) \otimes \mathbb{F}_2\{Sq^{j\cdot 2^{n+1}} \mid j \in \mathbb{Z}\} \xrightarrow{1\otimes\beta'} A(n) \otimes B(n) \longrightarrow B(n)$$

is an isomorphism of left $A(n)$-modules. Hence $B(n)$ is a free left $A(n)$-module with basis $\{Sq^{j\cdot 2^{n+1}} \mid j \in \mathbb{Z}\}$.

(b) The composite

$$B(0) \otimes A(n-1) \xrightarrow{\beta\otimes 1} B(n) \otimes A(n-1) \longrightarrow B(n)$$

is an isomorphism of right $A(n-1)$-modules. Hence $B(n)$ is a free right $A(n-1)$-module with basis $\{Sq^k \mid k \in \mathbb{Z}\}$.

3. The algebraic Singer construction

Recall that $B(n)$ is an $A(n)$-$A(n-1)$-bimodule. For any left $A(n-1)$-module M, the tensor product $B(n) \otimes_{A(n-1)} M$ becomes a left $A(n)$-module. If M is a left $A(n)$-module, the inclusion $B(n) \rightarrowtail B(n+1)$ of $A(n)$-$A(n-1)$-bimodules induces a left $A(n)$-module homomorphism

$$B(n) \otimes_{A(n-1)} M \longrightarrow B(n+1) \otimes_{A(n)} M \,. \tag{1}$$

It is compatible with the isomorphisms

$$B(0) \otimes M \cong B(0) \otimes A(n-1) \otimes_{A(n-1)} M \xrightarrow{\cong} B(n) \otimes_{A(n-1)} M$$

and

$$B(0) \otimes M \cong B(0) \otimes A(n) \otimes_{A(n)} M \xrightarrow{\cong} B(n+1) \otimes_{A(n)} M$$

from Proposition 2.3(b), hence the left $A(n)$-module homomorphism (1) is in fact an isomorphism.

Definition 3.1. For any left $\mathscr{A}$-module M let the *Singer construction*

$$R_+(M) = \operatorname*{colim}_n \big(B(n) \otimes_{A(n-1)} M\big)$$

be the $\mathscr{A}$-module whose underlying $A(n)$-module structure respects the isomorphism $B(n) \otimes_{A(n-1)} M \xrightarrow{\cong} R_+(M)$, for each $n \geq 0$.

For any $\mathscr{A}$-module M, the composite $A(n)$-$A(n-1)$-bimodule homomorphism $B(n) \twoheadrightarrow C(n) \rightarrowtail \mathscr{A}$ induces an $A(n)$-module homomorphism

$$B(n) \otimes_{A(n-1)} M \longrightarrow \mathscr{A} \otimes_{A(n-1)} M \,.$$

For varying n these are compatible with the isomorphisms in the colimit system defining $R_+(M)$ and the evident surjective homomorphisms

$$\mathscr{A} \otimes_{A(n-1)} M \twoheadrightarrow \mathscr{A} \otimes_{A(n)} M \,,$$

hence induce a homomorphism

$$R_+(M) \longrightarrow \operatorname*{colim}_n \big(\mathscr{A} \otimes_{A(n)} M\big) \cong \mathscr{A} \otimes_{\mathscr{A}} M \cong M \,.$$

Definition 3.2. For each left $\mathscr{A}$-module M let the *augmentation*

$$\epsilon \colon R_+(M) \longrightarrow M$$

be the $\mathscr{A}$-module homomorphism whose underlying $A(n)$-module homomorphism is given by the composite

$$B(n) \otimes_{A(n-1)} M \longrightarrow \mathscr{A} \otimes_{A(n-1)} M \longrightarrow M \,.$$

Theorem 3.3 (W. H. Lin)**.** *The augmentation ϵ is a* Tor*-isomorphism, in the sense that*

$$\epsilon_\# \colon \operatorname{Tor}^{\mathscr{A}}_{*,*}(\mathbb{F}_2, R_+(M)) \longrightarrow \operatorname{Tor}^{\mathscr{A}}_{*,*}(\mathbb{F}_2, M)$$

is an isomorphism in all bidegrees, for all left $\mathscr{A}$-modules M.

Lemma 3.4. *(a) The functor R_+ from left $\mathscr{A}$-modules to left $\mathscr{A}$-modules is exact.*

(b) If M is free then $R_+(M)$ is flat.

(c) If M is free then $1 \otimes \epsilon \colon \mathbb{F}_2 \otimes_{\mathscr{A}} R_+(M) \longrightarrow \mathbb{F}_2 \otimes_{\mathscr{A}} M$ is an isomorphism.

Proof. (a) Let $0 \to M' \to M \to M'' \to 0$ be a short exact sequence of left $\mathscr{A}$-modules. In particular, it is a short exact sequence of left $A(n-1)$-modules. Since $B(n)$ is free, hence flat, as a right $A(n-1)$-module, the sequence

$$0 \to B(n) \otimes_{A(n-1)} M' \to B(n) \otimes_{A(n-1)} M \to B(n) \otimes_{A(n-1)} M'' \to 0$$

of left $A(n)$-modules remains exact. Hence

$$0 \to R_+(M') \to R_+(M) \to R_+(M'') \to 0$$

is a short exact sequence of left $\mathscr{A}$-modules.

(b) It suffices to prove this when $M = \mathscr{A}$. For each $n \geq 0$ we have that $\mathscr{A}$ is free as a left $A(n-1)$-module, hence $R_+(\mathscr{A})$ is a direct sum of suspensions of copies of $B(n)$ as a left $A(n)$-module. Since $B(n)$ is free as a left $A(n)$-module, it follows that $R_+(\mathscr{A})$ is free, hence flat, as a left $A(n)$-module. Thus

$$\mathrm{Tor}^{\mathscr{A}}_{s,t}(N, R_+(\mathscr{A})) = \operatorname*{colim}_n \mathrm{Tor}^{A(n)}_{s,t}(N, R_+(\mathscr{A})) = 0$$

for each right $\mathscr{A}$-module N and every positive $s \geq 1$, which implies that $R_+(\mathscr{A})$ is flat as left $\mathscr{A}$-module.

(c) It suffices to prove this when $M = \mathscr{A}$. When viewed as a left $A(n-1)$-module, $\mathscr{A} = \mathrm{colim}_{m \geq n-1} A(m)$, so we can realize the source $\mathbb{F}_2 \otimes_{\mathscr{A}} R_+(\mathscr{A})$ of $1 \otimes \epsilon$ as the colimit

$$\begin{aligned}
\operatorname*{colim}_n \big(\mathbb{F}_2 \otimes_{A(n)} R_+(\mathscr{A})\big) &= \operatorname*{colim}_n \big(\mathbb{F}_2 \otimes_{A(n)} B(n) \otimes_{A(n-1)} \mathscr{A}\big) \\
&= \operatorname*{colim}_{m \geq n-1} \big(\mathbb{F}_2 \otimes_{A(n)} B(n) \otimes_{A(n-1)} A(m)\big) \\
&= \operatorname*{colim}_n \big(\mathbb{F}_2 \otimes_{A(n)} B(n) \otimes_{A(n-1)} A(n-1)\big) \\
&\cong \operatorname*{colim}_n \big(\mathbb{F}_2 \otimes_{A(n)} B(n)\big)
\end{aligned}$$

since the pairs (m, n) with $m = n-1$ are cofinal among the pairs (m, n) with $m \geq n-1$ in the product partial ordering.

By Proposition 2.3(a), we know that the left $A(n)$-module $B(n)$ is free on the generators Sq^k for $2^{n+1}|k$. The homomorphism

$$\mathbb{F}_2 \otimes_{A(n)} B(n) \longrightarrow \mathbb{F}_2 \otimes_{A(n+1)} B(n+1)$$

is induced by the left $A(n)$-module homomorphism $B(n) \rightarrowtail B(n+1)$ followed by the canonical surjection $\mathbb{F}_2 \otimes_{A(n)} B(n+1) \twoheadrightarrow \mathbb{F}_2 \otimes_{A(n+1)} B(n+1)$. It can therefore be rewritten as a homomorphism

$$\mathbb{F}_2\{Sq^k \mid 2^{n+1}|k\} \longrightarrow \mathbb{F}_2\{Sq^k \mid 2^{n+2}|k\}$$

sending Sq^k to 0 when k is of the form $j \cdot 2^{n+1}$ with j odd, and mapping Sq^0 to Sq^0. Hence

$$\operatorname*{colim}_n \big(\mathbb{F}_2 \otimes_{A(n)} B(n)\big) \cong \mathbb{F}_2\,,$$

sending $1 \otimes Sq^0$ to the generator.

Clearly the target of ϵ is $\mathbb{F}_2 \otimes_{\mathscr{A}} \mathscr{A} \cong \mathbb{F}_2$. To check that $1 \otimes \epsilon \colon \mathbb{F}_2 \to \mathbb{F}_2$ is an isomorphism it suffices to check that the class $1 \otimes Sq^0 \in \mathbb{F}_2 \otimes_{A(0)} B(0)$, which maps to the generator in the source, also maps to the generator in the target, but this is clear since $Sq^0(1) = 1$. □

Proof of Theorem 3.3. Let

$$\cdots \to F_2 \to F_1 \to F_0 \to M \to 0$$

be a resolution of M by free left $\mathscr{A}$-modules. Then by exactness and flatness

$$\cdots \to R_+(F_2) \to R_+(F_1) \to R_+(F_0) \to R_+(M) \to 0$$

is a resolution of $R_+(M)$ by flat left $\mathscr{A}$-modules. Hence $\mathrm{Tor}^{\mathscr{A}}_{*,*}(\mathbb{F}_2, R_+(M))$ is the homology of the complex

$$\cdots \to \mathbb{F}_2 \otimes_{\mathscr{A}} R_+(F_2) \to \mathbb{F}_2 \otimes_{\mathscr{A}} R_+(F_1) \to \mathbb{F}_2 \otimes_{\mathscr{A}} R_+(F_0) \to 0\,.$$

Since each F_s is free, ϵ induces an isomorphism from the latter chain complex to the complex

$$\cdots \to \mathbb{F}_2 \otimes_{\mathscr{A}} F_2 \to \mathbb{F}_2 \otimes_{\mathscr{A}} F_1 \to \mathbb{F}_2 \otimes_{\mathscr{A}} F_0 \to 0\,,$$

whose homology computes $\mathrm{Tor}^{\mathscr{A}}_{*,*}(\mathbb{F}_2, M)$. Hence the induced homomorphism $\epsilon_\#$ of homology groups is also an isomorphism. □

[[Recognize $R_+(\mathbb{F}_2) = \operatorname{colim}_n B(n) \otimes_{A(n-1)} \mathbb{F}_2$ as $\Sigma P(x^{\pm 1})$.]]

References

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