Chromatic Redshift

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Outline

1. K-Theoretical Redshift
   - Iteration, Height and Nesting
   - Redshift in Algebraic K-Theory
   - Spectral Lichtenbaum–Quillen Conjectures
2. Topological Cyclic Redshift
   - The Cyclotomic Trace Map
   - Circle-Equivariant Redshift
   - Beyond Elliptic Cohomology
3. Variations
   - Waldhausen's Localization Tower
   - Infinite Complexity
   - Higher Redshift

Outline

Iterated Modulation

$R, +, \times$	commutative ring
$\mathrm{Mod}(R), \oplus, \otimes$	bipermutative category
$\mathrm{Mod}(\mathrm{Mod}(R))$	ring-like 2-category
$\dots$	
$\mathrm{Mod}^{(n)}(R)$	ring-like n-category
$\dots$	

Iterated K-Theory

B	commutative S-algebra
$K(B)$	algebraic K-theory spectrum
$K(K(B))$	double algebraic K-theory
$\dots$	
$K^{(n)}(B)$	n-fold algebraic K-theory
$\dots$	

Height of Formal Group Laws

- Hesselholt–Madsen '97: Chromatic filtration of iterated topological cyclic homology?
- Chromatic filtration of iterated algebraic K-theory?
- Formal coproduct on $K^{(n)}(B)^*(\mathbb{C}P^\infty)$?
- $$F(x_1, x_2) = x_1 + x_2 + \dots \qquad \text{formal group law}$$
  $$[p]_F(x) \doteq x^{p^n} + \dots \qquad \text{height } n$$

Redshift

B	$\overset{K}{\longmapsto}$	$K(B)$
ring spectrum		ring spectrum
FGL of height n		FGL of height $n+1$
v_n-periodic		v_{n+1}-periodic
$\lvert v_n \rvert = 2p^n - 2$	$<$	$\lvert v_{n+1} \rvert = 2p^{n+1} - 2$
		longer wavelength, less energy

(Co-)Homological Incarnation

Nested Hopf subalgebras

$$0 \subset \cdots \subset \mathscr{E}(n) = E(Q_0, \dots, Q_n) \subset \cdots$$

in Steenrod algebra $\mathscr{A} = H^*(H)$.

Nested $\mathscr{A}_*$-comodule subalgebras

$$\mathscr{A}_* \supset \cdots \supset (\mathscr{A}//\mathscr{E}(n))^* = P(\bar{\xi}_k \mid k \geq 1) \otimes E(\bar{\tau}_k \mid k > n) \supset \cdots$$

invariant under Dyer–Lashof operations on $\mathscr{A}_* = H_*(H)$.

Outline

K of Finite Fields

Finite field k, characteristic $p > 0$

Theorem (Quillen '72)

$H_i(BGL(\bar{k}); \mathbb{F}_p) = 0$ for $i > 0$.

- $K(\bar{k})_p \simeq H\mathbb{Z}_p$
- $\pi_* K(k)_p = \pi_* K(\bar{k})_p^{hG_k}$ for $* \geq 0$
- $K(k)_p \simeq H\mathbb{Z}_p$
- Mult. by p injective on $\pi_* K(\bar{k})_p$ and $\pi_* K(k)_p$
- $p \in \pi_0 S$ lifts $v_0 \in \pi_0 BP$.

Lichtenbaum Conjecture

Separably closed field $\bar{F}$, characteristic $\neq p$

Conjecture (Lichtenbaum)

$\pi_t K(\bar{F})_p = \mathbb{Z}_p$ for $t \geq 0$ even, 0 for t odd.

- Proved by Suslin '84
- $K(\bar{F})_p \simeq ku_p$
- $\hat{L}_1 K(\bar{F}) \simeq KU_p$ where $\hat{L}_n = L_{K(n)}$
- Mult. by $u \in \pi_2 ku_p$ bijective on $\pi_* K(\bar{F})_p$ for $* \geq 0$.

K of Number Rings

Number field F, ring of S-integers A

$$\begin{array}{ccccc} A & \longrightarrow & & \longrightarrow & F \\ \uparrow & & & & \uparrow \\ \mathbb{Z} & \longrightarrow & \mathbb{Z}[1/p] & \longrightarrow & \mathbb{Q} \end{array}$$

Quillen Conjecture

Conjecture (Quillen '75)

$$E^2_{s,t} = H^{-s}_{ét}(\mathrm{Spec}\, A; \mathbb{Z}_p(t/2)) \Longrightarrow \pi_{s+t} K(A)_p$$

converging for $s + t \geq 1$.

- $\mathbb{Z}_p(t/2) = \pi_t K(\bar{F})_p$
- $\pi_* K(F)_p = \pi_* K(\bar{F})_p^{hG_F}$ for $* \geq 1$
- Mult. by $\beta \in \pi_{2p-2}(S/p)$ bijective on $\pi_*(K(A); \mathbb{Z}/p)$, for $* \geq 1$
- β lifts $u^{p-1} \in \pi_*(ku; \mathbb{Z}/p)$ and $v_1 \in \pi_* BP$.

Partial and Full Verifications

- Thomason '85:

$$\pi_*(K(F); \mathbb{Z}/p)[1/\beta] = \pi_*(K(\bar{F})^{hG_F}; \mathbb{Z}/p)$$

  for $* \geq 2$.
- Waldhausen '84: Is $K(A) \to L_1 K(A)$ a p-adic equivalence in high degrees, where $L_n = L_{E(n)}$?
- Bökstedt–Madsen '94, '95, R. '99, Hesselholt–Madsen '03: Confirmed for local number fields.
- Voevodsky '03, '11: Confirmed for (global number) fields by proof of Milnor and Bloch–Kato conjectures.

K of Topological K-Theory

Adams summand $L = E(1)$ of $KU_{(p)}$, conn. cover $\ell = BP\langle 1\rangle$

Theorem (Ausoni-R. '02)

$$V(1)_*K(\ell_p) = P(v_2) \otimes \Big[E(\lambda_1, \lambda_2, \partial) \\ \oplus E(\lambda_2) \otimes \mathbb{F}_p\{\lambda_1 t^d \mid 0 < d < p\} \\ \oplus E(\lambda_1) \otimes \mathbb{F}_p\{\lambda_2 t^{dp} \mid 0 < d < p\}\Big]$$

in degrees $\geq 2p-2$, where $(\dots)$.

K of Topological K-Theory, II

Adams summand $L = E(1)$ of $KU_{(p)}$, conn. cover $\ell = BP\langle 1 \rangle$

Theorem (Ausoni-R. '02, Ausoni '10)

$V(1)_*K(\ell_p) = (\dots)$ *and* $V(1)_*K(ku_p) = (\dots)$.

- $V(1) = S/(p, v_1) = S \cup_p e^1 \cup_{\alpha_1} e^{2p-1} \cup_p e^{2p}$
- Blumberg–Mandell '08: Also $V(1)_*K(L_p)$ and $V(1)_*K(KU_p)$
- Mult. by $v_2 \in \pi_{2p^2-2}V(1)$ bijective on each answer, for $* \geq 2p - 2$
- v_2 lifts $v_2 \in \pi_*BP$.

Outline

Quillen Conjecture, II

ℓ_p-algebra of S-integers B

Conjecture (à la Quillen/Voevodsky)

$$E^2_{s,t} = H^{-s}_{\mathrm{mot}}(\mathrm{Spec}\, B; \mathbb{F}_{p^2}(t/2)) \Longrightarrow V(1)_{s+t}K(B)$$

converging for $s + t \gg 0$.

- $H^*_{\mathrm{mot}}(-)$ motivic cohomology for commutative S-algebras?
- $\mathbb{F}_{p^2}(t/2) = V(1)_t E_2$, where $\pi_* E_2 = \mathbb{W}\mathbb{F}_{p^2}[[u_1]][u^{\pm 1}]$
- Need $\mathbb{F}_{p^2}$ due to sign in Ausoni's relation $b^{p-1} = -v_2$.

ℓ_p-Algebra of Integers

Definition (ℓ_p-algebra of integers B)

A connected commutative ℓ_p-algebra over a Galois extension of $L_p[1/p]$, semi-finite as ℓ_p-module.

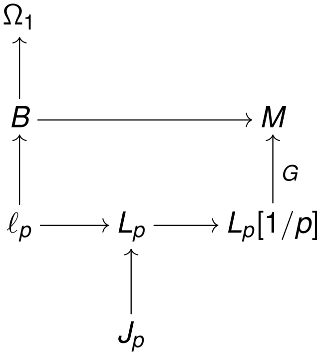
$$\begin{array}{ccccc} B & \longrightarrow & & \longrightarrow & M \\ \uparrow & & & & \uparrow G \\ \ell_p & \longrightarrow & L_p & \longrightarrow & L_p[1/p] \end{array}$$

Examples!? For S-integers, allow localizations.

ℓ_p-Algebraic Integers

Definition (ℓ_p-algebraic integers Ω_1)

p-completed homotopy colimit of all such B.



Schloß Ringberg '99 Conjecture

Conjecture (à la Lichtenbaum/Suslin)

- *Mult. by v_2 bijective on $V(1)_* K(B)$ for $* \gg 0$.*
- $V(1)_* K(\Omega_1) = V(1)_* E_2$ *for* $* \gg 0$.
- $\hat{L}_2 K(\Omega_1) \simeq E_2$.

- For G-Galois $B \to \Omega_1$, expect $V(1)_* K(B) = V(1)_* K(\Omega_1)^{hG}$ for $* \gg 0$
- Ring spectrum map $K(ku) \to E_2$?

e_n-Algebra of Integers

Lubin–Tate spectrum E_n, connective cover e_n

Definition (e_n-algebra of integers B)

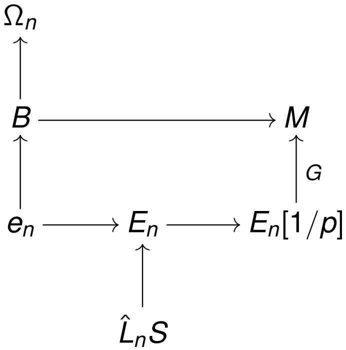
A connected commutative e_n-algebra over a Galois extension of $E_n[1/p]$, semi-finite as e_n-module.

$$\begin{array}{ccccc} B & \longrightarrow & & \longrightarrow & M \\ \uparrow & & & & \uparrow{\scriptstyle G} \\ e_n & \longrightarrow & E_n & \longrightarrow & E_n[1/p] \end{array}$$

e_n-Algebraic Integers

Definition (e_n-algebraic integers Ω_n)

p-completed homotopy colimit of all such B.



Finite Localizations

- Hopkins–Smith '98: Finite p-local spectrum F, with v_{n+1} self map $v \colon \Sigma^d F \to F$.
- Miller '92: Finite localization L^f_{n+1} annihilates finite $E(n+1)$-acyclic spectra.
- $F_* L^f_{n+1} X = F_* X[1/v]$ for any X.

Schloß Ringberg ’99 Conjecture, II

$e_n \to B \to \Omega_n$ and (F, v) as above

Conjecture

- *Mult. by v bijective on* $F_*K(B)$ *for* $* \gg 0$.
- $K(B) \to L^f_{n+1}K(B)$ *a p-adic equivalence in high degrees.*
- $F_*K(\Omega_n) = F_*E_{n+1}$ *for* $* \gg 0$.
- $\hat{L}_{n+1}K(\Omega_n) \simeq E_{n+1}$.

Outline

Topological Hochschild Homology

- B commutative S-algebra
  $\sim$ functions on X
- $THH(B) = B \otimes S^1$ topological Hochschild homology
  $\sim$ functions on free loop space $\mathscr{L}X$
- $THH(B)^{hS^1} = F(ES^1_+, THH(B))^{S^1}$ homotopy fixed points
  $\sim$ functions on Borel construction $ES^1_+ \wedge_{S^1} \mathscr{L}X$
- $THH(B)^{tS^1} = [\widetilde{ES}^1 \wedge F(ES^1_+, THH(B))]^{S^1}$ Tate construction
  $\sim$ functions on periodicized Borel construction.

Trace Maps

$$
\begin{array}{ccccc}
 & & THH(B)^{hC_{p^n}} & & \\
 & \nearrow & \uparrow & & \\
K(B) & \longrightarrow & THH(B)^{C_{p^n}} & \longrightarrow & THH(B) \\
 & \searrow & \downarrow & & \\
 & & THH(B)^{tC_{p^{n+1}}} & &
\end{array}
$$

Trace Maps, II

$$
\begin{array}{ccccc}
 & \nearrow & THH(B)_p^{hS^1} & & \\
 & & \uparrow & & \\
K(B) & \longrightarrow & TF(B;p) & \longrightarrow & THH(B) \\
 & & \downarrow & & \\
 & \searrow & THH(B)_p^{tS^1} & &
\end{array}
$$

Traces of Redshift

- No redshift in $THH(B)$.
- All redshift yet seen in $K(B)$ also visible in $THH(B)^{tS^1}$.
- Detectable in $\mathscr{A}_*$-coaction on homology.

Outline

Homological Approach

e_n-algebra of integers B

$$\begin{array}{ccccccc} & & E(n) & \longrightarrow & E_n & & \\ & & \uparrow & & \uparrow & & \\ H & \longleftarrow & BP\langle n\rangle & \longrightarrow & e_n & \longrightarrow & B \end{array}$$

- $H_*(e_n)$ awkward for $n \geq 2$
- $H_*(BP\langle n\rangle) = P(\bar{\xi}_k \mid k \geq 1) \otimes E(\bar{\tau}_k \mid k > n)$
- Subalgebra of $\mathscr{A}_* = P(\bar{\xi}_k \mid k \geq 1) \otimes E(\bar{\tau}_k \mid k \geq 0)$
- $H_*(B)$ commutative $H_*(BP\langle n\rangle)$-algebra

v_n-Periodic Input

Adams spectral sequence

$$E_2^{s,t}(B) = \mathrm{Ext}_{\mathscr{A}_*}^{s,t}(\mathbb{F}_p, H_*(B)) \Longrightarrow \pi_{t-s}(B)$$

algebra over

$$E_2^{*,*} = P(v_0, \dots, v_n)$$

converging to

$$\pi_* BP\langle n \rangle_p = \mathbb{Z}_p[v_1, \dots, v_n]$$

generating v_n-periodicity in $\pi_*(B)$.

Add Circle Action

Circle action gives suspension operator σ.
Bökstedt spectral sequence

$$E^2_{s,t}(B) = HH_s(H_*(B))_t \Longrightarrow H_{s+t}(THH(B))$$

algebra over

$$E^2_{*,*} = H_*(BP\langle n\rangle) \otimes E(\sigma\bar{\xi}_k \mid k \geq 1) \otimes \Gamma(\sigma\bar{\tau}_k \mid k > n)$$

converging to $H_*(THH(BP\langle n\rangle))$.

Differentials and Extensions

Dyer–Lashof operations

$$Q^{p^k}(\bar{\tau}_k) = \bar{\tau}_{k+1}$$

imply multiplicative extensions $(\sigma\bar{\tau}_k)^p = \sigma\bar{\tau}_{k+1}$ and differentials

$$d^{p-1}(\gamma_j\sigma\bar{\tau}_k) \doteq \sigma\bar{\xi}_{k+1} \cdot \gamma_{j-p}\sigma\bar{\tau}_k$$

for $k > n$, $j \geq p$, leaving $E^p_{*,*} = E^\infty_{*,*}$ converging to

$$H_*(THH(BP\langle n\rangle)) = H_*(BP\langle n\rangle) \otimes E(\sigma\bar{\xi}_1, \dots, \sigma\bar{\xi}_{n+1}) \otimes P(\sigma\bar{\tau}_{n+1}) .$$

Remove Circle Action

Homological Tate spectral sequence

$$E^2_{s,t}(B) = \hat{H}^{-s}(S^1; H_t(THH(B))) \Longrightarrow H^c_{s+t}(THH(B)^{tS^1})$$

algebra over

$$E^2_{*,*} = P(t^{\pm 1}) \otimes H_*(THH(BP\langle n\rangle))$$

converging to $H^c_*(THH(BP\langle n\rangle)^{tS^1})$.

Differentials, II

Circle invariance gives differentials

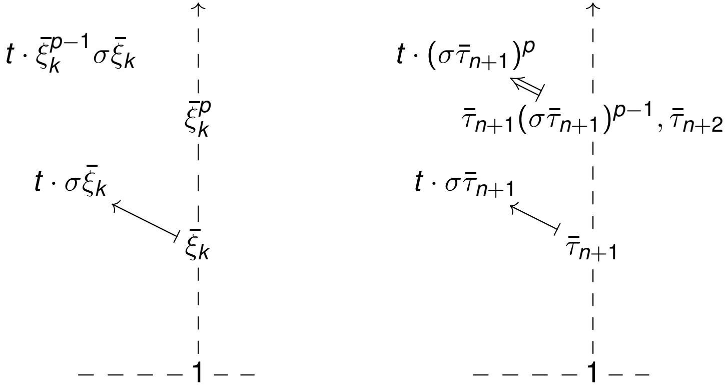
$$d^2(t^i \cdot x) = t^{i+1} \cdot \sigma x$$

leaving

$$E^3_{*,*} = P(t^{\pm 1}) \otimes P(\bar\xi_1^p, \dots, \bar\xi_{n+1}^p, \bar\xi_k \mid k > n+1) \otimes E(\tau_k' \mid k > n+1) \otimes E(\bar\xi_1^{p-1}\sigma\bar\xi_1, \dots, \bar\xi_{n+1}^{p-1}\sigma\bar\xi_{n+1})$$

where $\tau_k' = \bar\tau_k - \bar\tau_{k-1}(\sigma\bar\tau_{k-1})^{p-1}$ for $k > n+1$.

(E^2, d^2)-Charts



$1 \leq k \leq n+1$

Extensions, II

- Bruner–R. '05: Often collapses at $E^3 = E^\infty$.
- Lunøe-Nielsen–R. '11, '12, Knut Berg: Often $\mathscr{A}_*$-comodule extensions combining copies of

$$P(\bar\xi_1^p, \dots, \bar\xi_{n+1}^p, \bar\xi_k \mid k > n+1) \otimes E(\tau_k' \mid k > n+1)$$

to

$$P(\bar\xi_k \mid k \geq 1) \otimes E(\tau_k' \mid k > n+1) \cong H_*(BP\langle n+1\rangle)\,.$$

- Lose $\bar\tau_{n+1}$ to kill $\sigma\bar\tau_{n+1}$.

v_{n+1}-Periodic Output

- $H_*^c(THH(B)^{tS^1})$ algebra over $H_*^c(THH(BP\langle n\rangle)^{tS^1})$, typically with associated graded

$$P(t^{\pm p^{n+1}}) \otimes H_*(BP\langle n+1\rangle) \otimes E(\nu_1, \dots, \nu_{n+1})\,.$$

- Limit of Adams spectral sequences

$$E_2^{s,t}(B) = \mathrm{Ext}_{\mathscr{A}_*}^{s,t}(\mathbb{F}_p, H_*^c(THH(B)^{tS^1})) \Longrightarrow \pi_{t-s} THH(B)^{tS^1}$$

algebra over $E_2^{*,*}(BP\langle n\rangle)$, containing factors like

$$\mathrm{Ext}_{\mathscr{A}_*}^{*,*}(\mathbb{F}_p, H_*(BP\langle n+1\rangle)) = P(v_0, \dots, v_n, v_{n+1})\,.$$

State of Affairs

- Room for differentials due to exterior factor $E(\nu_1, \ldots, \nu_{n+1})$.
- Might truncate the periodic v_{n+1}-action.
- Empirically this does not happen.
- A general explanation is currently lacking.

Outline

Hyperelliptic Cohomology

- Topological modular forms *tmf*.
- $\pi_*(tmf)$ is v_2-periodic.
- Do $\pi_* K(tmf)$ and $\pi_* THH(tmf)^{tS^1}$ detect v_3-periodic families in $\pi_* S$?
- Work in progress for $p = 2$ with Bruner '08.

Outline

Chromatic and Telescopic Localizations

$$
\begin{array}{ccccccccccc}
 & & & & E_n & & & & KU_p & & \\
 & & & & \uparrow{\scriptstyle \mathbb{G}_n} & & & & \uparrow{\scriptstyle \mathbb{Z}_p^\times} & & \\
 & & & & \hat{L}_n S & & & & J_p & & H\mathbb{Q} \\
 & & & & \uparrow & & & & \uparrow & & \uparrow{\scriptstyle \simeq} \\
S_{(p)} & \longrightarrow & \dots & \longrightarrow & L_n^f S & \longrightarrow & L_{n-1}^f S & \longrightarrow \dots \longrightarrow & L_1^f S & \longrightarrow & L_0^f S
\end{array}
$$

- Ravenel's Telescope Conjecture '84: $L_n^f S \xrightarrow{\simeq} L_n S$?
- True for $n \leq 1$.
- Mahowald–Ravenel–Shick '01: Probably false for $n \geq 2$.

K-Theory Localization Tower

Theorem (Waldhausen '84, trading L_n for L_n^f)

Tower of fiber sequences for $n \geq 1$

$$\begin{array}{ccccccccc} & & & & K(\mathscr{C}_n, w_n) & & \dots & & \\ & & & & \downarrow & & & & \\ K(S_{(p)}) & \longrightarrow & \dots & \longrightarrow & K(L_n^f S) & \longrightarrow & K(L_{n-1}^f S) & \longrightarrow \dots \longrightarrow & K(\mathbb{Q}) \end{array}$$

- $\mathscr{C}_n$ category of finite spectra of type $\geq n$
- w_n subcategory of $E(n)$-equivalences

Monochromatic K-Theory

- $\mathscr{K}_n^{sm}$ category of small $K(n)$-local spectra
- $\mathscr{E}_n^{df}$ category of E_n-module spectra with each π_i finite

Proposition (Hovey–Strickland '99)

$(\mathscr{C}_n, w_n) \to (\mathscr{K}_n^{sm}, h)$ is an idempotent completion.

$$\pi_i K(\mathscr{C}_n, w_n) \cong \pi_i K(\mathscr{K}_n^{sm}) \qquad \text{for } i > 0.$$

Base change along $\hat{L}_n S \to E_n$ takes $\mathscr{K}_n^{sm}$ to $\mathscr{E}_n^{df}$.

Conjecture (à la wishful thinking)

$$K(\mathscr{K}_n^{sm}) \longrightarrow K(\mathscr{E}_n^{df})^{h\mathbb{G}_n}$$

close to an equivalence.

Broken Dévissage

- Express $K(\mathscr{E}_n^{df})$ using K of E_n and localizations.
- $n = 1$ for simplicity.
- Barwick: Fiber sequence

$$K(\mathscr{E}_1^{df}) \longrightarrow K(KU_p) \longrightarrow K(KU_p[1/p])\,.$$

- Ausoni–R. '12: Transfer map

$$K(KU/p) \longrightarrow K(\mathscr{E}_1^{df})$$

is far from an equivalence.

Congregation

$$
\begin{array}{ccccccc}
 & & K(\mathscr{E}_n^{df}) & \longrightarrow & K(E_n) & & \\
 & & \uparrow{\scriptstyle \mathbb{G}_n} & & & & \\
K(\mathscr{C}_n, w_n) & \longrightarrow & K(\mathscr{K}_n^{sm}) & & & & \\
 & \searrow & & & & & \\
\dots \longrightarrow & & K(L_n^f S) & \longrightarrow & K(L_{n-1}^f S) & \longrightarrow & \dots
\end{array}
$$

- $K(E_n)$ governs change in K-theory along $L_n^f S \to L_{n-1}^f S$.

Outline

K of the Sphere Spectrum

- Waldhausen et al.: $K(S)$ geometrically important.
- R. '02: Compute $\mathscr{A}$-module $H^*(K(S))$ for $p = 2$.
- R. '03: Compute $\mathscr{A}$-module $H^*(K(S))$ for p regular, up to an extension.
- Gives $\pi_* K(S)_p$ in a range.
- As complicated as $\pi_* S$.

K of Complex Bordism

- $S \to MU \to H$ halfway house.
- S is totalization of cosimplicial spectrum

$$[q] \mapsto MU \wedge MU^{\wedge q} .$$

- Is $K(S)$ close to totalization of

$$[q] \mapsto K(MU \wedge MU^{\wedge q}) ?$$

- Seek conceptual understanding of $K(S)$ by Hopf–Galois descent from $K(MU)$.
- Pursued by Bruner–R. '05, R. '08, R. '09 and Lunøe-Nielsen–R. '11.

Outline

Higher Redshift

- B commutative S-algebra, G Lie group of rank k.
- Study
$$B \longmapsto (B \otimes G)^{hG}$$
or Tate-like construction.
- Expect shift from v_n- to v_{n+k}-periodicity.
- Carlsson–Dundas et al. '10, '11: $G = T^k$.
- R. '11: General G.
- R. '08, Torleif Veen '13: Partial verifications.