Algebraic K-theory of group rings and topological cyclic homology

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Outline

Credits

This is an overview of joint work with

- Wolfgang Lück,
- Holger Reich and
- Marco Varisco.

Outline

Borel conjecture

Conjecture (Borel (1953))

Let G be any discrete group. Any two closed manifolds of the homotopy type of BG are homeomorphic.

This is an analogue of the Poincaré conjecture for aspherical manifolds.

Novikov conjecture

The integral Pontryagin classes $p_i(TM) \in H^{4i}(M;\mathbb{Z})$ are not topological invariants, but the rational Pontryagin classes are. Consider a map $u\colon M \to BG$ from an n-manifold to a classifying space. The *higher x-signature*, for $x \in H^{n-4i}(BG;\mathbb{Q})$, is the rational number

$$\mathrm{sign}_x(M,u) = \langle L_i(TM) \cup u^*(x), [M] \rangle .$$

Conjecture (Novikov (1970))

If $h\colon M' \to M$ is an orientation-preserving homotopy equivalence, then $\mathrm{sign}_x(M,u) = \mathrm{sign}_x(M,uh)$.

L-theory assembly map

Conjecture (Novikov, reformulated)

The L-theory assembly map

$$a^L \colon BG_+ \wedge \mathbb{L}(\mathbb{Z}) \longrightarrow \mathbb{L}(\mathbb{Z}[G])$$

is rationally injective, i.e., the induced homomorphism

$$a^L_* \otimes \mathbb{Q} \colon H_*(BG; L_*(\mathbb{Z})) \otimes \mathbb{Q} \longrightarrow L_*(\mathbb{Z}[G]) \otimes \mathbb{Q}$$

is injective in each degree.

Hsiang conjecture

Conjecture (Hsiang (1983))

If G is a torsion-free group, and BG has the homotopy type of a finite CW complex, then the K-theory assembly map

$$a^K : BG_+ \wedge K(\mathbb{Z}) \longrightarrow K(\mathbb{Z}[G])$$

is a rational equivalence.

Families

- A *family* $\mathscr{F}$ of subgroups of G is a collection of subgroups, closed under conjugation with elements of G and passage to subgroups.
- Let $E\mathscr{F}$ denote the universal G-CW space with stabilizers in $\mathscr{F}$. Universality amounts to the condition that $E\mathscr{F}^H$ is contractible for each $H \in \mathscr{F}$.

The orbit category

Definition

The *orbit category* Or G has as objects the homogeneous G-spaces G/H, and as morphisms the G-maps.

- The rule $G/H \mapsto E\mathscr{F}^H$ defines a contravariant functor $E\mathscr{F}^?$ from Or G to spaces.
- The rule $G/H \mapsto K(\mathbb{Z}[H])$ can be extended to a covariant functor $K(\mathbb{Z}[?])$ from Or G to spectra.

Family assembly map

The smash product

$$E\mathscr{F}_+ \wedge_{\mathrm{Or}\,G} K(\mathbb{Z}[-]) = E\mathscr{F}_+^? \wedge_{\mathrm{Or}\,G} K(\mathbb{Z}[?])$$

is a spectrum defined as a homotopy coend.
The G-map $E\mathscr{F} \to *$ induces a natural map

$$a^K \colon E\mathscr{F}_+ \wedge_{\mathrm{Or}\,G} K(\mathbb{Z}[-]) \longrightarrow *_+ \wedge_{\mathrm{Or}\,G} K(\mathbb{Z}[-]) \simeq K(\mathbb{Z}[G])\,,$$

which we call the K-theory *assembly map* for $\mathscr{F}$.

Farrell–Jones conjecture

A group is called *virtually cyclic* if it contains a (finite or infinite) cyclic subgroup of finite index. Let G be any discrete group, and let $\mathscr{V}\mathscr{C}yc$ be the family of virtually cyclic subgroups of G.

Conjecture (Farrell-Jones (1993))

The K-theory assembly map for $\mathscr{V}\mathscr{C}yc$,

$$a^K : E\mathscr{V}\mathscr{C}yc_+ \wedge_{\mathrm{Or}\,G} K(\mathbb{Z}[-]) \longrightarrow K(\mathbb{Z}[G])\,,$$

is an equivalence.

Outline

1. Conjectures
2. Theorems
3. Proofs

The Bökstedt–Hsiang–Madsen theorem

Theorem (Bökstedt–Hsiang–Madsen (1993))

Let G be a discrete group such that condition (H') holds.

(H') $H_*(BG; \mathbb{Z})$ *is of finite type.*

Then the connective K-theory assembly map

$$a^K : BG_+ \wedge K(\mathbb{Z}) \longrightarrow K(\mathbb{Z}[G])$$

is rationally injective.

Our theorem

Theorem (Lück–Reich–Rognes–Varisco)

Let G be a discrete group such that conditions (H) and (K) hold for each finite cyclic subgroup C of G:

(H) $H_*(BZ_G C; \mathbb{Z})$ *is of finite type, where* $Z_G C$ *is the centralizer of C in G;*

(K) *The canonical map* $K(\mathbb{Z}[C]) \longrightarrow \prod_p K(\mathbb{Z}_p[C])^\wedge_p$ *is rationally injective in each degree, where p ranges over all primes.*

Then the connective K-theory Farrell–Jones assembly map

$$a^K : E\mathscr{V}\mathscr{C}yc_+ \wedge_{\mathrm{Or}\, G} K(\mathbb{Z}[-]) \longrightarrow K(\mathbb{Z}[G])$$

is rationally injective.

Comments

- Condition (K) is known to hold when C is the trivial group, which is why there is no explicit condition (K') in the result of Bökstedt–Hsiang–Madsen.
- Condition (K) holds in degrees $t \leq 1$; in degrees $t \geq 2$ it is expected to hold in all cases, and would follow from the Schneider conjecture (1979), generalizing Leopoldt's conjecture from K_1 to K_t.
- Condition (H), which encompasses Condition (H'), appears to be an intrinsic limitation of the cyclotomic trace method as applied to this problem.

Outline

Reduction to finite subgroups

Let $\mathscr{F}in$ be the family of finite subgroups of G.

Proposition (Grunewald (2008))

The family comparison map

$$E\mathscr{F}in_+ \wedge_{\mathrm{Or}\,G} K(\mathbb{Z}[-]) \longrightarrow E\mathscr{V}\mathscr{C}yc_+ \wedge_{\mathrm{Or}\,G} K(\mathbb{Z}[-])$$

is a rational equivalence.

Passage to spherical group rings

Let $\mathbb{S}$ be the sphere spectrum.

Proposition

The linearization maps

$$E\mathscr{F}in_+ \wedge_{\mathrm{Or}\,G} K(\mathbb{S}[-]) \longrightarrow E\mathscr{F}in_+ \wedge_{\mathrm{Or}\,G} K(\mathbb{Z}[-])$$

and

$$K(\mathbb{S}[G]) \longrightarrow K(\mathbb{Z}[G])$$

are rational equivalences.

Summary of first reductions

$$
\begin{array}{ccc}
E\mathscr{VC}yc_+ \wedge_{\mathrm{Or}\,G} K(\mathbb{Z}[-]) & \xrightarrow{a^K} & K(\mathbb{Z}[G]) \\
\uparrow \simeq_{\mathbb{Q}} & & \| \\
E\mathscr{F}in_+ \wedge_{\mathrm{Or}\,G} K(\mathbb{Z}[-]) & \xrightarrow{a^K} & K(\mathbb{Z}[G]) \\
\uparrow \simeq_{\mathbb{Q}} & & \uparrow \simeq_{\mathbb{Q}} \\
E\mathscr{F}in_+ \wedge_{\mathrm{Or}\,G} K(\mathbb{S}[-]) & \xrightarrow{a^K} & K(\mathbb{S}[G])
\end{array}
$$

Topological cyclic homology

The *cyclotomic trace map* to *topological cyclic homology* gives a natural transformation

$$\mathrm{trc}\colon K(\mathbb{S}[-]) \longrightarrow TC(\mathbb{S}[-];p)$$

of functors from Or G to spectra.

$$\begin{array}{ccc} E\mathscr{F}in_+ \wedge_{\mathrm{Or}\,G} K(\mathbb{S}[-]) & \xrightarrow{a^K} & K(\mathbb{S}[G]) \\ \downarrow{\scriptstyle 1\wedge \mathrm{trc}} & & \downarrow{\scriptstyle \mathrm{trc}} \\ E\mathscr{F}in_+ \wedge_{\mathrm{Or}\,G} TC(\mathbb{S}[-];p) & \xrightarrow{a^{TC}} & TC(\mathbb{S}[G];p) \end{array}$$

The role of condition (K)

Proposition (Hesselholt–Madsen)

Let C be a finite group. If $K(\mathbb{Z}[C]) \to K(\mathbb{Z}_p[C])^\wedge_p$ is rationally injective, then so is $\mathrm{trc}\colon K(\mathbb{S}[C]) \longrightarrow TC(\mathbb{S}[C]; p)$.

Proposition (Lück)

If the above holds for each finite cyclic subgroup C of G, then

$$E\mathscr{F}in_+ \wedge_{\mathrm{Or}\,G} K(\mathbb{S}[-]) \longrightarrow E\mathscr{F}in_+ \wedge_{\mathrm{Or}\,G} TC(\mathbb{S}[-]; p)$$

is also rationally injective.

A warning

In the case of the trivial family $\mathscr{F} = \{e\}$, the lower horizontal map

$$a^{TC} \colon BG_+ \wedge TC(\mathbb{S}; p) \longrightarrow TC(\mathbb{S}[G]; p)$$

is the *TC*-assembly map considered by [BHM].
It does not split quite as claimed in Madsen's survey (1994).

The homotopy pullback square

There is a homotopy Cartesian square

$$\begin{array}{ccc} TC(\mathbb{S}[G];p) & \xrightarrow{\alpha} & C(\mathbb{S}[G];p) \\ \beta\downarrow & & \downarrow\text{trf} \\ THH(\mathbb{S}[G]) & \xrightarrow{1-\Delta_p} & THH(\mathbb{S}[G]) \end{array}$$

where the Bökstedt–Hsiang–Madsen functor

$$C(\mathbb{S}[G];p) = \operatorname*{holim}_{n\geq 1} THH(\mathbb{S}[G])_{hC_{p^n}}$$

is the homotopy limit over the transfer maps.

Explanations

- The composite $\beta \circ \mathrm{trc} \colon K(\mathbb{S}[G]) \to THH(\mathbb{S}[G])$ is the Waldhausen *trace map*, in the form given by Bökstedt.
- There is a natural equivalence

$$THH(\mathbb{S}[G]) \simeq \mathbb{S}[B^{cy}(G)]\,,$$

  where $B^{cy}(G)$ is the *cyclic bar construction* on G.
- $\Delta_p \colon B^{cy}(G) \to B^{cy}(G)$ is the *p-th power map*.

A decomposition

- There is a decomposition

$$B^{cy}(G) = \coprod_{[g]} B^{cy}_{[g]}(G)$$

  where $[g]$ ranges over the conjugacy classes of elements in G, and $B^{cy}_{[g]}(G)$ is the path component that contains the vertex g.
- The p-th power map Δ_p takes $B^{cy}_{[g]}(G)$ to $B^{cy}_{[g^p]}(G)$.

The difficulty

The *THH*-assembly map

$$a^{THH} \colon BG_+ \wedge THH(\mathbb{S}) \longrightarrow THH(\mathbb{S}[G])$$

is induced by the inclusion $BG \cong B^{cy}_e(G) \to B^{cy}(G)$.
It is split by the evident retraction $pr \colon B^{cy}(G)_+ \to BG_+$, but

$$pr \colon THH(\mathbb{S}[G]) \longrightarrow BG_+ \wedge THH(\mathbb{S})$$

is not in general compatible with the p-th power map Δ_p.
This does not produce a map

$$pr \colon TC(\mathbb{S}[G]; p) \longrightarrow BG_+ \wedge TC(\mathbb{S}; p)$$

splitting the *TC*-assembly map.

The solution

The original, correct, strategy of Bökstedt–Hsiang–Madsen, does not split the assembly map a^{TC} but the assembly map

$$a^C \colon BG_+ \wedge C(\mathbb{S}; p) \longrightarrow C(\mathbb{S}[G]; p)$$

for the functor C.
Hence we must construct a natural transformation

$$\alpha \colon TC(\mathbb{S}[-]; p) \longrightarrow C(\mathbb{S}[-]; p)$$

of functors from Or G to spectra.
This requires natural Segal–tom Dieck splittings and Adams transfer equivalences, constructed by Reich–Varisco (2014).

Reduction to C and THH

$$\begin{array}{ccc} E\mathscr{F}in_+ \wedge_{\mathrm{Or}\, G} TC(\mathbb{S}[-];p) & \xrightarrow{a^{TC}} & TC(\mathbb{S}[G];p) \\ \Big\downarrow {\scriptstyle 1\wedge(\alpha\vee\beta)} & & \Big\downarrow {\scriptstyle \alpha\vee\beta} \\ E\mathscr{F}in_+ \wedge_{\mathrm{Or}\, G} (C(\mathbb{S}[-];p) \vee T(\mathbb{S}[-])) & \xrightarrow{a^{C\vee T}} & C(\mathbb{S}[G];p) \vee T(\mathbb{S}[G]) \end{array}$$

(We sometimes abbreviate THH to T on this page, and the next.)

Rational injectivity of $\alpha \vee \beta$

Proposition ([BHM])

Let D be a finite group. The map

$$\alpha \vee \beta \colon TC(\mathbb{S}[D]; p) \longrightarrow C(\mathbb{S}[D]; p) \vee THH(\mathbb{S}[D])$$

is rationally injective in non-negative degrees.

Proposition (Lück)

The map

$$E\mathscr{F}in_+ \wedge_{\mathrm{Or}\,G} TC(\mathbb{S}[-]; p) \longrightarrow E\mathscr{F}in_+ \wedge_{\mathrm{Or}\,G} (C(\mathbb{S}[-]; p) \vee T(\mathbb{S}[-]))$$

is rationally injective in non-negative degrees.

The $\mathscr{F}$-part of THH

For any family $\mathscr{F}$ let

$$B^{cy}_{\mathscr{F}}(G) = \coprod_{\langle g \rangle \in \mathscr{F}} B^{cy}_{[g]}(G)$$

be the union of the path components in $B^{cy}(G)$ that contain the vertices (g) such that the cyclic group $\langle g \rangle$ is a member of the family $\mathscr{F}$.

The $\mathscr{F}$*-part* of $THH(\mathbb{S}[G])$ satisfies

$$THH_{\mathscr{F}}(\mathbb{S}[G]) \simeq \mathbb{S}[B^{cy}_{\mathscr{F}}(G)] \,.$$

Splitting the *THH*-assembly map for $\mathscr{F}$

The inclusion $B^{cy}_{\mathscr{F}}(G) \to B^{cy}(G)$, and the projection $pr_{\mathscr{F}} \colon B^{cy}(G)_+ \to B^{cy}_{\mathscr{F}}(G)_+$ make the $\mathscr{F}$-part a retract of $THH(\mathbb{S}[-])$.

Proposition

The left hand vertical map and the lower horizontal map in the commutative square

$$\begin{array}{ccc} E\mathscr{F}_+ \wedge_{\mathrm{Or}\,G} THH(\mathbb{S}[-]) & \xrightarrow{a^{THH}} & THH(\mathbb{S}[G]) \\ {\scriptstyle 1\wedge pr_{\mathscr{F}}}\downarrow{\scriptstyle \simeq} & & \downarrow{\scriptstyle pr_{\mathscr{F}}} \\ E\mathscr{F}_+ \wedge_{\mathrm{Or}\,G} THH_{\mathscr{F}}(\mathbb{S}[-]) & \xrightarrow[\simeq]{a^{THH_{\mathscr{F}}}} & THH_{\mathscr{F}}(\mathbb{S}[G]) \end{array}$$

are stable equivalences.

Splitting the C-assembly map for $\mathscr{F}$

$$\begin{array}{ccc}
E\mathscr{F}_+ \wedge_{\mathrm{Or}\,G} C(\mathbb{S}[-];p) & & \\
\downarrow \kappa & \searrow a^C & \\
\mathrm{holim}_n \big(E\mathscr{F}_+ \wedge_{\mathrm{Or}\,G} THH(\mathbb{S}[-])_{hC_{p^n}} \big) & \longrightarrow & C(\mathbb{S}[G];p) \\
\downarrow \simeq & & \downarrow pr_{\mathscr{F}} \\
\mathrm{holim}_n \big(E\mathscr{F}_+ \wedge_{\mathrm{Or}\,G} THH_{\mathscr{F}}(\mathbb{S}[-])_{hC_{p^n}} \big) & \xrightarrow{\simeq} & C_{\mathscr{F}}(\mathbb{S}[G];p)
\end{array}$$

End of proof

Proposition (Lück–Reich–Varisco (2003))

Assuming condition (H),

$$\kappa\colon E\mathscr{F}_+ \wedge_{\mathrm{Or}\,G} C(\mathbb{S}[-];p) \longrightarrow \operatorname*{holim}_n \big(E\mathscr{F}_+ \wedge_{\mathrm{Or}\,G} THH(\mathbb{S}[-])_{hC_{p^n}}\big)$$

is an equivalence for $\mathscr{F}$ the family of finite cyclic subgroups of G, hence also for the family $\mathscr{F}in$ of finite subgroups of G.

This implies that a^C for $\mathscr{F}in$ is split injective. Q.E.D.

Summary of all reductions ($T = THH$)

$$\begin{array}{ccc}
E\mathcal{VC}yc_+ \wedge_{\mathrm{Or}\,G} K(\mathbb{Z}[-]) & \xrightarrow{a^K} & K(\mathbb{Z}[G]) \\
\uparrow \simeq_{\mathbb{Q}} & & \| \\
E\mathcal{F}in_+ \wedge_{\mathrm{Or}\,G} K(\mathbb{Z}[-]) & \xrightarrow{a^K} & K(\mathbb{Z}[G]) \\
\uparrow \simeq_{\mathbb{Q}} & & \uparrow \simeq_{\mathbb{Q}} \\
E\mathcal{F}in_+ \wedge_{\mathrm{Or}\,G} K(\mathbb{S}[-]) & \xrightarrow{a^K} & K(\mathbb{S}[G]) \\
\downarrow \mathbb{Q}\text{-inj.} & & \downarrow \mathrm{trc} \\
E\mathcal{F}in_+ \wedge_{\mathrm{Or}\,G} TC(\mathbb{S}[-];p) & \xrightarrow{a^{TC}} & TC(\mathbb{S}[G];p) \\
\downarrow \text{non-neg. } \mathbb{Q}\text{-inj.} & & \downarrow \alpha\vee\beta \\
E\mathcal{F}in_+ \wedge_{\mathrm{Or}\,G} (C(\mathbb{S}[-];p) \vee T(\mathbb{S}[-])) & \xrightarrow[\text{inj.}]{a^{C\vee T}} & C(\mathbb{S}[G];p) \vee T(\mathbb{S}[G])
\end{array}$$