COHOMOLOGY OF THE SMOOTH WHITEHEAD SPECTRUM

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These are notes for the author's talk at Oberwolfach in September 1998.

1. Algebraic K-theory of spaces. Let $hR(*)$ be the category of finite based CW-complexes and homotopy equivalences between these. Let $|hR(*)|$ denote its geometric realization, also known as the nerve of this category. Waldhausen's algebraic K-theory space $A(*)$ is a kind of group completion of $|hR(*)|$, in the sense that it is an infinite loop space, and comes equipped with a map

$$e\colon |hR(*)| \to A(*)$$

such that for cofiber sequences $Y' \to Y \to Y''$ the relation

$$[Y'] + [Y''] = [Y]$$

holds in $\pi_0 A(*)$. Note that finite based CW-complexes Y represent points in $|hR(*)|$, and thus map to points in $A(*)$. In fact $\pi_0 A(*) \cong \mathbb{Z}$ is a receptacle for the universal additive homotopy invariant of finite based CW-complexes, namely the reduced Euler characteristic $Y \mapsto \tilde{\chi}(Y) = \chi(Y) - 1$. The infinite loop space $A(*)$ can likewise be viewed as a receptacle for the topological enrichment e of the Euler characteristic.

Hence if we want to study the category of finite CW-complexes and homotopy equivalences, but are only willing to work with infinite loop spaces, then $A(*)$ is the closest thing.

Given a space X there is a similar infinite loop space $A(X)$ obtained from the category $R(X)$ of finite relative CW-complexes (Y, X) containing the fixed space X as a retract. The association $X \mapsto A(X)$ is in fact a homotopy functor, called Waldhausen's algebraic K-theory of spaces.

It will be convenient to work with spectra rather than infinite loop spaces, so hereafter $A(X)$ will denote the algebraic K-theory spectrum rather than the algebraic K-theory space.

2. Whitehead spectra and concordances. Let M be a CAT manifold, where CAT is one of the three geometric categories TOP, PL and DIFF. The CAT concordance space $C^{CAT}(M)$ is the space of CAT automorphisms of $M \times I$ fixing a neighborhood of $\partial M \times I \cup M \times 0$, and there is a stabilization map $\sigma\colon C^{CAT}(M) \to C^{CAT}(I \times M)$. Waldhausen constructed a CAT Whitehead spectrum $Wh^{CAT}(M)$, such that there is a homotopy equivalence

$$\Omega^2 \Omega^\infty Wh^{CAT}(M) \simeq \lim_{n\to\infty} C^{CAT}(I^n \times M)$$

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between the double loop space of the underlying CAT Whitehead space, and the so-called stable CAT concordance space. This generalizes the Whitehead group, since $\pi_1 Wh^{CAT}(M) \cong Wh(\pi_1(M))$ in either category.

Waldhausen also constructed a cofiber sequence of spectra

$$A(*) \wedge M_+ \xrightarrow{\alpha} A(M) \to Wh^{CAT}(M)$$

when CAT is TOP or PL (the stable TOP and PL concordance spaces are homotopy equivalent), and a split cofiber sequence of spectra

$$\Sigma^\infty(M_+) \xrightarrow{\iota} A(M) \to Wh^{DIFF}(M)$$

in the smooth category. (So $DIFF$ is different.)

3. The case of a point. In the case $M = *$ the split cofiber sequence gives the splitting

$$A(*) \simeq S^0 \vee Wh^{DIFF}(*),$$

so knowing $A(*)$ is essentially equivalent to knowing the smooth Whitehead spectrum $Wh^{DIFF}(*)$ and the sphere spectrum. Also $A(*)$ plays a special role in the TOP and PL categories, as indicated by the upper cofiber sequence.

The aim of this talk is to describe the 2-primary homotopy type of $Wh^{DIFF}(*)$, and thus of $A(*)$. We give the mod 2 spectrum cohomology of either spectrum as a module over the Steenrod algebra $A = \mathcal{A}(2)$, and compute its homotopy groups modulo odd torsion in dimensions ≤ 20.

For an easy geometric interpretation of these results, we can consider the space of diffeomorphisms of a disc D^n, not necessarily fixing the boundary. This is homotopy equivalent to the product of the orthogonal group of isometries $O(n)$, and the smooth concordance space $C^{DIFF}(S^{n-1})$. ((Draw a picture of an annulus.)) The stabilization map

$$C^{DIFF}(S^{n-1}) \to \Omega^2\Omega^\infty Wh^{DIFF}(*)$$

is approximately $(n/3)$-connected by Igusa's stability theorem. Hence in this concordance stable range, the homotopy groups of $Wh^{DIFF}(*)$ shifted down two degrees are the homotopy groups of the "homogeneous space" of diffeomorphisms of D^n modulo the isometries. Ignoring odd torsion, these begin

$$\mathbb{Z}/2, 0, \mathbb{Z}, 0, \mathbb{Z}/2, 0, \mathbb{Z} \oplus \mathbb{Z}/2, \mathbb{Z}/8 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2, \mathbb{Z}/2, \mathbb{Z}/4$$

in dimensions $1 \leq * \leq 10$. The first $\mathbb{Z}/2$ was found by Igusa using the Lee–Sczcarba calculation of $K_3(\mathbb{Z})$, the second is an integral result using $K_4(\mathbb{Z}) = 0$, and the rest of the groups are new.

4. Algebraic K-theory. Let X be a based connected space, and let $R = \mathbb{Z}[\pi_1(X)]$ be the group ring of its fundamental group. There is a linearization map $L\colon A(X) \to K(R)$ from Waldhausen's algebraic K-theory of spaces to Quillen's algebraic K-theory of the ring R. When $X = *$ is a point, this map $L\colon A(*) \to K(\mathbb{Z})$ is a rational equivalence, and combined with Borel's rational calculation of the algebraic K-groups of rings of integers in number fields, such as $\mathbb{Z}$, this determines $A(*)$ rationally. As an application, Farrell and Hsiang were able to determine the rational homotopy type of the diffeomorphism groups of discs and spheres, in the concordance stable range.

To study the torsion information in the homotopy of $A(*)$, we need to understand the torsion in the algebraic K-theory $K(\mathbb{Z})$, as well as a precise measure of the difference between $A(*)$ and $K(\mathbb{Z})$.

For a ring R of integers in a number field there are conjectures due to Lichtenbaum and Quillen, regarding what the torsion in the homotopy groups of $K(R)$ should look like. These are based on the idea that algebraic K-theory of number fields should satisfy Galois descent, i.e., that for a Galois extension E/F with group G the algebraic K-theory $K(F)$ should agree in positive dimensions with the homotopy fixed points for G acting on $K(E)$. However these conjectures have remained open for a long time, and only last year was the 2-primary homotopy type of $K(\mathbb{Z})$ completely determined by Rognes and Weibel, based on Voevodsky's proof of the Milnor conjecture and a spectral sequence due to Bloch and Lichtenbaum. (A gap concerning real embeddings in Weibel's 1996 Comptes Rendus paper is fixed in our joint 1997 preprint, using the calculation of $TC(\mathbb{Z})^\wedge_2$ mentioned below, and a reciprocity argument.)

Thus one ingredient going into the study of $A(*)$ is our new understanding of $K(\mathbb{Z})$ at the prime 2.

5. Topological cyclic homology. The second ingredient is the topological cyclic homology of Bökstedt, Hsiang and Madsen. For each prime p there is a topological cyclic homology of spaces, $TC(X)^\wedge_p$, and of rings, $TC(R)^\wedge_p$, and by a theorem of Dundas the p-completed fiber of the K-theory linearization map $L\colon A(X) \to K(R)$ is homotopy equivalent to the fiber of a TC-theory linearization map $L\colon TC(X) \to TC(R)$, when X is a connected based space and $R = \mathbb{Z}[\pi_1(X)]$. Stated differently there is a homotopy Cartesian square of spectra

$$\begin{array}{ccc} A(X)^\wedge_p & \xrightarrow{L} & K(R)^\wedge_p \\ \downarrow{\scriptstyle \mathrm{trc}_X} & & \downarrow{\scriptstyle \mathrm{trc}_R} \\ TC(X)^\wedge_p & \xrightarrow{L} & TC(R)^\wedge_p \end{array}$$

where $\mathrm{trc}_X\colon A(X) \to TC(X)$ and $\mathrm{trc}_R\colon K(R) \to TC(R)$ are special cases of the cyclotomic trace map of Bökstedt, Hsiang and Madsen from algebraic K-theory to topological cyclic homology. Hence topological cyclic homology provides a perfect relative invariant detecting the difference between $A(X)$ and $K(R)$.

In the case $X = *$ and $p = 2$, it then remains to understand $TC(*)^\wedge_2$, $TC(\mathbb{Z})^\wedge_2$ and the maps in the diagram.

6. A Thom spectrum. Bökstedt, Hsiang and Madsen computed the topological cyclic homology of spaces, $TC(X)^\wedge_p$, in terms of relatively standard homotopy theoretic constructions. For $X = *$ there is a splitting

$$TC(*)^\wedge_p \simeq S^0 \vee \mathrm{fib}(\mathrm{trf}_{S^1})$$

where

$$\mathrm{trf}_{S^1}\colon \Sigma\mathbb{C}P^\infty_+ \to S^0$$

is the S^1-transfer map. On the level of framed bordism, this map takes a framed manifold $M^n \to \mathbb{C}P^\infty$ to the total space of the S^1-bundle over M induced by pullback of the universal S^1-bundle over $\mathbb{C}P^\infty$, with the obvious framing.

$$\pi^S_n(\Sigma\mathbb{C}P^\infty_+) = \Omega^{fr}_n(\mathbb{C}P^\infty) \to \pi^S_{n+1}(S^0) = \Omega^{fr}_{n+1}\,.$$

There is a more convenient identification of the homotopy fiber of this S^1-transfer map, which we now recall.

The Thom complex $Th(kH_n)$ of $k \geq 0$ times the canonical complex line bundle over $\mathbb{C}P^n$ is the truncated complex projective space $\mathbb{C}P_k^{n+k} = \mathbb{C}P^{n+k}/\mathbb{C}P^{k-1}$. By James periodicity we can also make sense of this for $k < 0$, provided we interpret $\mathbb{C}P_k^{n+k}$ as a spectrum, i.e., the Thom spectrum of kH. Passing to the limit as $n \to +\infty$, we obtain the Thom spectrum $\mathbb{C}P_{-1}^{\infty} = Th(-H)$ of minus the canonical line bundle over $\mathbb{C}P^{\infty}$. There is a cofiber sequence of spectra

$$S^{-2} \to \mathbb{C}P_{-1}^{\infty} \to \mathbb{C}P_{+}^{\infty} \xrightarrow{\partial} S^{-1}$$

and Knapp proved that the connecting map ∂ is the desuspension of the S^1-transfer map. Hence

$$\mathrm{fib}(\mathrm{trf}_{S^1}) \simeq \Sigma \mathbb{C}P_{-1}^{\infty} .$$

Hence we have a diagram of two homotopy Cartesian squares, where we omit the implicit completion at a prime p.

$$\begin{array}{ccccc} Wh^{DIFF}(*) & \longrightarrow & A(*) & \xrightarrow{L} & K(\mathbb{Z}) \\ \downarrow{\scriptstyle \widetilde{\mathrm{trc}}} & & \downarrow{\scriptstyle \mathrm{trc}_*} & & \downarrow{\scriptstyle \mathrm{trc}_{\mathbb{Z}}} \\ \Sigma\mathbb{C}P_{-1}^{\infty} & \longrightarrow & TC(*) & \xrightarrow{L} & TC(\mathbb{Z}) \end{array}$$

In particular the three vertical homotopy fibers are all homotopy equivalent, and we denote either of them simply by fib(trc).

Proposition. *There is a cofiber sequence*

$$\mathbb{C}P_{-1}^{\infty} \to \mathrm{fib}(\mathrm{trc}) \to Wh^{DIFF}(*) .$$

We shall see that for $p = 2$ we completely understand the fiber of the cyclotomic trace map, fib(trc), so our understanding of the smooth Whitehead spectrum $Wh^{DIFF}(*)$, or $A(*)$, is as good as our understanding of the Thom spectrum $\mathbb{C}P_{-1}^{\infty}$.

7. Two-primary homotopy of $\mathbb{C}P_{-1}^{\infty}$. The spectrum cohomology of $\mathbb{C}P_{-1}^{\infty}$ is easy to describe.

Definition. Let $J \subset A$ be the left ideal with additive basis the set of monomials Sq^I for which the admissible sequence $I = (i_1, \dots, i_n)$ has length $n \geq 2$, or $I = (i)$ with i odd. Then A/J has additive basis the Sq^i with i even.

Let us briefly write $H^*(X) = H^*_{spec}(X; \mathbb{F}_2)$ for the mod 2 spectrum cohomology of a spectrum X.

Proposition. $H^*(\mathbb{C}P_{-1}^{\infty}) \cong \Sigma^{-2} A/J$ *as a graded left A-module.*

We can use the spectral sequence

$$E^1_{s,t} = \pi_{s+t}(\mathbb{C}P_{-1}^{s}/\mathbb{C}P_{-1}^{s-1}) \cong \pi_{t-s}^S \Longrightarrow \pi_{s+t}(\mathbb{C}P_{-1}^{\infty})$$

associated to the filtration of $\mathbb{C}P_{-1}^{\infty}$ by the subspectra $\mathbb{C}P^s{}_{-1}$, as well as the Adams spectral sequence

$${}_cE_2^{s,t} = \mathrm{Ext}_A^{s,t}(H^*(\mathbb{C}P_{-1}^{\infty}), \mathbb{F}_2) \Longrightarrow \pi_{t-s}(\mathbb{C}P_{-1}^{\infty})^{\wedge}_2$$

to study the homotopy groups of $\mathbb{C}P_{-1}^{\infty}$. Comparison of these two spectral sequences, combined with calculations by Mosher and Mukai of the stable homotopy of $\mathbb{C}P^{\infty}$ and of the image of the S^1-transfer map, allows us to determine the differentials in both of these spectral sequences in dimensions ≤ 20.

Theorem (Rognes). *The* 2*-primary homotopy groups of* $\mathbb{C}P^\infty_{-1}$ *are known (up to some extensions) in dimensions* $* \le 20$*, and begin*

$$\mathbb{Z}, 0, \mathbb{Z}, 0, \mathbb{Z}, \mathbb{Z}/8, \mathbb{Z}, \mathbb{Z}/2, \mathbb{Z}/2 \oplus \mathbb{Z}, \mathbb{Z}/16, \mathbb{Z}/2 \oplus \mathbb{Z}, \mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/8$$

in dimensions $-2 \le * \le 9$.

8. The fiber of the cyclotomic trace map. At the prime 2, we now give a precise description of the homotopy fiber of the cyclotomic trace map

$$\mathrm{trc}_{\mathbb{Z}} \colon K(\mathbb{Z})^\wedge_2 \to TC(\mathbb{Z})^\wedge_2 \,.$$

Bökstedt and Madsen computed the homotopy type of $TC(\mathbb{Z})^\wedge_p$ for p odd, and Rognes extended the calculation to the case $p = 2$. Then by a theorem of Hesselholt and Madsen, using results of McCarthy, $K(\mathbb{Z}_p)^\wedge_p$ is the connected cover of $TC(\mathbb{Z})^\wedge_p$, so there is a cofiber sequence

$$K(\mathbb{Z}_p)^\wedge_p \to TC(\mathbb{Z})^\wedge_p \to \Sigma^{-1} H\mathbb{Z}_p \,.$$

Theorem (Rognes). *There are cofiber sequences*

$$K^{\mathrm{red}}(\mathbb{Z})^\wedge_2 \to K(\mathbb{Z}_2)^\wedge_2 \xrightarrow{\mathrm{red}} K(\mathbb{F}_3)^\wedge_2$$

and

$$\Sigma K(\mathbb{F}_3)^\wedge_2 \to K^{\mathrm{red}}(\mathbb{Z})^\wedge_2 \to \Sigma bu \,,$$

with explicitly known connecting maps.

Using the proof of the Milnor conjecture by Voevodsky, the Bloch–Lichtenbaum spectral sequence, and the result above, Rognes and Weibel proved:

Theorem (Rognes and Weibel). *There is a cofiber sequence*

$$\Sigma bo \to K(\mathbb{Z})^\wedge_2 \xrightarrow{\pi_3} K(\mathbb{F}_3)^\wedge_2$$

with explicitly known connecting map.

From the homotopy equivalence $TC(\mathbb{Z})^\wedge_p \simeq TC(\mathbb{Z}_p)^\wedge_p$ and the theorem of Hesselholt and Madsen, one can recover the homotopy fiber of $\mathrm{trc}_{\mathbb{Z}}$ from the homotopy fiber of $j \colon K(\mathbb{Z}) \to K(\mathbb{Z}_2)$ completed at 2. The composite $\mathrm{red} \circ j$ is homotopic to π_3 followed by a homotopy equivalence of $K(\mathbb{F}_3)$, so $\mathrm{fib}(j)$ is homotopy equivalent to the fiber of

$$j^{\mathrm{red}} \colon \Sigma bo \to K^{\mathrm{red}}(\mathbb{Z}_2) \,.$$

By a theorem of Rognes, the composite $\Sigma bo \to K^{\mathrm{red}}(\mathbb{Z}_2) \to \Sigma bu$ may be taken to be Σc, i.e., the suspended complexification map. Knowledge of the connecting maps for the cofiber sequences above, together with calculations in the K-local category, leads to the following description.

Theorem (Rognes). *There is a cofiber sequence*

$$\Sigma^3 ko \to \mathrm{fib}(\mathrm{trc}) \to \Sigma^{-2} ku \,.$$

The connecting map $\delta \colon \Sigma^{-2} ku \to \Sigma^4 ko$ *is characterized up to homotopy by its* K*-localization, which is the composite*

$$\Sigma^4 r \circ \beta^{-2} \circ (\psi^3 - 1) \circ \beta^{-1} \colon \Sigma^{-2} KU \to \Sigma^4 KO \,.$$

9. Cohomology of $Wh^{DIFF}(*)$. We obtain the following diagram of horizontal and vertical cofiber sequences, where $\epsilon\colon \mathbb{C}P^\infty_{-1} \to \Sigma^{-2}ku$ is the composite making the upper square commute.

$$
\begin{array}{ccccccc}
 & & \mathbb{C}P^\infty_{-1} & = & \mathbb{C}P^\infty_{-1} & & \\
 & & \downarrow & & \downarrow{\scriptstyle \epsilon} & & \\
\Sigma^3 ko & \longrightarrow & \mathrm{fib}(\mathrm{trc}) & \longrightarrow & \Sigma^{-2}ku & \xrightarrow{\delta} & \Sigma^4 ko \\
\| & & \downarrow & & \downarrow & & \\
\Sigma^3 ko & \longrightarrow & Wh^{DIFF}(*) & \longrightarrow & \mathrm{cofib}(\epsilon) & &
\end{array}
$$

Applying spectrum cohomology, we find that $\delta^* = 0$ and ϵ^* is surjective in dimension -2, thus in all dimensions because the image is a cyclic A-module. We obtain the following diagram of horizontal and vertical extensions of left graded A-modules.

$$
\begin{array}{ccccc}
 & & H^*(\mathbb{C}P^\infty_{-1}) & = & H^*(\mathbb{C}P^\infty_{-1}) \\
 & & \uparrow & & \uparrow{\scriptstyle \epsilon^*} \\
H^*(\Sigma^3 ko) & \longleftarrow & H^*\,\mathrm{fib}(\mathrm{trc}) & \longleftarrow & H^*(\Sigma^{-2}ku) \\
\| & & \uparrow & & \uparrow \\
H^*(\Sigma^3 ko) & \longleftarrow & H^* Wh^{DIFF}(*) & \longleftarrow & H^*\,\mathrm{cofib}(\epsilon)
\end{array}
$$

Here $H^*(\Sigma^3 ko) = \Sigma^3 A/A(Sq^1, Sq^2)$, $H^*(\Sigma^{-2}ku) = \Sigma^{-2}A/A(Sq^1, Sq^3)$ and $H^*(\mathbb{C}P^\infty_{-1}) = \Sigma^{-2}A/J$. So $H^*\,\mathrm{cofib}(\epsilon) = \Sigma^{-2}J/A(Sq^1, Sq^3)$, which begins in degree 4. This leads us to our main theorem.

Theorem (Rognes). *$H^*_{spec}(Wh^{DIFF}(*); \mathbb{F}_2)$ is the unique nontrivial extension of A-modules*

$$\Sigma^{-2}J/A(Sq^1, Sq^3) \to H^*_{spec}(Wh^{DIFF}(*); \mathbb{F}_2) \to \Sigma^3 A/A(Sq^1, Sq^2)\,.$$

Also

$$H^*_{spec}(A(*); \mathbb{F}_2) \cong \mathbb{F}_2[0] \oplus H^*_{spec}(Wh^{DIFF}(*); \mathbb{F}_2)$$

as A-modules, where $\mathbb{F}_2[0]$ denotes the trivial A-module in dimension 0.

That there are precisely two such extensions of A-modules follows by a change-of-rings calculation. To see that the extension is nontrivial, one can use the known calculation that $\pi_3 Wh^{DIFF}(*) = \mathbb{Z}/2$ is the bottom homotopy group of this spectrum, so there is a nonzero Sq^1 from H^3 to H^4.

10. Homotopy of $Wh^{DIFF}(*)$. There are Adams spectral sequences for the 2-primary homotopy of each of the spectra in the middle column above.

$$
\begin{aligned}
{}_cE_2^{s,t} &= \mathrm{Ext}_A^{s,t}(H^*(\mathbb{C}P^\infty_{-1}), \mathbb{F}_2) \Longrightarrow \pi_{t-s}(\mathbb{C}P^\infty_{-1})^\wedge_2 \\
{}_fE_2^{s,t} &= \mathrm{Ext}_A^{s,t}(H^*\,\mathrm{fib}(\mathrm{trc}), \mathbb{F}_2) \Longrightarrow \pi_{t-s}(\mathrm{fib}(\mathrm{trc}))^\wedge_2 \\
E_2^{s,t} &= \mathrm{Ext}_A^{s,t}(H^*(Wh^{DIFF}(*)), \mathbb{F}_2) \Longrightarrow \pi_{t-s} Wh^{DIFF}(*)^\wedge_2
\end{aligned}
$$

The short exact sequence in cohomology induces a long exact sequence of E_2-terms

$$\dots \to {}_cE_2^{s,t} \to {}_fE_2^{s,t} \to E_2^{s,t} \xrightarrow{\partial} {}_cE_2^{s+1,t} \to \dots$$

Here the Adams spectral sequence for $\mathbb{C}P^\infty_{-1}$ is understood up to dimension 20, or as far as we understand the homotopy of this Thom spectrum. The Adams spectral sequence for fib(trc) is completely understood, in that the E_2-term and all differentials are known. We can compute the E_2-term in the Adams spectral sequence for $Wh^{DIFF}(*)$, either by hand with a minimal resolution, or by Bruner's Ext-calculating programs. Then we can use the above long exact sequence to translate differentials in either of the two other spectral sequences to the one for $Wh^{DIFF}(*)$.

Fortunately the spectral sequence ${}_fE_*$ for fib(trc) is concentrated mainly in high Adams filtrations, while the h_0-torsion in ${}_cE_*$ for $\mathbb{C}P^\infty_{-1}$ is concentrated mainly in low Adams filtrations. Hence one can argue that all differentials in the spectral sequence E_* for $Wh^{DIFF}(*)$ can be found this way.

Theorem (Rognes). *The* 2*-primary homotopy groups of* $Wh^{DIFF}(*)$ *and* $A(*)$ *are known in dimensions* $* \leq 17$*, and up to extensions in dimensions* $18 \leq * \leq 21$*. They begin*

$$\mathbb{Z}/2, 0, \mathbb{Z}, 0, \mathbb{Z}/2, 0, \mathbb{Z}/2 \oplus \mathbb{Z}, \mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/8, \mathbb{Z}/2, \mathbb{Z}/4,$$
$$\mathbb{Z}, \mathbb{Z}/4, \mathbb{Z}/2 \oplus \mathbb{Z}/2, \mathbb{Z}/2 \oplus \mathbb{Z}/8, \mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}$$

in dimensions $3 \leq * \leq 17$*.*

The 2*-primary* k*-invariants of* $Wh^{DIFF}(*)$ *are also partially known, starting with* $k^5 = \beta Sq^2 \in H^6(K(\mathbb{Z}/2,3);\mathbb{Z})$ *and* $k^7 = p^*Sq^5 \in H^8(P^5Wh^{DIFF}(*);\mathbb{Z}/2)$ *with* $p\colon P^5Wh^{DIFF}(*) \to K(\mathbb{Z}/2,3)$ *the essential map.*

Corollary (Rognes). *The Hatcher–Waldhausen map* $G/O \to \Omega Wh^{DIFF}(*)$ *is precisely* 7*- or* 8*-connected after* 2*-adic completion.*

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