HOMOLOGY AND COHOMOLOGY OPERATIONS

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1. Based spaces

1.1 Homology and cohomology of spaces.

1.1.1. We will work in the category of cofibrantly based compactly generated weak Hausdorff spaces.

1.1.2. Let all chains, cochains, homology and cohomology groups have implicit coefficients $\mathbb{Z}/2$. Hence we briefly write $H_*(X) = H_*(X;\mathbb{Z}/2)$ and $H^*(X) = H^*(X;\mathbb{Z}/2)$.

1.1.3. The evaluation of cochains on chains $S^*(X)\otimes S_*(X) \to \mathbb{Z}/2$ induces a perfect pairing $\langle\ ,\ \rangle\colon H^*(X)\otimes H_*(X) \to \mathbb{Z}/2$, or equivalently a natural isomorphism

$$H^*(X) \cong \operatorname{Hom}(H_*(X), \mathbb{Z}/2).$$

This is a case of the universal coefficient theorem.

1.1.4. The Eilenberg–Zilber shuffle homomorphism $S_*(X)\otimes S_*(Y) \to S_*(X\times Y)$ provides an associative and commutative chain homotopy inverse to the Alexander–Whitney map $S_*(X\times Y) \to S_*(X)\otimes S_*(Y)$. Both maps are chain homotopy equivalences, and induce the Künneth isomorphisms

$$H_*(X)\otimes H_*(Y) \cong H_*(X\times Y)$$

and

$$H^*(X)\otimes H^*(Y) \cong H^*(X\times Y).$$

We write $x\times y$ for the image of $x\otimes y$ under either of these isomorphisms. See [McL].

1.1.5. The diagonal map $\Delta\colon X\to X\times X$ and the collapse map $c\colon X\to *$ induce a coproduct

$$\psi\colon H_*(X) \xrightarrow{\Delta_*} H_*(X\times X) \cong H_*(X)\otimes H_*(X)$$

and counit

$$\epsilon\colon H_*(X) \xrightarrow{c_*} H_*(*) = \mathbb{Z}/2$$

which give $H_*(X)$ the structure of a coalgebra. It is cocommutative, and coaugmented by $\eta\colon \mathbb{Z}/2 = H_*(*) \to H_*(X)$ induced by the inclusion $i\colon * \to X$ of the base point.

1.1.6. Dually $H^*(X) \cong \operatorname{Hom}(H_*(X),\mathbb{Z}/2)$ is an algebra, with cup product

$$\phi\colon H^*(X)\otimes H^*(X) \to H^*(X)$$

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derived from Δ^* and unit $\eta\colon \mathbb{Z}/2 \to H^*(X)$ derived from c^*. It is commutative, and augmented by $\epsilon\colon H^*(X) \to \mathbb{Z}/2$ derived from i^*. We write $\phi(x \otimes y) = x \smile y = xy$ for the cup product.

1.1.7. Our algebras, coalgebras and Hopf algebras are always associative and unital (or coassociative and counital, etc.).

1.2. The Steenrod squares.

1.2.1. Let $C_2 = \{1, T\}$ be the group of order 2, and let EC_2 be a free contractible C_2-space. Its orbit space $BC_2 = EC_2/C_2$ is the classifying space for principal C_2-bundles. One model for EC_2 is $S^\infty = \bigcup_n S^n$ with the antipodal action.

Let $\epsilon\colon W_* \to \mathbb{Z}/2$ be the periodic resolution of $\mathbb{Z}/2$ by free $\mathbb{Z}/2[C_2]$-modules. Specifically, let $W_i = \mathbb{Z}/2[C_2]\{e_i\}$ be the free module of rank 1 on a generator e_i and let $\partial(e_{i+1}) = 1 \cdot e_i + T \cdot e_i$, for all $i \geq 0$. Then $H_i(BC_2) \cong H_i(W_* \otimes_{C_2} \mathbb{Z}/2) = \mathbb{Z}/2\{e_i\}$ for all $i \geq 0$.

Dually $H^*(BC_2) = \mathbb{Z}/2[t]$, with $t \in H^1(BC_2)$ dual to e_1. Then t^n is dual to e_n.

1.2.2. There is a chain homotopy equivalence $H_*(X) \to S_*(X)$, where $H_*(X)$ is viewed as a chain complex with zero boundary maps, given by choosing a basis for $H_*(X)$ and choosing representing cycles in $Z_*(X) \subset S_*(X)$ for these homology classes.

Likewise there is a C_2-equivariant chain homotopy equivalence $W_* \to S_*(EC_2)$. Viewing EC_2 as S^∞ as the union of the S^n's with $S^n = S^{n-1} \cup D^n_+ \cup D^n_-$, one model takes e_n and Te_n to chains representing D^n_+ and D^n_-, respectively.

1.2.3. We can now construct the Steenrod squaring operations. The diagonal Δ induces a C_2-equivariant map $1 \times \Delta\colon EC_2 \times X \to EC_2 \times X^2$. Here C_2 acts by $T(e, x) = (Te, x)$ on the left, and by $T(e, x, y) = (Te, y, x)$ on the right. Hence there is an induced map of orbit spaces

$$1 \times_{C_2} \Delta\colon BC_2 \times X \to EC_2 \times_{C_2} X^2.$$

We can compute the cohomology of the target. There are isomorphisms

$$H^*(EC_2 \times_{C_2} X^2) \cong H^*(S_*(EC_2) \otimes_{C_2} S_*(X)^{\otimes 2}) \cong H^*(W_* \otimes_{C_2} H_*(X)^{\otimes 2})$$

induced by shuffle homomorphisms, by the chain equivalence $W_* \simeq S_*(EC_2)$, and the chain equivalence $S_*(X) \simeq H_*(X)$.

Hence we get a homomorphism

$$\theta^*\colon H^*(W_* \otimes_{C_2} H_*(X)^{\otimes 2}) \to H^*(BC_2 \times X)$$

by composing the isomorphisms above with $(1 \times_{C_2} \Delta)^*$.

1.2.4. Now suppose we are given a class $x \in H^n(X)$. View it as an n-cocycle $\xi\colon H_n(X) \to \mathbb{Z}/2$. Also view the augmentation $\epsilon\colon W_* \to \mathbb{Z}/2$ as a 0-cocycle. The product $\epsilon \otimes \xi \otimes \xi\colon W_* \otimes H_*(X)^{\otimes 2} \to \mathbb{Z}/2$ is C_2-equivariant, hence descends to define a cohomology class

$$[\epsilon \otimes \xi \otimes \xi] \in H^*(W_* \otimes_{C_2} H_*(X)^{\otimes 2}).$$

By the isomorphism above this gives a cohomology class in $H^*(EC_2 \times_{C_2} X^2)$. Applying $(1 \times_{C_2} \Delta)^*$ we obtain a cohomology class

$$\theta^*[\epsilon \otimes \xi \otimes \xi] \in H^*(BC_2 \times X) \cong H^*(BC_2) \otimes H^*(X).$$

This class has degree $2n$, and can be uniquely written as a sum

$$\sum_{i=0}^{n} t^{n-i} \otimes Sq^i(x) \in \bigoplus_{i=0}^{n} H^{n-i}(BC_2) \otimes H^{n+i}(X)$$

where the terms $Sq^i(x) \in H^{n+i}(X)$ are determined by this formula. This defines the Steenrod squaring operation

$$Sq^i \colon H^n(X) \to H^{n+i}(X).$$

1.2.5. We write $(i,j) = \binom{i+j}{i}$ for the binomial coefficient. Thus $(i,j) = (j,i)$. It is $(i+j)!/i!j!$ if $i \geq 0$ and $j \geq 0$, and 0 if $i < 0$ or $j < 0$. In the formulas below, the binomial coefficient needs only be considered mod 2.

1.2.6. Theorem. *There are homomorphisms $Sq^i \colon H^*(X) \to H^*(X)$ for all $n, i \geq 0$, which satisfy:*

(1) The Sq^i are natural with respect to maps $f \colon X \to Y$ of spaces.

(2) Sq^i raises degrees by i.

(3) $Sq^i(x) = 0$ if $i > \deg(x)$.

(4) $Sq^i(x) = x^2$ if $i = \deg(x)$.

(5) $Sq^0(x) = x$ for all x.

(6) $Sq^1(x) = \beta(x)$ is the Bockstein homomorphism in cohomology associated to the short exact sequence of coefficients $0 \to \mathbb{Z}/2 \to \mathbb{Z}/4 \to \mathbb{Z}/2 \to 0$.

(7) The external and internal Cartan formulas hold:

$$Sq^k(x \times y) = \sum_{i+j=k} Sq^i(x) \times Sq^j(y)$$

in $H^*(X \times Y)$ for $x \in H^*(X)$, $y \in H^*(Y)$, and

$$Sq^k(xy) = \sum_{i+j=k} Sq^i(x) Sq^j(y)$$

in $H^*(X)$ for $x, y \in H^*(X)$.

(8) The Adem relations hold: If $a < 2b$ then

$$Sq^a Sq^b = \sum_i (a-2i, b-a+i-1) Sq^{a+b-i} Sq^i.$$

The binomial coefficient can also be written $\binom{b-i-1}{a-2i}$. The sum runs over i with $a-b+1 \leq i \leq a/2$.

(9) The Sq^i are stable and the Kudo transgression theorem holds: $Sq^i \sigma^*(x) = \sigma^* Sq^i(x)$ where $\sigma^* \colon \tilde{H}^{n+1}(X) \to \tilde{H}^n(\Omega X)$ is the cohomology suspension. Hence also $Sq^i \Sigma(x) = \Sigma Sq^i(x)$ where $\Sigma \colon \tilde{H}^n(X) \to \tilde{H}^{n+1}(\Sigma X)$ is the suspension isomorphism.

If X is simply-connected and if $x \in H^n(\Omega X)$ transgresses to $y \in H^{n+1}(X)$ in the Serre spectral sequence of the path space fibration $\Omega X \to PX \to X$, then $Sq^i(x) \in H^{n+i}(\Omega X)$ transgresses to $Sq^i(y) \in H^{n+i+1}(X)$.

For proofs, see [ES] and [May2].

1.3. The Steenrod algebra A.

1.3.1. Consider sequences $I = (i_1, \dots, i_k)$ of integers with $i_s > 0$. Define the *degree*, *length* and *excess* of I by

$$d(I) = \sum_{s=1}^{k} i_s, \qquad \ell(I) = k \qquad \text{and}$$
$$e(I) = \sum_{s=1}^{k-1} (i_s - 2i_{s+1}) + i_k = i_1 - \sum_{s=2}^{k} i_s.$$

The sequence I determines the cohomology operation

$$Sq^I = Sq^{i_1} \circ \cdots \circ Sq^{i_k} : H^*(X) \to H^*(X)$$

which increases degrees by $d(I)$. I is said to be *admissible* if $i_s \geq 2i_{s+1}$ for $1 \leq s < k$. The empty sequence $I = ()$ is admissible and satisfies $d(I) = 0$, $\ell(I) = 0$ and $e(I) = \infty$; it determines the identity cohomology operation $Sq^{()} = 1$.

1.3.2. Define the *mod* 2 *Steenrod algebra* A as the quotient of the free (associative, unital) algebra F generated by the Steenrod squaring operations Sq^i for $i > 0$, by the two-sided ideal J generated by the Adem relations 1.2.6(8). Here $Sq^0 = 1$.

Then J coincides with the two-sided ideal K of elements in F that annihilate $H^*(X)$ for every space X. Hence A equals the algebra generated by the Steenrod squares of cohomology operations on spaces. In turn, every stable mod 2 cohomology operation $H^n(X) \to H^{n+i}(X)$ is of this form. Hence we may refer to A as precisely the algebra of stable mod 2 cohomology operations.

1.3.3. The admissible monomials Sq^I form a $\mathbb{Z}/2$-module basis for A. These begin:

$$1; Sq^1; Sq^2; Sq^3, Sq^2Sq^1; Sq^4, Sq^3Sq^1; Sq^5, Sq^4Sq^1;$$
$$Sq^6, Sq^5Sq^1, Sq^4Sq^2; Sq^7, Sq^6Sq^1, Sq^5Sq^2, Sq^4Sq^2Sq^1; \dots$$

1.3.4. The cohomology of any space X is naturally a left module over the Steenrod algebra A: The pairing

$$\lambda \colon A \otimes H^*(X) \to H^*(X)$$

takes $Sq^i \otimes x$ to the value $Sq^i(x)$, and more generally $Sq^I \otimes x$ to $Sq^I(x)$.

The cohomology of a space is an *unstable* left A-algebra, in the sense that $Sq^i(x) = x^2$ for $i = \deg(x)$ and $Sq^i(x) = 0$ for $i > \deg(x)$.

1.3.5. The Steenrod algebra A admits a unique structure of Hopf algebra, with coproduct $\psi \colon A \to A \otimes A$ determined by $\psi(Sq^k) = \sum_{i+j} Sq^i \otimes Sq^j$, and counit determined by $\epsilon(Sq^i) = 0$ for $i > 0$. The coproduct is cocommutative.

1.3.6. The cohomology algebra of any space X is naturally a (commutative augmented) left A-module algebra. This means that the cup product map

$$\phi \colon H^*(X) \otimes H^*(X) \to H^*(X)$$

taking $x \otimes y$ to xy is a left A-module homomorphism. (And similarly for the unit map η.) Here the left A-module structure on $H^*(X) \otimes H^*(X)$ uses the Hopf algebra coproduct on A as follows:

$$a \cdot (x \otimes y) = \sum a'x \otimes a''y$$

where $\psi(a) = \sum a' \otimes a''$. With $a = Sq^k$ and $\psi(Sq^k) = \sum_{i+j=k} Sq^i \otimes Sq^j$ the assertion that the cup product map is a left A-module homomorphism amounts to the internal Cartan formula 1.2.6(7).

1.3.7. Let $X = K(\mathbb{Z}/2, n)$ be the Eilenberg–Mac Lane space with $\pi_n(X) = \mathbb{Z}/2$ as its only nonzero homotopy group. For any set S let $P\{S\}$ and $E\{S\}$ denote the polynomial and exterior algebras over $\mathbb{Z}/2$ generated by S, respectively.

1.3.8. Theorem (Serre). *Let $\iota_n \in H^n(K(\mathbb{Z}/2, n)) = \mathbb{Z}/2$ be the fundamental class, dual to the spherical class in $H_n(K(\mathbb{Z}/2, n)) \cong \pi_n(K(\mathbb{Z}/2, n)) = \mathbb{Z}/2$. Then*

$$H^*(K(\mathbb{Z}/2, n)) \cong P\{Sq^I(\iota_n) \mid I \text{ admissible, } e(I) < n\}.$$

as an algebra over A.

1.3.9. Let $A(n)$ be the quotient algebra of A by the two-sided ideal generated by Sq^I with $e(I) > n$. Then the admissible monomials Sq^I with $e(I) \le n$ form a $\mathbb{Z}/2$-module basis for $A(n)$.

For $X = \prod_{n \ge 0} K(\pi_n(X), n)$ a product of Eilenberg–Mac Lane spaces such that each $\pi_n(X)$ is a $\mathbb{Z}/2$-module, let

$$D(X) = \bigoplus_{n \ge 0} A(n) \otimes \pi_n(X).$$

Then $H^*(X)$ is isomorphic to the polynomial algebra on $D(X)$ over $\mathbb{Z}/2$, modulo the relations $Sq^i(x) = x^2$ for $i = \deg(x)$.

1.4. The dual Steenrod operations.

1.4.1. By duality, the Steenrod squaring operation $Sq^i \colon H^n(X) \to H^{n+i}(X)$ induces a homomorphism

$$Sq^i_* \colon H_{n+i}(X) \to H_n(X).$$

(This is clear when X has finite type so $H_*(X) \cong \operatorname{Hom}(H^*(X), \mathbb{Z}/2)$, and follows in general by naturality.) So

$$\langle Sq^i(x), a\rangle = \langle x, Sq^i_*(a)\rangle$$

for all $x \in H^n(X)$ and $a \in H_{n+i}(X)$. Here $\langle x, c\rangle$ denotes the Kronecker pairing, evaluating a cohomology class x on a homology class c.

1.4.2. Note that $(Sq^i Sq^j)_* = Sq^j_* Sq^i_*$. The dual action makes $H_*(X)$ a right module over the Steenrod algebra

$$\rho \colon H_*(X) \otimes A \to H_*(X)$$

by mapping $x \otimes Sq^i$ to $Sq^i_*(x)$, or equivalently a left module over the opposite algebra A^{op}.

We remark that since the Steenrod operations are stable, this dual action of Sq^i_* commutes with suspensions, and hence also acts on the spectrum homology

$$H_n^{\mathrm{spec}}(\mathbf{X}) = \operatorname{colim}_k H_{n+k}(X_k)$$

of a spectrum $\mathbf{X} = (X_k)_k$.

1.4.3. We can give $H_*(X) \otimes H_*(X)$ a right module action by A using the coproduct ψ. Then $H_*(X)$ is a (cocommutative coaugmented) right A-module coalgebra.

1.5. The dual Steenrod algebra A^*.

1.5.1. The Steenrod algebra A with coproduct ψ is a cocommutative Hopf algebra. Hence the dual

$$A^* = \mathrm{Hom}(A, \mathbb{Z}/2)$$

is a commutative Hopf algebra, with coproduct dual to the composition product on A. Its structure was analyzed in [Mil].

1.5.2. Theorem (Milnor). *There is an algebra isomorphism*

$$A^* \cong P\{\xi_1, \xi_2, \dots\}$$

with $\deg(\xi_i) = 2^i - 1$ *for all* $i \geq 1$. *The coproduct on* A^* *is given by*

$$\psi(\xi_k) = \sum_{i+j=k} \xi_i^{2^j} \otimes \xi_j .$$

Here $\xi_0 = 1$.

To identify the ξ_i, note that $A \cong H^*_{\mathrm{spec}}(\mathbf{H}\mathbb{Z}/2; \mathbb{Z}/2)$ so $A^* \cong H_*^{\mathrm{spec}}(\mathbf{H}\mathbb{Z}/2; \mathbb{Z}/2)$ receives a map from $\Sigma^{-1} H_*(K(\mathbb{Z}/2, 1); \mathbb{Z}/2)$. Then ξ_i is the image of the generator of $H_{2^i}(K(\mathbb{Z}/2, 1); \mathbb{Z}/2) \cong \mathbb{Z}/2$.

1.5.3. The ξ_i are the indecomposable elements of A^* (i.e., generate $I(A^*)/I(A^*)^2$ where $I(A^*) \subset A^*$ is the augmentation ideal $\ker(\epsilon)$), and are dual to the primitive elements Q_{i-1} in A (i.e., those $x \in A$ such that $\psi(x) = x \otimes 1 + 1 \otimes x$). These Milnor primitives can be inductively defined by $Q_0 = Sq^1$ and $Q_n = [Q_{n-1}, Sq^{2^n}]$ for $n \geq 1$.

1.5.4. Directly dual to the A-module action λ on $H^*(X)$ is a homomorphism

$$\lambda^* \colon H_*(X) \to H_*(X) \otimes A^*$$

making $H_*(X)$ a (cocommutative coaugmented) right A^*-comodule coalgebra.

The two dualized viewpoints are related as follows. With I running through the admissible sequences the $\{Sq^I\}_I$ form a basis for A. There is a dual basis $\{(Sq^I)^*\}_I$ for A^*, and the homology operations $\{Sq^I_*\}_I$ give a basis for A^{op}. Then $\rho(x \otimes Sq^I_*) = Sq^I_*(x)$ and

$$\lambda^*(x) = \sum_I Sq^I_*(x) \otimes (Sq^I)^*$$

for $x \in H_*(X)$.

2. Infinite loop spaces

2.1. Iterated loop spaces.

2.1.1. Let $n \geq 1$. A space X is an *n-fold loop space* if there is a space X_n and a homotopy equivalence $X \simeq \Omega^n X_n$. We call X_n an *n-fold delooping* of X. Then for $0 \leq k \leq n$ there are spaces $X_k = \Omega^{n-k} X_n$, with $\Omega X_{k+1} \cong X_k$ for $0 \leq k < n$, and $X \simeq X_0$.

2.1.2. Any loop space is naturally an H-group, i.e., a group up to homotopy. If $X \simeq \Omega Y$ then composition of loops defines a product

$$\mu\colon X \times X \simeq \Omega Y \times \Omega Y \to \Omega Y \simeq X$$

and the base point defines a unit map $\eta\colon * \to X$. These satisfy associativity and left and right unit axioms up to homotopy. Furthermore, reversing the orientation of a loop determines a homotopy inverse

$$\iota\colon X \simeq \Omega Y \to \Omega Y \simeq X$$

which acts as a left and right inverse, up to homotopy. Thus X represents a group valued functor $Z \mapsto [Z, X]$.

2.1.3. A space X is an *infinite loop space*, or Ω^∞*-space*, if there is a space X_n for each $n \geq 0$, with $\Omega X_{k+1} \cong X_k$ for all $k \geq 0$, and $X \simeq X_0$. The sequence of spaces $(X_n)_{n\geq 0}$ is called a *spectrum*, and X_0 is its *underlying space*. Hence an infinite loop space is any space homotopy equivalent to the underlying space of a spectrum. See [May1].

2.1.4. A spectrum $\mathbf{E} = (E_n)_{n\geq 0}$ (with $\Omega E_{n+1} \cong E_n$ for all $n \geq 0$) represents a generalized cohomology theory $\tilde{E}^*$, defined on CW-complexes X by

$$\tilde{E}^n(X) = [X, E_n]$$

(based homotopy classes) for all $n \geq 0$. This is a contravariant homotopy functor with natural suspension isomorphisms $\tilde{E}^n(X) \cong \tilde{E}^{n+1}(\Sigma X)$ for all n and spaces X. Equivalently, $\tilde{E}^*$ satisfies excision, admits long exact sequences for pairs, etc.

2.1.5. Dually, $\mathbf{E}$ represents a generalized homology theory $\tilde{E}_*$, defined on CW-complexes by

$$\tilde{E}_n(X) = \operatorname{colim}_k \pi_{n+k}(E_k \wedge X)$$

for all integers n. This is a covariant homotopy functor with natural suspension isomorphisms $\tilde{E}_n(X) \cong \tilde{E}_{n+1}(\Sigma X)$, as above. Again $\tilde{E}_*$ satisfies excision, admits long exact sequences for pairs, etc. The *coefficient groups* of the generalized homology theory are the groups $\tilde{E}_n = \tilde{E}_n(S^0)$.

2.2. Examples of infinite loop spaces.

2.2.1. Infinite loop spaces are ubiquitous, i.e., constantly encountered.

2.2.2. For each Abelian group A and $k \geq 0$, the Eilenberg–Mac Lane space $K(A, k)$ is an infinite loop space. Its n-th delooping is a $K(A, n+k)$-space. The spectrum $\mathbf{H}A$ with n-th space $K(A, n)$ has underlying space $K(A, 0) \simeq A$, and represents ordinary singular homology and cohomology with coefficients in A.

2.2.3. Let

$$Q(X) = \operatorname{colim}_n \Omega^n \Sigma^n X,$$

where $\Omega^n \Sigma^n X \to \Omega^{n+1}\Sigma^{n+1} X$ is obtained by applying Ω^n for $Y = \Sigma^n X$ to the map $Y \to \Omega\Sigma Y$ that takes a point $y \in Y$ to the loop $t \mapsto y \wedge t$ in $\Sigma Y = Y \wedge S^1$. Then $Q(X) \cong \Omega Q(\Sigma X)$.

Then $Q(X)$ is the free infinite loop space on X, in the sense that the functor Q from spaces to infinite loop spaces is left adjoint to the forgetful functor mapping the other way. Equivalently, there is a natural bijective correspondence between infinite loop maps $Q(X) \to Y$ and space level maps $X \to Y$, when Y is an infinite loop space.

2.2.4. One important example is $QS^0 = \mathrm{colim}_n \Omega^n S^n$. It is the underlying space in the sphere spectrum $\mathbf{S}$, with n-th space $Q(S^n)$. It represents stable homotopy and cohomotopy as generalized homology and cohomology theories. Its coefficient groups $\pi_* QS^0$ are the stable homotopy groups of spheres.

2.2.5. By Bott periodicity, $\Omega^2(\mathbb{Z} \times BU) \simeq \mathbb{Z} \times BU$, while $\Omega^8(\mathbb{Z} \times BO) \simeq \mathbb{Z} \times BO$ and $\Omega^8(\mathbb{Z} \times BSp) \simeq \mathbb{Z} \times BSp$. Here BG denotes the classifying space of the group G, with U, O and Sp the infinite unitary, orthogonal and symplectic groups.

Hence $\mathbb{Z} \times BU$ is an infinite loop space, with n-th delooping homotopy equivalent to $\mathbb{Z} \times BU$ for $n \geq 0$ even, and $\Omega(\mathbb{Z} \times BU) \simeq U$ for $n \geq 1$ odd. It is the underlying space of the complex K-theory spectrum $\mathbf{KU}$, representing complex topological K-theory and K-homology as generalized cohomology and homology theories, respectively.

Likewise $\mathbb{Z} \times BO$ and $\mathbb{Z} \times BSp$ are infinite loop spaces, underlying the spectra $\mathbf{KO}$ and $\mathbf{KSp}$ representing real, resp. quaternionic, topological K-theory.

2.2.6. The connected, 1-connected and 2-connected covers of $\mathbb{Z} \times BO$ are BO, BSO and $BSpin$. The connected and 2-connected covers of $\mathbb{Z} \times BU$ are BU and BSU. The connected cover of $\mathbb{Z} \times BSp$ is BSp. Here SO and SU are the infinite special (determinant 1) orthogonal and unitary groups, and $Spin$ is the infinite spin group, which is the universal (double) cover of SO.

2.3. The little cubes operad.

2.3.1. To discuss the geometry of n-fold loop spaces, or infinite loop spaces, we need to consider certain spaces of operations on such iterated loop spaces. These combine to form a structure called an *operad*, for which we omit the precise definition. See [May3]. These spaces were introduced by Boardman and Vogt.

2.3.2. Let $I^n = I \times \cdots \times I$ denote the unit n-cube. A *little n-cube* f in I^n is a linear embedding $f\colon I^n \to I^n$ with parallel axes, i.e., it is of the form $f = f_1 \times \cdots \times f_n$ where each $f_s\colon I \to I$ is a linear function $f_s(t) = (1-t)x_s + ty_s$ with $0 \leq x_s < y_s \leq 1$.

Let $\mathcal{C}_n(j)$ be the set of those ordered j-tuples

$$c = \langle c_1, \dots, c_j \rangle$$

of little n-cubes whose images have pairwise disjoint interiors in I^n. Topologize $\mathcal{C}_n(j)$ as a subspace of the space of maps $\coprod^j I^n \to I$. This is the space of little n-cubes.

Multiplication by the identity little 1-cube $1\colon I \to I$ defines stabilization maps $\mathcal{C}_n(j) \to \mathcal{C}_{n+1}(j)$, replacing each little n-cube f by the little $(n+1)$-cube $f \times 1$. Let $\mathcal{C}_\infty(j) = \mathrm{colim}_n \mathcal{C}_n(j)$ be the union of these spaces, i.e., the space of j little ∞-cubes.

2.3.3. As an example, consider $f\colon I \to I$ and $g\colon I \to I$ given by $f(t) = t/2$ and $g(t) = (t+1)/2$. Then $\langle f, g \rangle$ is a point in $\mathcal{C}_1(2)$. We shall see in the next section that this point corresponds to the usual multiplication on loop spaces given by composition of loops. Similarly $\langle f \times 1^{n-1}, g \times 1^{n-1} \rangle$ is a little n-cube, representing the loop sum on n-fold loop spaces.

2.3.4. The symmetric group Σ_j acts on $\mathcal{C}_n(j)$ from the right, by permuting the ordering of the j little n-cubes.

For example, the order two element $T \in C_2 \cong \Sigma_2$ takes $\langle f, g \rangle$ to $\langle g, f \rangle$, which represents multiplication of loops in the opposite order. Then $\langle f, g \rangle$ and $\langle g, f \rangle$ are in different components of $\mathcal{C}_1(2)$, but their images in $\mathcal{C}_n(2)$ are in the same component for all $n \geq 2$. This reflects the homotopy commutativity of loop sum on n-fold loop spaces for $n \geq 2$, and that $\pi_n(X)$ is Abelian for $n \geq 2$.

2.3.5. Let $F(\mathbb{R}^n; j)$ be the *configuration space* of j distinct points $(x_1, \dots, x_j)$ in $\mathbb{R}^n$. Then Σ_j acts freely on $F(\mathbb{R}^n; j)$ by permuting the points, and $F(\mathbb{R}^n; j)$ is $(n-2)$-connected. The inclusion $\mathbb{R}^n \to \mathbb{R}^{n+1}$ induces Σ_j-equivariant maps $F(\mathbb{R}^n; j) \to F(\mathbb{R}^{n+1}; j)$, with union

$$F(\mathbb{R}^\infty; j) = \operatorname{colim}_n F(\mathbb{R}^n; j).$$

Then $F(\mathbb{R}^\infty; j)$ is a free contractible Σ_j-space, hence has the Σ_j-equivariant homotopy type of $E\Sigma_j$.

2.3.6. There is a Σ_j-equivariant homotopy equivalence $\mathcal{C}_n(j) \to F(\mathbb{R}^n; j)$, which takes a j-tuple $\langle c_1, \dots, c_j \rangle$ to the configuration $(x_1, \dots x_j)$ of center points

$$x_s = c_s(1/2, \dots, 1/2)$$

of the little n-cubes. Hence $\mathcal{C}_\infty(j) \simeq E\Sigma_j$ for all $j \geq 0$.

2.3.7. When $j = 2$, $\mathcal{C}_n(2)$ is the space of pairs $\langle c_1, c_2 \rangle$ of little cubes in I^n with disjoint interiors, and $T \in C_2$ acts by transposing their numbering. By the center point map, this is equivariantly homotopy equivalent to the configuration space $F(\mathbb{R}^n; 2)$ of pairs of disjoint points (x_1, x_2) in $\mathbb{R}^n$, and T again transposes their numbering. This is in turn equivariantly homotopy equivalent to S^{n-1} with the antipodal action, by the map taking (x_1, x_2) to the unit vector pointing from x_1 to x_2. These identifications are all compatible with stabilization in n, and provides an equivariant homotopy equivalence from $\mathcal{C}_\infty(2)$ to S^∞, which is our preferred model for EC_2.

2.4. The action on iterated loop spaces.

2.4.1. Let X be an n-fold loop space, with n-th delooping X_n. So $X \simeq \Omega^n X_n$. Assume that in fact X is an n-fold loop space in the strict sense, i.e., that $X = \Omega^n X_n$. We will show that each point in $\mathcal{C}_n(j)$ defines a map

$$X^j = X \times \cdots \times X \to X,$$

i.e., an operation on X with j inputs and 1 output. In the next section we will use this map to produce homology operations from $H_*(X)$ to itself. Thereafter we may replace X by any homotopy equivalent space, and hence obtain homology operations also for n-fold loop spaces in the weaker sense, with $X \simeq \Omega^n X_n$.

2.4.2. For $c = \langle c_1, \dots c_j \rangle \in \mathcal{C}_n(j)$ and $X = \Omega^n Y$, define a map

$$\theta_{n,j}(c)\colon X^j \to X$$

as follows. Take points $x_1, \dots, x_j \in X = \Omega^n Y$, viewed as maps $y_s \colon I^n \to S^n \to Y$. (Here $I^n \to S^n$ collapses $\partial I^n \subset I^n$ to the base point, and identifies $I^n/\partial I^n \cong S^n$ by a standard homeomorphism. For example, fix a homeomorphism $I^1/\partial I^1 \cong S^1$ and smash with itself n times.)

Define $\theta_{n,j}(c) \colon I^n \to Y$ to be $y_s \circ c_s^{-1}$ on the subspace $c_s(I^n) \subseteq I^n$, for all $1 \leq s \leq j$, and map the remainder of I^n to the base point. The resulting map $I^n \to Y$ takes ∂I^n to the base point, hence descends to a map $S^n \to Y$, or equivalently a point in $X = \Omega^n Y$.

We rephrase the definition. A j-tuple of little n-cubes $\langle c_1, \dots, c_j \rangle$ and a j-tuple of maps $(I^n, \partial I^n) \to (Y, *)$ are paired up to define a map from the union

$$\bigcup_{s=1}^{j} c_s(I^n) \subseteq I^n$$

to Y, which is extended by mapping the complement to the base point. This defines a map $(I^n, \partial I^n) \to (Y, *)$. Thinking of such maps as points in $X = \Omega^n Y$ defines $\theta_{n,j}(c)(x_1, \dots, x_j)$.

2.4.3. Letting $c = \langle c_1, \dots, c_j \rangle$ vary, we obtain a continuous action map

$$\theta_{n,j} \colon \mathcal{C}_n(j) \times X^j \to X,$$

which is natural in n-fold loop spaces $X = \Omega^n Y$. Each point in $\mathcal{C}_n(j)$ thus parametrizes a map $X^j \to X$.

The permutation group Σ_j acts diagonally on the left hand side, by renumbering j-tuples of little n-cubes in $\mathcal{C}_n(j)$ and by permuting the factors in X^j. The map $\theta_{n,j}$ ignores this action, and hence descends to a continuous map

$$\theta'_{n,j} \colon \mathcal{C}_n(j) \times_{\Sigma_j} X^j \to X.$$

2.4.4. Now suppose X is an infinite loop space in the strict sense, i.e., that there are spaces $(X_n)_{n \geq 0}$ with $X = X_0$ and $X_n \cong \Omega X_{n+1}$ for all $n \geq 0$. Then the maps $\theta_{n,j}$ and $\theta_{n+1,j}$ are compatible with the stabilization $\mathcal{C}_n(j) \to \mathcal{C}_{n+1}(j)$, and so we obtain maps

$$\theta_{\infty,j} \colon \mathcal{C}_\infty(j) \times X^j \to X$$

and

$$\theta'_{\infty,j} \colon \mathcal{C}_\infty(j) \times_{\Sigma_j} X^j \to X.$$

Combined with the Σ_j-equivariant homotopy equivalence $E\Sigma_j \simeq \mathcal{C}_\infty(j)$, we get the structure map

$$\theta_j \colon X^j_{h\Sigma_j} = E\Sigma_j \times_{\Sigma_j} X^j \to X.$$

When $j = 2$ we may use S^∞ with the antipodal action as our model for $E\Sigma_2$, and get a map

$$\theta_2 \colon S^\infty \times_{C_2} (X \times X) \to X$$

where $T \in C_2$ transposes the factors in $X \times X$.

2.5. A monad.

2.5.1. Let X be any space. Define

$$C_\infty X = \coprod_{j\geq 0} \mathcal{C}_\infty(j) \times_{\Sigma_j} X^j / \sim$$

where $(d_i(c), x) \sim (c, d_i(x))$ for all $c = \langle c_1, \dots, c_j \rangle$, $x = (x_1, \dots, x_{j-1})$ and $0 \leq i \leq j$. Here $d_i(c)$ omits the i-th little cube c_i from c, while $d_i(x)$ inserts the base point $*$ in the i-th position of x, shifting $x_i, \dots, x_{j-1}$ one step to the right.

Then C_∞ is a *monad*, in the sense that there is a product $\mu\colon C_\infty(C_\infty X) \to C_\infty X$ induced by composition of little ∞-cubes, and a unit $\eta\colon X \to C_\infty X$ mapping to the $(j = 1)$-summand. These satisfy associativity and unit axioms.

2.5.2. There is a natural map for any X

$$\alpha_\infty\colon C_\infty X \to Q(X) = \operatorname{colim}_n \Omega^n \Sigma^n X.$$

When X is an infinite loop space, the structure maps $\theta'_{\infty,j}$ respect the identifications in $C_\infty X$ and combine to define a map

$$\theta\colon C_\infty X \to X.$$

Then θ is the composite of α_∞ with the counit map $Q(X) \to X$ obtained from the infinite loop space structure on X. The map θ satisfies transitivity and unit axioms with respect to the monad product μ. We say that X is a C_∞-algebra.

2.5.3. Approximation theorem (May). *$\alpha_\infty\colon C_\infty X \to Q(X)$ is a group completion for every space X. In particular, for X path connected it is a weak homotopy equivalence.*

2.5.4. Recognition principle (May). *Every infinite loop space is a C_∞-algebra, and every connected C_∞-algebra has the weak homotopy type of an infinite loop space.*

See [May3] and [May4].

3. Homology of infinite loop spaces

3.1. The Pontryagin product.

3.1.1. An *H-space* X is a space with a homotopy associative and unital map $\mu\colon X \times X \to X$. Then μ induces a product

$$\phi\colon H_*(X) \otimes H_*(X) \cong H_*(X \times X) \xrightarrow{\mu_*} H_*(X)$$

which, together with the unit

$$\eta\colon \mathbb{Z}/2 = H_*(*) \to H_*(X)$$

induced by the base point inclusion $i\colon * \to X$, makes $H_*(X)$ an algebra. We call ϕ the *Pontryagin product* induced by the H-space structure map μ.

3.1.2. For an H-space X, the algebra and coalgebra structures on $H_*(X)$ induced by the Pontryagin product and the diagonal map are compatible, in the sense that the algebra product and unit are coalgebra homomorphisms, and dually the coalgebra coproduct and counit are algebra homomorphisms. This structure on $H_*(X)$ is called a *Hopf algebra.*

Since the diagonal is cocommutative, $H_*(X)$ is always a cocommutative Hopf algebra.

3.1.3. When the H-space X admits a homotopy inverse $\iota\colon X \to X$, we call X an *H-group.* The inverse induces a *conjugation*

$$\chi\colon H_*(X) \xrightarrow{\iota_*} H_*(X)$$

which satisfies $\phi(1 \otimes \chi)\psi = \eta\epsilon$.

3.1.4. Loop spaces $X = \Omega X_1$ provide examples of H-groups. The product $\mu\colon \Omega X_1 \times \Omega X_1 \to \Omega X_1$ is given by loop sum, the unit $\eta\colon * \to \Omega X_1$ maps to the constant loop, and the inverse $\iota\colon \Omega X_1 \to \Omega X_1$ reverses the parametrization of a loop. Then $H_*(X) = H_*(\Omega X_1)$ is a cocommutative Hopf algebra with conjugation.

We will write $ab = \phi(a \otimes b)$ for the Pontryagin product on $H_*(X)$ when X is a loop space.

3.1.5. Second loop spaces $X = \Omega^2 X_2$ provide examples of homotopy commutative H-groups. This follows since the space of 2 little n-cubes $\mathcal{C}_n(2)$ is connected for $n \geq 2$, and so the multiplication maps μ and μT are homotopic. (Here $T\colon X \times X \to X \times X$ denotes the twist map.)

Hence $H_*(X)$ is a commutative and cocommutative Hopf algebra with conjugation, when X is an n-fold loop space, with $n \geq 2$. This applies in particular when X is an infinite loop space.

3.2. Structure theorems for Hopf algebras.

3.2.1. For a discussion of algebras, coalgebras, Hopf algebras and the associated module structures, see [MM]. That paper also proves the following structure theorems for Hopf algebras, due to Hopf, Leray and Borel.

3.2.2. Let A be a graded Hopf algebra over $\mathbb{Z}/2$, with product ϕ, unit and coaugmentation η, coproduct ψ, and counit and augmentation ϵ.

A Hopf algebra A is *connected* if the unit map η is an isomorphism in degree 0. It is of *finite type* if the underlying vector space in each degree is finite dimensional.

The *height* of an element $x \in A$ is the least integer q such that $x^q = 0$; or, if no such integer exists, the height of x is infinity.

3.2.3. Theorem (Borel). *If A is a commutative connected Hopf algebra of finite type over $\mathbb{Z}/2$, then there is an isomorphism of algebras*

$$A \cong \bigotimes_{i \in I} A_i$$

where each A_i is a Hopf algebra with a single algebra generator x_i.

3.2.4. Proposition. *If A is a connected Hopf algebra over $\mathbb{Z}/2$ which has one algebra generator x, then the height of x is either a power of two or infinity. Thus $A \cong \mathbb{Z}/2[x]/(x^{2^e})$ or $A \cong \mathbb{Z}/2[x]$ as algebras.*

3.2.5. Let $I(A) = \ker(\epsilon)$ be the augmentation ideal, and dually let $J(A) = \mathrm{cok}(\eta)$. There are direct sum decompositions $A \cong I(A) \oplus \mathbb{Z}/2$ and $A \cong \mathbb{Z}/2 \oplus J(A)$ since $\epsilon\eta = 1$ on $\mathbb{Z}/2$.

Let the vector space

$$P(A) = \{x \in A \mid \psi(x) = x \otimes 1 + 1 \otimes x\}$$

be the *primitives* in A, and let

$$Q(A) = I(A)/I(A) \cdot I(A)$$

be the *indecomposables* of A. We think of $P(A)$ as a subspace of A, and $Q(A)$ as a quotient space of A.

We say that A is *primitively generated* if it is generated by $P(A)$ as an algebra.

3.2.6. Theorem. *If A is a commutative, primitively generated, connected Hopf algebra of finite type over $\mathbb{Z}/2$, then there is an isomorphism of Hopf algebras*

$$A \cong \bigotimes_{i \in I} A_i$$

where each A_i is a Hopf algebra with a single algebra generator x_i.

See [MM].

3.2.7. When A has one algebra generator x as in Proposition 3.2.4, $\psi(x) = x \otimes 1 + 1 \otimes x$ for degree reasons, and this determines the Hopf algebra structure on A in either of the cases $A = \mathbb{Z}[x]/(x^{2^e})$ and $A = \mathbb{Z}/2[x]$.

3.3. The Araki–Kudo/Dyer–Lashof operations.

3.3.1. Araki and Kudo [AK] defined homology operations Q^i in the mod 2 homology of infinite loop spaces. Their work was extended by Browder in [Br]. Dyer and Lashof [DL] introduced the corresponding operations in mod p homology, for p an odd prime. The algebra of operations is known as the Dyer–Lashof algebra, and we will refer to the Q^i as Dyer–Lashof operations, in spite of the fact that we restrict to the mod 2 case, where the operations were first defined by Araki and Kudo.

The mod 2 Steenrod operations are also known as *reduced* squares, as they take values in degrees below the degree of the cup product square. Dually the mod 2 Dyer–Lashof operations are *increased* squares, as they take values in degrees above the degree of the Pontryagin product square.

3.3.2. We extend our notations from section 1.2. Let $C_2 = \{1, T\}$ be the group of order 2. The space of 2 little ∞-cubes $\mathcal{C}_\infty(2)$ is a free contractible C_2-space, and serves as a model for EC_2. The augmented singular complex $\epsilon\colon S_*(\mathcal{C}_\infty(2)) \to \mathbb{Z}/2$ is a C_2-free resolution of $\mathbb{Z}/2$.

Also let $\epsilon\colon W_* \to \mathbb{Z}/2$ be the periodic C_2-free resolution of $\mathbb{Z}/2$ with $W_i = \mathbb{Z}/2[C_2]\{e_i\}$ and $\partial(e_{i+1}) = 1 \cdot e_i + T \cdot e_i$ for all $i \geq 0$. There is a chain homotopy equivalence $W_* \to S_*(\mathcal{C}_\infty(2))$ over $\mathbb{Z}/2$.

Finally there is a chain homotopy equivalence $H_*(X) \to S_*(X)$, where $H_*(X)$ is viewed as a chain complex with zero boundary maps, given by choosing a basis for $H_*(X)$ and choosing representing cycles in $Z_*(X) \subset S_*(X)$ for these homology classes.

3.3.3. Now suppose X is a strict infinite loop space. Thus there are spaces X_n and homeomorphisms $X \cong \Omega^n X_n$ for all $n \geq 0$. We obtained operad actions

$$\theta'_{\infty,2}\colon \mathcal{C}_\infty(2) \times_{C_2} X^2 \to X$$

in 2.4.4. There are isomorphisms

$$H_*(W_* \otimes_{C_2} H_*(X)^{\otimes 2}) \cong H_*(S_*(\mathcal{C}_\infty(2)) \otimes_{C_2} S_*(X)^{\otimes 2}) \cong H_*(\mathcal{C}_\infty(2) \times_{C_2} X^2)$$

induced by the chain equivalences $W_* \simeq S_*(\mathcal{C}_\infty(2))$ and $H_*(X) \simeq S_*(X)$, and the shuffle homomorphism.

Hence we get a homomorphism

$$\theta_*\colon H_*(W_* \otimes_{C_2} H_*(X)^{\otimes 2}) \to H_*(X)$$

by composing the isomorphisms above with $(\theta'_{\infty,2})_*$.

3.3.4. Now suppose given a class $x \in H_n(X)$. When $i \geq n$ we form

$$Q^i(x) = \theta_*(e_{i-n} \otimes x \otimes x)$$

in $H^{n+i}(X)$. This makes sense, because $e_{i-n} \otimes x \otimes x$ defines a cycle in $W_* \otimes_{C_2} H_*(X)^{\otimes 2}$. Therefore, this formula defines the Araki–Kudo/Dyer–Lashof squaring operation

$$Q^i\colon H_n(X) \to H_{n+i}(X)$$

for $i \geq n = \deg(x)$. When $i < n$ we let $Q^i(x) = 0$.

3.3.5. For example, $Q^n(x) = x^2$ when $x = \deg(n)$ is the Pontryagin square, since $\theta_*(e_0 \otimes x \otimes x) = \phi(x \otimes x)$. Next, $Q^{n+1}(x) = \theta_*(e_1 \otimes x \otimes x)$ with $n = \deg(x)$ is represented by the closed path from $x^2 = \phi_*(x \otimes x)$ to itself, parametrized by a path in $\mathcal{C}_\infty(2)$ from the point μ representing loop sum to the (antipodal) point μT representing loop sum in the opposite order.

3.3.6. Theorem. *Let X be an infinite loop space. There are homomorphisms $Q^i\colon H_*(X) \to H_*(X)$ for all $n, i \geq 0$, which satisfy:*

(1) The Q^i are natural with respect to maps $f\colon X \to Y$ of infinite loop spaces.

(2) Q^i raises degrees by i.

(3) $Q^i(x) = 0$ if $i < \deg(x)$.

(4) $Q^i(x) = x^2$ if $i = \deg(x)$, where $x^2 = \phi(x \otimes x)$ is the Pontryagin product square.

(5) $Q^i(1) = 0$ for all $i > 0$, where $1 = [*] \in H_0(X)$ is the algebra unit element.

(6) The external, internal and diagonal Cartan formulas hold:

$$Q^k(x \times y) = \sum_{i+j=k} Q^i(x) \times Q^j(y)$$

in $H_*(X \times Y)$ *for* $x \in H_*(X)$, $y \in H_*(Y)$,

$$Q^k(xy) = \sum_{i+j=k} Q^i(x) Q^j(y)$$

in $H_*(X)$ *for* $x, y \in H_*(X)$, *and*

$$\psi(Q^k(x)) = \sum_{i+j=k} \sum Q^i(x') \otimes Q^j(x'')$$

if $\psi(x) = \sum x' \otimes x''$, $x \in H_*(X)$.

(7) The Q^i *are stable and the Kudo transgression theorem holds:* $Q^i \sigma_*(x) = \sigma_* Q^i(x)$ *where* $\sigma_* \colon \tilde{H}_{n-1}(\Omega X) \to \tilde{H}_n(X)$ *is the homology suspension.*

If X *is simply-connected and if* $x \in H_n(X)$ *transgresses to* $y \in H_{n-1}(\Omega X)$ *in the Serre spectral sequence of the path space fibration* $\Omega X \to PX \to X$, *then* $Q^i(x) \in H_{n+i}(X)$ *transgresses to* $Q^i(y) \in H_{n+i-1}(\Omega X)$.

(8) The Adem relations hold: If $a > 2b$ *then*

$$Q^a Q^b = \sum_i (2i - a, a - b - i - 1) Q^{a+b-i} Q^i.$$

The binomial coefficient can also be written $\binom{i-b-1}{2i-a}$. *The sum runs over* i *with* $a/2 \le i \le a - b - 1$.

(9) The Nishida relations hold: Recall that Sq^a_* *is the homology operation dual to* Sq^a. *Then*

$$Sq^a_* Q^b = \sum_i (a - 2i, b - 2a + 2i) Q^{b-a+i} Sq^i_*.$$

The binomial coefficient can also be written $\binom{b-a}{a-2i}$. *The sum runs over* i *with* $a - b/2 \le i \le a/2$.

For a proof, see [May5, §1]. $Sq^1_* Q^b = (b-1) Q^{b-1}$ is a useful special case of the Nishida relations.

3.3.7. The composite Dyer–Lashof operation $Q^a Q^b(x)$ is defined by the upper composite in the following diagram, where we briefly write θ_j for $\theta'_{\infty,j}$:

$$\begin{array}{ccccc}
\mathcal{C}_\infty(2) \times_{C_2} (\mathcal{C}_\infty(2) \times_{C_2} X^2)^2 & & \xrightarrow{1 \times_{C_2} (\theta_2)^2} & & \mathcal{C}_\infty(2) \times_{C_2} X^2 \\
\gamma \downarrow \simeq & & & & \downarrow \theta_2 \\
\mathcal{C}_\infty(4) \times_{C_2 \ltimes C_2^2} X^4 & \xrightarrow{\pi} & \mathcal{C}_\infty(4) \times_{\Sigma_4} X^4 & \xrightarrow{\theta_4} & X
\end{array}$$

The diagram commutes. Note the covering map π, which is induced by the inclusion $C_2 \ltimes C_2^2 \subset \Sigma_4$ of a Sylow 2-subgroup in Σ_4. The induced homomorphism $H_*(B(C_2 \ltimes C_2^2)) \to H_*(B\Sigma_4)$ is a surjection, split by a transfer map. Hence the elements in the kernel of this map will correspond to relations among the $Q^a Q^b$. These are the Adem relations.

3.3.8. The Dyer–Lashof operations Q^i commute with the conjugation χ in the homology of an infinite loop space. This follows by stability 3.3.6(7) and naturality with respect to the inverse map ι on $X \cong \Omega X_1$.

3.4. The Dyer–Lashof algebra R.

3.4.1. Consider sequences $I = (i_1, \dots, i_k)$ of integers with $i_s \geq 0$. Define the *degree*, *length* and *excess* of I as before by

$$d(I) = \sum_{s=1}^{k} i_s, \qquad \ell(I) = k \qquad \text{and}$$
$$e(I) = i_k - \sum_{s=2}^{k}(2i_s - i_{s-1}) = i_1 - \sum_{s=2}^{k} i_s.$$

The sequence I determines the homology operation

$$Q^I = Q^{i_1} \circ \cdots \circ Q^{i_k} \colon H_*(X) \to H_*(X)$$

which increases degrees by $d(I)$. I is said to be *admissible* if $i_s \leq 2i_{s+1}$ for $1 \leq s < k$. The empty sequence $I = ()$ is admissible and satisfies $d(I) = 0$, $\ell(I) = 0$ and $e(I) = \infty$; it determines the identity homology operation $Q^{()} = 1$.

3.4.2. Let F denote the free (associative, unital) algebra generated by $\{Q^i\}_{i \geq 0}$. For $q \geq 0$ define $J(q)$ to be the two-sided ideal of F generated by the Adem relations 3.3.6(8), and by the relations $Q^I = 0$ if $e(I) < q$. Define $R(q)$ to be the quotient algebra $F/J(q)$, and observe that there are successive quotient maps $R(q) \to R(q+1)$. Here $Q^0 \neq 1$. Let $R = R(0)$; R will be called the *mod* 2 *Dyer–Lashof algebra.*

Then $J(q)$ coincides with the two-sided ideal $K(q)$ of elements in F that annihilate every homology class of degree $\geq q$ of every infinite loop space. Hence $R(q)$ equals the algebra generated by the Dyer–Lashof operations of homology operations in degrees $\geq q$ on infinite loop spaces.

3.4.3. The admissible monomials Q^I of excess $e(I) \geq q$ form a $\mathbb{Z}/2$-module basis for $R(q)$. For $R = R(0)$ these begin:

$$1, Q^0, Q^0Q^0, \dots; Q^1; Q^2, Q^1Q^1; Q^3, Q^2Q^1; Q^4, Q^2Q^2, Q^2Q^1Q^1;$$
$$Q^5, Q^3Q^2; Q^6, Q^4Q^2, Q^3Q^3, Q^3Q^2Q^1; Q^7, Q^4Q^3, Q^4Q^2Q^1; \dots$$

3.4.4. The homology of an infinite loop space X is a left module over R: The pairing

$$R \otimes H_*(X) \to H_*(X)$$

takes $Q^i \otimes x$ to $Q^i(x)$, and more generally $Q^I \otimes x$ to $Q^I(x)$. This left R-module is *allowable* in the sense that $J(q)H_q(X) = 0$ for all $q \geq 0$.

3.4.5. The Dyer–Lashof algebra R admits a unique right A-module structure such that the Nishida relations 3.3.6(9) are satisfied. The coproduct ψ defined on generators by

$$\psi(Q^k) = \sum_{i+j=k} Q^i \otimes Q^j$$

and the counit ϵ with $\epsilon(Q^0) = 1$ make R into a Hopf algebra, as well as a right A-module coalgebra.

3.4.6. A *component coalgebra* C is a coaugmented coalgebra which is a direct sum of connected coalgebras. Let

$$\pi C = \{g \in C \mid \psi(g) = g \otimes g,\ g \neq 0\}.$$

Then πC is a basis for C_0.

A component Hopf algebra B is *monomial* if πB is a monoid under the product of B.

If X is a based space, $H_*(X) = \bigoplus_{g\in\pi_0(X)} H_*(X_g)$ is a component coalgebra, where $X_g \subseteq X$ is the path component $g \in \pi_0(X)$. The base point determines the coaugmentation $\eta\colon \mathbb{Z}/2 \to H_*(X)$. In this case, $\pi H_*(X) \cong \pi_0(X)$.

The homology of an H-space is a monomial component Hopf algebra.

3.4.7. An *allowable R-algebra* is a commutative allowable left R-module algebra such that $Q^i(x) = x^2$ if $i = \deg(x)$. An *allowable R-coalgebra* is a cocommutative component allowable left R-module coalgebra. An *allowable R-Hopf algebra (with conjugation)* is a monoidal left R-module Hopf algebra (with conjugation) which is allowable both as an R-algebra and as an R-coalgebra.

For any of these structures, an *allowable AR-structure* is an allowable left R-structure and an unstable right A-structure of the same type, such that the A- and R-operations satisfy the Nishida relations. Here an unstable right A-structure in homology is the dual of an unstable left A-structure in cohomology.

See [May5, §2].

3.5. Some free functors.

3.5.1. Consider the following categories:

$$\begin{array}{c}
\text{Allowable } AR\text{-Hopf algebras with conjugation} \\
G \uparrow \ \downarrow \text{forget} \\
\text{Allowable } AR\text{-Hopf algebras} \\
W \uparrow \ \downarrow \text{forget} \\
\text{Allowable } AR\text{-coalgebras} \\
E \uparrow \ \downarrow \text{forget} \\
\text{Cocommutative component unstable right } A\text{-module coalgebras}
\end{array}$$

The homology of an infinite loop space lives in the uppermost category. There are forgetful functors going down the list. These admit left adjoints G, W and E, i.e., functors going up the list, yielding free objects in the respective categories.

3.5.2. We now describe the effect of the free functor E on the prime example of an object in the lowermost category, namely $H_*(X)$ for a based space X. Let $JH_*(X) = \operatorname{cok}(\eta)$ be the quotient of $H_*(X)$ by $\mathbb{Z}/2 = H_*(*)$, which is isomorphic to $\tilde{H}_*(X)$. Then

$$EH_*(X) = \mathbb{Z}/2 \oplus \bigoplus_{q\geq 0} R(q) \otimes JH_q(X).$$

Here $EH_*(X)$ is a free allowable AR-coalgebra, with left R-action induced by the surjections $\epsilon\colon R \to \mathbb{Z}/2$ and $R \to R(q)$. The coproduct is defined using the diagonal Cartan formula 3.3.6(6).

As an example, $EH_*(S^0) = \mathbb{Z}/2 \oplus R$.

3.5.3. Next we describe the free functor W on the object $EH_*(X)$. Note that $JEH_*(X) = \mathrm{cok}(\eta)$ is the sum $\bigoplus_{q\geq 0} R(q) \otimes JH_q(X)$. Then

$$WEH_*(X) = P(\bigoplus_{q\geq 0} R(q) \otimes JH_q(X))/K$$

where $P(-)$ denotes the polynomial algebra generated by its argument, and K is the two-sided ideal generated by the relations $Q^i(x) = x^2$ for $i = \deg(x)$.

Then $WEH_*(X)$ is a free allowable AR-Hopf algebra. The product is that obtained from the polynomial algebra, the coproduct is determined by that of $EH_*(X)$. The A- and R-module actions are determined by the internal Cartan formulas 1.2.6(7) and 3.3.6(7).

For example, $WEH_*(S^0) = P(R)/K$ is the polynomial algebra on the admissible Dyer–Lashof operations Q^0 and Q^I of excess $e(I) > 0$, as an algebra. Its components are $\mathbb{N}_0 = \{0, 1, 2, \dots\}$, with $(Q^0)^k$ in the kth component.

3.5.4. A connected allowable AR-Hopf algebra admits a unique conjugation. In general the free functor G to allowable AR-Hopf algebras with conjugation amounts to a localization, group completing the monoid πC of components.

For example, when X is connected $GWEH_*(X) = WEH_*(X)$. When $X = S^0$, $GWEH_*(S^0)$ is a sum indexed of integers $k \in \mathbb{Z}$ of copies of the polynomial algebra $P(R(1))$ on the admissible Q^I with $e(I) > 0$.

3.6 The homology of $Q(X)$.

3.6.1. The homology of the space $C_\infty X$ from 2.5.1 also admits Dyer–Lashof operations, and is an allowable AR-Hopf algebra. The unit inclusion $\eta\colon X \to C_\infty X$ induces a homomorphism $\eta_*\colon H_*(X) \to H_*(C_\infty X)$, whose left adjoint is a natural transformation

$$\bar{\eta}_*\colon WEH_*(X) \to H_*(C_\infty X)$$

extending η_* over the inclusion $H_*(X) \to WEH_*(X)$.

3.6.2 Theorem (May). *For every space X, $\bar{\eta}_*\colon WEH_*(X) \to H_*(C_\infty X)$ is a natural isomorphism of allowable AR-Hopf algebras.*

3.6.3. The homology of the free infinite loop space $Q(X)$ from 2.2.3 certainly admits Dyer–Lashof operations, and is an allowable AR-Hopf algebra with conjugation. The unit inclusion $\eta\colon X \to Q(X)$ now induces a homomorphism $\eta_*\colon H_*(X) \to H_*(Q(X))$, whose left adjoint is a natural transformation

$$\tilde{\eta}_*\colon GWEH_*(X) \to H_*(Q(X))$$

extending η_* over the inclusion $H_*(X) \to GWEH_*(X)$.

3.6.4 Theorem (Dyer–Lashof, May). *For every space X, $\tilde{\eta}_*\colon GWEH_*(X) \to H_*(Q(X))$ is a natural isomorphism of allowable AR-Hopf algebras with conjugation.*

3.6.5 Corollary (Nakaoka).

$$\bigoplus_{n\geq 0} H_*(B\Sigma_n) \cong H_*(C_\infty S^0) \cong WEH_*(S^0) \cong P(R)/K.$$

Here a product $Q^{I_1} \cdot \ldots \cdot Q^{I_k}$ of admissible monomials in R contributes to the summand $H_*(B\Sigma_n)$ where $n = \sum_{s=1}^{k} 2^{\ell(I_s)}$. So Q^i generates $H_i(B\Sigma_2)$. Multiplication by $Q^{()}$ induces the injections $H_*(B\Sigma_n) \to H_*(B\Sigma_{n+1})$ for all n.

3.7 Proof of Theorem 3.6.4.

3.7.1. We consider the proof of 3.6.4, concerning $H_*(Q(X))$. For simplicity, let us assume X is path connected. The general case involves additional bookkeeping to account for the various components.

Let tX be a basis for $JH_*(X) \cong \tilde{H}_*(X)$, and let $P\{TX\}$ by the polynomial algebra generated by the set

$$TX = \{Q^I(x) \mid x \in tX,\ I \text{ admissible, } e(I) > \deg(x)\}.$$

Then as algebras

$$GWEH_*(X) = WEH_*(X) \cong P\{TX\}.$$

3.7.2. If X is $(q-1)$-connected, $q > 1$, then $\eta_* \colon H_*(X) \to H_*(Q(X))$ is an isomorphism for $* < 2q$, by Freudenthal's stability theorem. Likewise the inclusion $H_*(X) \to GWEH_*(X)$ is an isomorphism for $* < 2q$. Thus the theorem holds in degrees $< 2q$ is X is $(q-1)$-connected. We claim that if the theorem holds for the suspension ΣX in degrees $< n$, then it holds for X in degrees $< n-1$. This will finish the proof, since for any $q > 1$ then the theorem for $\Sigma^q X$ in degrees $< 2q$ implies the theorem for X in degrees $< q$. Letting q grow to infinity, the theorem follows.

3.7.3. Let $\{E^r\} = \{E^r_{**}\}_r$ be the Serre spectral sequence in homology for the path space fibration

$$Q(X) \cong \Omega Q(\Sigma X) \to PQ(\Sigma X) \to Q(\Sigma X).$$

Then

$$E^2_{**} = H_*(Q(\Sigma X)) \otimes H_*(Q(X))$$

since the base $Q(\Sigma X)$ is simply-connected. Clearly $E^\infty_{**} = \mathbb{Z}/2$ concentrated in bidegree $(0,0)$.

3.7.4. Take $t\Sigma X = \{\Sigma_* x \mid x \in tX\}$ as the basis for $JH_*(\Sigma X)$. Here $\Sigma_* \colon \tilde{H}_*(X) \cong \tilde{H}_*(\Sigma X)$. Construct a model spectral sequence $\{'E^r\} = \{'E^r_{**}\}_r$ with

$$'E^r_{**} = WEH_*(\Sigma X) \otimes WEH_*(X).$$

The differentials of $\{'E^r\}$ are specified by requiring $\{'E^r\}$ to be a spectral sequence of differential bigraded algebras, such that if $Q^I(x) \in TX$ then there is a transgressive differential

$$\tau(Q^I(\Sigma_* x)) = Q^I(x).$$

Then $\{'E^r\}$ is isomorphic to a tensor product of spectral sequences of the form

$$E\{y\} \otimes P\{\tau(y)\}.$$

Here y runs through

$$\{Q^I(\Sigma_* x) \mid x \in tX\,,\ I \text{ admissible},\ e(I) > \deg(x)\}.$$

This is because to the eyes of $\{'E^r\}$, the base $WEH_*(\Sigma X)$ behaves like a tensor product of exterior algebras on these y.

Clearly $'E^\infty_{**} = \mathbb{Z}/2$ concentrated in bidegree $(0,0)$.

3.7.5. By construction, there is a unique morphism of algebras $f\colon 'E^2_{**} \to E^2_{**}$ such that

$$f = \tilde{\eta}_* \otimes \tilde{\eta}_* \colon WEH_*(\Sigma X) \otimes WEH_*(X) \to H_*(Q(\Sigma X)) \otimes H_*(Q(X)).$$

By 3.3.6(7) and the definition of $\{'E^r_{**}\}_r$, f induces a morphism of spectral sequences.

Hence f is a morphism of first quadrant algebra spectral sequences with E^2-terms of the form (base) $\otimes$ (fiber), both converging to $\mathbb{Z}/2$ in bidegree $(0,0)$. By the Moore–Zeeman comparison theorem, then, the hypothesis that f induces an isomorphism $\tilde{\eta}_*\colon WEH_*(\Sigma X) \to H_*(Q(\Sigma X))$ in degrees $< n$ on the base implies that f induces an isomorphism $\tilde{\eta}_*\colon WEH_*(X) \to H_*(Q(X))$ in degrees $< n-1$ on the fiber.

This completes the proof.

3.7.6. In effect, the proof shows that for $y \in H_*(\Sigma X)$, the classes $Q^I(y)$ in $H_*(Q(\Sigma X))$ for $e(I) \geq \deg(y)$ transgress to classes $Q^I(x)$ in $H_*(Q(X))$, for $e(I) > \deg(x)$, with $x = \tau(y)$. In addition, the differential algebra structure generates powers $(Q^I(x))^k$ of these classes in $H_*(Q(X))$, but these are now products of terms $Q^J(x)$ with excess one less: $e(J) \geq \deg(x)$.

4. Calculations

4.1 E_∞ ring spaces.

Let X be an infinite loop space. The little cubes action maps

$$\theta_{\infty,j}\colon \mathcal{C}_\infty(j) \times X^j \to j$$

satisfy certain compatibility conditions, such that they form a structure called an E_∞-action on X. We say that X is an E_∞-space, or a homotopy-everything space.

In particular, an E_∞-space is a homotopy commutative H-space. For any chosen point of $\mathcal{C}_\infty(2)$ determines a multiplication $\mu\colon X \times X \to X$, which is unital, commutative and associative up to homotopy by the contractibility of each $\mathcal{C}_\infty(j)$, $j \leq 3$. However, the presence of the higher $\theta_{\infty,j}$ and contractibility of $\mathcal{C}_\infty(j)$ for higher j ensures that an E_∞-structure is a more restrictive notion than just a homotopy commutative H-space. All 'higher coherence homotopies' are also available.

The structure maps above allow us to define the Pontryagin product and Dyer–Lashof operations on $H_*(X)$ when X is any E_∞-space.

For example,

$$C_\infty S^0 = \coprod_{j\ge 0} \mathcal{C}_\infty(j)/\Sigma_j \simeq \coprod_{j\ge 0} B\Sigma_j$$

is such an E_∞-space. Its homology is

$$\bigoplus_{j\ge 0} H_*(B\Sigma_j) \cong H_*(C_\infty S^0) \cong WEH_*(S^0) \cong P(R)/K$$

where $P(R)$ is the polynomial algebra on the Dyer–Lashof algebra R, and K is the two-sided ideal of relations generated by $Q^i(x) = x^2$ for $i = \deg(x)$. The isomorphism takes $Q^I \in R$ to the class $Q^I([1])$, where $[1] \in H_0(B\Sigma_1)$ is the nonzero class.

This example shows that E_∞-spaces need not be H-groups, i.e., they need not be grouplike H-spaces. However, any grouplike E_∞-space is an infinite loop space. E_∞-spaces are the homotopical generalization of Abelian monoids, and grouplike E_∞-spaces (=infinite loop spaces) are the homotopical generalization of Abelian groups.

E_∞-spaces arise, for example, as the classifying space $B\mathcal{C}$ of a permutative category $\mathcal{C}$, or more generally of a symmetric monoidal category. Grouplike E_∞-spaces arise as the group completions of such classifying spaces, which equals the K-theory space $K(\mathcal{C}) = \Gamma B\mathcal{C}$ of the category. The homotopy groups of this infinite loop space are the K-groups of the category.

We also encounter the homotopical generalization of semi-rings (no additive inverse assumed) and rings. The former arise as the classifying spaces of bi-permutative categories, or more generally of symmetric bi-monoidal categories.

As an example, consider the category $\mathcal{O}$ of finite dimensional real inner product spaces. It admits two symmetric monoidal pairings: direct sum $\oplus$ and tensor product $\otimes$ of vector spaces. Its classifying space $B\mathcal{O} \simeq \coprod_{n\ge 0} BO(n)$ admits two induced H-space structures: one induced by direct sum $BO(n) \times BO(m) \to BO(n+m)$, and one by tensor product $BO(n) \times BO(m) \to BO(nm)$.

As a different example, points of QS^0 are represented by maps $f\colon S^n \to S^n$ and $g\colon S^m \to S^m$ for n, m large. The loop sum using a suspension coordinate in the source defines one (additive) pairing on QS^0, but the smash product of such maps, $f \wedge g\colon S^{n+m} \to S^{n+m}$, defines another (multiplicative) pairing.

Such spaces, then, have two E_∞-structures, and thus two associated H-space structures. We think of one of these as additive and the other as multiplicative, and assume they are related by suitable distributivity conditions. This structure is known as an *E_∞-ring space.*

The homology $H_*(X)$ of an E_∞-ring space is thus a Hopf algebra with Pontryagin product $*$ and Dyer–Lashof operations Q^i derived from the additive E_∞-structure, but also a Hopf algebra with Pontryagin product # and Dyer–Lashof operations $\tilde{Q}^i$ derived from the multiplicative E_∞-structure. Note that we denote the multiplicative Dyer–Lashof operations with a tilde. The common coproduct and right A-module structure combines with these two related products to yield an (allowable) *AR-Hopf bialgebra.*

There are formulas in [May7, §§1–3] expressing distributivity of # over the additive $*$ and Q^i, the mixed Cartan formula for $\tilde{Q}^i(x * y)$, and the mixed Adem relations for $\tilde{Q}^a(Q^b(x))$. In principle, they inductively reduce the determination of the multiplicative Pontryagin product $\#$ and Dyer–Lashof operations $\tilde{Q}^i$ in an E_∞ ring space X to their evaluation on additive R-algebra generators for $H_*(X)$.

4.2. The homology of SO, BO etc..

Calculations begin with $O(1) = C_2 = \Sigma_2$. Its classifying space $BO(1) \simeq \mathbb{R}P^\infty$ has $H^*(BO(1)) = P\{x\}$ with $\deg(x) = 1$, and dually $H_*(BO(1)) = \mathbb{Z}/2\{e_i \mid i \geq 1\}$, with e_i dual to x^i.

The diagonal inclusion $O(1)^n \to O(n)$ induces maps $BO(1)^n \to BO(n)$ of classifying spaces. The induced map on cohomology

$$H^*(BO(n)) \to H^*(BO(1))^{\otimes n} \cong P\{x_1, \dots x_n\}$$

is an injection, with image the symmetric polynomials in the $x_1, \dots, x_n$. Hence

$$H^*(BO(n)) \cong P\{w_1, \dots, w_n\}$$

with w_i the ith Stiefel–Whitney class, mapping to the ith elementary symmetric polynomial $\sigma_i(x_1, \dots, x_n)$ by the injection above. Letting $n \to \infty$, we have

$$H^*(BO) \cong P\{w_i \mid i \geq 1\}.$$

The coproduct tied to Whitney sum is $\psi(w_k) = \sum_{i+j=k} w_i \otimes w_j$.

Dually, the map on homology

$$H_*(BO(1))^{\otimes n} \to H_*(BO(n))$$

is surjective, identifying $H_*(BO(n))$ with the coinvariants of the n-fold tensor power of $\mathbb{Z}/2\{e_i\}_i$ under the permutation action. Thus

$$H_*(BO(n)) \cong \bigoplus_{d \leq n} P\{e_1, \dots, e_n\}_d$$

where $P\{e_1, \dots, e_n\}_d$ denotes the degree d part of the polynomial algebra. Letting $n \to \infty$, we have

$$H_*(BO) = P\{e_i \mid i \geq 1\}$$

where each e_i is the image of $e_i \in H_1(BO(1))$ under the inclusion $BO(1) \to BO$.

Let $Q_0 = Sq^1$ and $Q_n = [Q_{n-1}, Sq^{2^n}]$ be the Milnor primitives in the Steenrod algebra A. Recall that $H^*(K(\mathbb{Z}/2, 2)) \cong P\{\iota_2, Q_n(\iota_2) \mid n \geq 0\}$ where ι_2 is the fundamental class, and

$$Q_n(\iota_2) = Sq^{2^n} \circ \dots \circ Sq^1(\iota_2).$$

The Serre spectral sequences for the 1- and 2-connected coverings of BO give:

$$H^*(BSO) = P\{w_i \mid i \geq 2\}$$
$$H^*(BSpin) = H^*(BSO)/(w_2, Q_n(w_2) \mid n \geq 0).$$

Dually

$$H_*(BSO) = P\{e_2, e_3, \dots\}$$
$$H_*(BSpin) = (?).$$

The Serre spectral sequences for the path space fibration for BSO and $BSpin$ yields:

$$H^*(SO) = P\{\sigma^*(w_{2i}) \mid i \geq 1\}$$
$$H^*(Spin) = H^*(SO)/(\sigma^*(w_2))$$

where $\sigma^*(w_{2i})$ in degree $2i-1$ transgresses to w_{2i}. Dually

$$H_*(SO) = \Gamma\{a'_{2i-1} \mid i \geq 1\}$$
$$H_*(Spin) = \Gamma\{a'_{2i-1} \mid i \geq 2\}$$

(divided power algebras) with a'_{2i-1} dual to $\sigma^*(w_{2i})$. This gives a different basis for $H_*(SO)$, compared to

$$H_*(SO) = E\{a_i \mid i \geq 1\}$$

where a_i is the image of e_i under the reflection map $BO(1) \to SO$ taking a line L in $\mathbb{R}^\infty$ to the reflection in L followed by reflection in a fixed line.

4.3. The homology of SG, G/O and BSG.

There is a unit map $QS^0 \to BO \times \mathbb{Z}$ of E_∞ ring spaces.

One way to construct it is as an induced map of K-theory spaces: Let $\mathcal{E}$ be the symmetric bimonoidal category of finite sets, under disjoint union $\coprod$ and Cartesian product $\times$. Likewise let $\mathcal{O}$ be the symmetric bimonoidal category of finite dimensional real inner product spaces, under direct sum $\oplus$ and tensor product $\otimes$. There is a symmetric bimonoidal functor $\mathcal{E} \to \mathcal{O}$ taking a finite set U to the real vector space $V = \mathbb{R}\{U\}$ with U as an orthonormal basis. Then on classifying spaces we have a map

$$B\mathcal{E} \simeq \coprod_{n\geq 0} B\Sigma_n \to B\mathcal{O} \simeq \coprod_{n \geq 0} BO(n)$$

of E_∞ ring spaces. Group completing, we obtain the unit map

$$K(\mathcal{E}) \simeq QS^0 \to K(\mathcal{O}) \simeq BO \times \mathbb{Z}$$

by way of the Barratt–Priddy–Quillen equivalence on the left.

On π_0, the unit map is an isomorphism $\mathbb{Z} \cong \mathbb{Z}$. Let $[i] \in H_0(QS^0)$ denote the homology class of a point in the degree i component $Q_i(S^0) \subset QS^0$. Restricting the unit map to the 0-components, we obtain an additive infinite loop map $Q_0(S^0) \to BO = BO_\oplus$. Restricting instead to the 1-components, we get a multiplicative infinite loop map $SG = Q_1(S^0) \to BO_\otimes$. (These connected E_∞-spaces are actually infinite loop spaces, as noted above.)

Consider the infinite loop space fiber sequence

$$SO \xrightarrow{j} SG \to G/O \to BSO \xrightarrow{Bj} BSG.$$

Here j is the natural inclusion (the j-*map*), taking an isometry $\mathbb{R}^n \to \mathbb{R}^n$ in $SO(n)$ to its induced map on 1-point compactifications $S^n \to S^n$ in SG. Then j is an infinite loop map from the additive infinite loop space structure on $SO \simeq \Omega BSO_\oplus$ to the multiplicative infinite loop space structure on $SG = Q_1(S^0)$. Next, Bj is the classifying map of j, and is G/O the homotopy fiber of Bj.

The calculation of $H_*(QS^0)$ gives $H_*(Q_0(S^0)) \cong P(X_0)$ as an algebra under $*$, where

$$X_0 = \{Q^I([1]) * [2 - 2^{\ell(I)}] \mid I \text{ admissible, } \ell(I) \geq 1 \text{ and } e(I) > 0\}.$$

We can define a shifted $*$-product $\underline{*}$ on $H_*(SG)$ by $x \underline{*} y = (x*[-1])*(y*[-1])*[1]$. Then $H_*(SG) \cong P(X_1)$ as an algebra under $\underline{*}$, where

$$X_1 = \{Q^I([1]) * [1 - 2^{\ell(I)}] \mid I \text{ admissible, } \ell(I) \geq 1 \text{ and } e(I) > 0\}.$$

However, we are interested in the multiplicative infinite loop space structure on SG, which is related to that on G/O, and therefore rather want to know the algebra structure of $H_*(SG)$ under $\#$.

For an admissible sequence I with $d(I) > 0$, define

$$x_I = Q^I([1]) * [1 - 2^{\ell(I)}] \in H_*(SG).$$

In particular $x_i = Q^i([1]) * [-1]$ for $i \geq 1$.

Theorem.

(1) $H_*(SO) \cong E\{a_i \mid i \geq 1\}$ *as an algebra. Here* a_i *is the image of the generator in* $H_i(BO(1))$ *under the reflection map* $BO(1) \to SO$.

(2) $H_*(SG) \cong E\{x_i \mid i \geq 1\} \otimes P(X)$ *as an algebra under* $\#$, *where*

$$X = \{x_I \mid \ell(I) > 2 \text{ and } e(I) > 0, \text{ or } \ell(I) = 2 \text{ and } e(I) \geq 0\}.$$

Here X *is the union of the* $x_{(i,i)}$ *with* $i \geq 1$ *corresponding to* $I = (i,i)$ *of excess* 0, *and the* x_I *with* $\ell(I) \geq 2$ *and* $e(I) > 0$.

(3) $j_*(a_i) = x_i$ *for all* $i \geq 1$, *so* $\mathrm{im}(j_*) = E\{x_i \mid i \geq 1\}$ *and*

$$H_*(G/O) \cong H_*(SG)//H_*(SO) \cong P(X).$$

(4) $H_*(BO) \cong P\{v_i \mid i \geq 1\}$, *where* v_i *is the image of the generator* e_i *in* $H_i(BO(1))$ *under the map* $BO(1) \to BO$.

(5) $H_*(BSO) \cong P\{v_i \mid i \geq 2\}$, *where the* v_i*'s correspond under the map* $BSO \to BO$. *Here* $v_{i+1} = \sigma_*(a_i)$ *where* σ_* *is the homology suspension, so* v_{i+1} *transgresses to* $a_i \in H_i(SO)$ *in the Serre spectral sequence for the path space fibration for* BSO.

(6) $H_*(BG) \cong H_*(BO) \otimes E\{\sigma_*(x_{(i,i)})\} \otimes P(BX)$ *as a Hopf algebra. Here*

$$BX = \{\sigma_*(x_I) \mid \ell(I) > 2 \text{ and } e(I) > 1, \text{ or } \ell(I) = 2 \text{ and } e(I) \geq 1\}.$$

(7) $H_*(BSG) \cong H_*(BSO) \otimes E\{\sigma_*(x_{(i,i)})\} \otimes P(BX)$ *as a Hopf algebra, with* BX *as above.*

See [May7, §5]. The various non-classical parts are due to Milgram, May and Madsen.

Comparing the sets of polynomial generators X and X_1, note that passing from the product $\underline{*}$ to $\#$ we delete the elements x_i and adjoin their squares $x_{(i,i)}$ under the $\underline{*}$ product. Thus the appearance of these generators in $H_*(SG)$ is forced by the relations $x_i^2 = 0$, arising in $H_*(SG)$ since $H_*(SO)$ is an exterior algebra.

4.4. The R-algebra structure of $H_*(SG)$.

In the previous section we described the multiplicative $\#$-algebra structure on $H_*(SG)$ in terms of the generators x_I, but these were defined by means of the additive homology operations Q^i. We would rather have a description in terms of the multiplicative homology operations $\tilde{Q}^i$.

Theorem (Madsen). $H_*(SG) \cong E\{x_i \mid i \geq 1\} \otimes P(\tilde{X})$ *as an algebra under* $\#$, *where*

$$\tilde{X} = \{\tilde{Q}^J(x_K) \mid \ell(K) = 2 \text{ and } x_{(J,K)} \in X\}.$$

Both $E\{x_i \mid i \geq 1\} = H_*(SO)$ *and* $P(\tilde{X})$ *are sub* AR*-Hopf algebras of* $H_*(SG)$*. Thus* $H_*(G/O) \cong P(\tilde{X})$*.*

For $I = (J, K)$, $\ell(K) = 2$, such that $x_I \in X$, write $\tilde{x}_I = \tilde{Q}^J(x_K)$ for the corresponding element of $\tilde{X}$.

Corollary. *As Hopf algebras*

$$H_*(BSG) \cong H_*(BSO) \otimes E\{\sigma_*(x_{(i,i)}) \mid i \geq 1\} \otimes P(B\tilde{X})$$

and

$$H_*(B(G/O)) = H_*(BSG)//H_*(BSO) \cong E\{\sigma_*(x_{(i,i)}) \mid i \geq 1\} \otimes P(B\tilde{X})$$

where

$$B\tilde{X} = \{\sigma_*(\tilde{x}_I) \mid \ell(I) > 2 \text{ and } e(I) > 1 \text{ or } \ell(I) = 2 \text{ and } e(I) \geq 1\}.$$

Thus the admissible x_i and x_K with $\ell(K) = 2$ generate $H_*(SG)$ as a (multiplicative) R-algebra. Furthermore, operations $\tilde{Q}^i(x_K)$ with (i, K) inadmissible can decompose many of the x_K with $\ell(K) = 2$:

Theorem (Madsen). *The following set is a basis for the* $\mathbb{Z}/2$*-module of* R*-algebra indecomposable elements of* $H_*(SG)$*:*

$$\{x_{2^k} \mid k \geq 0\} \cup \{x_{(2^k,2^k)} \mid k \geq 0\} \cup \{x_{(2^k n+2^k,2^k n)} \mid n \geq 1 \text{ and } k \geq 0\}$$

See [Mad] and [May7, §6, §13]. The set $\tilde{X}$ contains precisely one element from the set above in each degree ≥ 2.

4.5. Homology operations for orthogonal and unitary groups.
The symmetric bi-monoidal category of real inner product spaces has classifying space $B\mathcal{O} \simeq \coprod_{n\geq 0} BO(n)$, with group completion $\Gamma B\mathcal{O} \simeq BO \times \mathbb{Z}$. We identify BO with its 0-component $\Gamma_0 B\mathcal{O} \subset \Gamma B\mathcal{O}$.

In $H_*(BO(1))$ let $\{e_i\}$ be the standard basis. Let

$$BO(1) \xrightarrow{\eta} \coprod_{n\geq 0} BO(n) \xrightarrow{\iota} BO \times \mathbb{Z}$$

be the standard inclusions, and let $v_i = \iota_*\eta_*(e_i)$ and $\bar{v}_i = v_i * [-1] \in H_i(BO)$. Then $H_*(BO) \cong P\{\bar{v}_i \mid i \geq 1\}$ as an algebra under $*$.

Theorem (Priddy).

$$Q^a(v_b) = \sum_i (a-b-1, b-i) v_i * v_{a+b-i}.$$

The binomial coefficient can also be written $\binom{a-i-1}{b-i}$.

This determines $Q^a(\bar{v}_b)$ via the Cartan formula

$$Q^k(\bar{v}_b) = \sum_{i+j=k} Q^i(v_b) * Q^j([-1]).$$

The $Q^j([-1])$ can be inductively determined by relating the Q^i to the conjugation χ. Modulo $*$ decomposable elements,

$$Q^a(\bar{v}_b) \equiv (a-b-1, b-i)\bar{v}_{a+b},$$

but the precise decomposables appearing are not so easy to list.

The symmetric bi-monoidal category of complex inner product spaces has classifying space $B\mathcal{U} \simeq \coprod_{n\geq 0} BU(n)$, with group completion $\Gamma B\mathcal{U} \simeq BU \times \mathbb{Z}$. We identify BU with its 0-component $\Gamma_0 B\mathcal{U} \subset \Gamma B\mathcal{U}$.

Let $f_i \in H_i(BU(1))$ be the image of e_i under the complexification map $BO(1) \to BU(1)$. Then $f_{2i-1} = 0$ and $\{f_{2i} \mid i \geq 0\}$ is a basis for $H_*(BU(1))$. Let $\eta \colon BU(1) \to \coprod_{n\geq 0} BU(n) \to BU \times \mathbb{Z}$ be the standard inclusion, and let $v_i = \eta_*(f_i)$ and $\bar{f}_i = f_i * [-1] \in H_i(BU)$. As before, $v_{2i-1} = \bar{v}_{2i-1} = 0$. Then $H_*(BU) \cong P\{\bar{v}_{2i} \mid i \geq 1\}$ as an algebra under $*$.

Theorem (Priddy).

$$Q^{2a}(v_{2b}) = \sum_i (a-b-1, b-i) v_{2i} * v_{2a+2b-2i}.$$

The binomial coefficient can also be written $\binom{a-i-1}{b-i}$.

Again this determines $Q^{2a}(\bar{v}_{2b})$ via the Cartan formula.
See [May7, §§7–8].

4.6. The complex image of J space, JU.

The K-theory $K(\mathbb{F}_3)$ of the category $\mathcal{GL}(\mathbb{F}_3)$ of finite dimensional $\mathbb{F}_3$-vector spaces is the group completion of the classifying space $B\mathcal{GL}(\mathbb{F}_3) \simeq \coprod_{n\geq 0} BGL_n(\mathbb{F}_3$. It is homotopy equivalent to the homotopy fiber $JU \times \mathbb{Z}$ of

$$\psi^3 - 1 \colon BU \times \mathbb{Z} \to BU,$$

also known as the *complex image of J space.* We identify JU with the 0-component $\Gamma_0 B\mathcal{GL}(\mathbb{F}_3)$.

Consider the standard inclusions

$$BO(1) \xrightarrow{\eta} \coprod_{n\geq 0} BGL_n(\mathbb{F}_3) \xrightarrow{\iota} JU \times \mathbb{Z}.$$

The first map identifies $O(1)$ and $GL_1(\mathbb{F}_3)$. Let $v_i = \iota_*\eta_*(e_i)$ and $\bar{v}_i = v_i * [-1] \in H_i(JU)$.

Theorem (Quillen).

$$H_*(JU) \cong P\{\bar{v}_{2i} \mid i \geq 1\} \otimes E\{\bar{v}_{2i-1} \mid i \geq 1\}$$

as an algebra under $*$.

Again Priddy's theorem above allows the determination of the operations Q^i on $H_*(JU)$, in much the same way as for $H_*(BO)$.

The elements

$$p_{2i-1} = \bar{v}_{2i-1} + \sum_{j=1}^{i-1} \bar{v}_{2j} * p_{2i-2j-1}$$

$$p_{4i} = i\bar{v}_{2i} * \bar{v}_{2i} + \sum_{j=1}^{i-1} \bar{v}_{2j} * \bar{v}_{2j} * p_{4i-4j}$$

for $i \geq 1$ form a basis for the primitive elements in $H_*(JU)$.

4.7. The orthogonal image of J-space, JO.

The K-theory of the category $\mathcal{O}(\mathbb{F}_3)$ of finite dimensional inner product spaces over $\mathbb{F}_3$ might be called the orthogonal K-theory $KO(\mathbb{F}_3)$ of the finite field. It is homotopy equivalent to the homotopy fiber $JO \times \mathbb{Z}$ of

$$\psi^3 - 1 \colon BO \times \mathbb{Z} \to BSO.$$

We identify $JO \cong \Gamma_0 B\mathcal{O}(\mathbb{F}_3)$.

There is an involution Φ on the subcategory of even dimensional inner product spaces in $\mathcal{O}(\mathbb{F}_3)$, acting on 2×2-matrices by conjugation with $\gamma = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}$. Thus Φ_* is an involution on $H_*(JO)$.

Theorem.

$$H_*(JO) \cong P\{\bar{v}_i \mid i \geq 1\} \otimes E\{\bar{u}_i \mid i \geq 1\}$$

as an algebra under $*$, *where* $(\Phi - 1)_*(\bar{v}_i) = \bar{u}_i$. *The homology operations satisfy:*

$$Q^a(v_b) = \sum_{i,j,k} (a-b-1, b-j) v_i * v_k * \bar{u}_{j-i} * \bar{u}_{a+b-j-k}$$

$$Q^a(\bar{u}_b) = \sum_{i,j} (a-b-i-1, b-j) \bar{u}_i * \bar{u}_j * \bar{u}_{a+b-i-j}$$

4.8. The real image of J-space, J.

The K-theory of the category $\mathcal{N}(\mathbb{F}_3)$ of finite dimensional inner product spaces over $\mathbb{F}_3$ with morphisms of spinor norm equal to their determinant is denoted $KN(\mathbb{F}_3)$. It is homotopy equivalent to the homotopy fiber $J \times \mathbb{Z}$ of

$$\psi^3 - 1 \colon BO \times \mathbb{Z} \to BSpin.$$

We identify $J \cong \Gamma_0 B\mathcal{N}(\mathbb{F}_3)$. This is the *real image of J space.*

There is a unit map $e \colon QS^0 \to J \times \mathbb{Z}$ of E_∞ ring spaces, lifting the unit map $QS^0 \to BO \times \mathbb{Z}$. Write $J_\otimes$ for the 1-component of $J \times \mathbb{Z}$, then $e \colon SG \to J_\otimes$ is an infinite loop map, related to the Adams e-invariant. Let $C = \text{hofib}(e)$ be the homotopy fiber of e, called the *cokernel of J space.* Hence there is a fiber sequence of infinite loop spaces

$$C \to SG \xrightarrow{e} J_\otimes.$$

A solution to the Adams conjecture yields a map $\alpha \colon J \to SG$, such that the composite $e\alpha \colon J \to J_\otimes$ is a homotopy equivalence. Such a solution exists as a space level map, but not as an H-map. Hence as groups, $\pi_*(SG) \cong \pi_*(J) \oplus \pi_*(C)$, with the summand $\pi_*(J)$ closely related to the image of the j-homomorphisms $j_* \colon \pi_*(SO) \to \pi_*(SG)$. This is why J is called the (real) image of J space, and C the cokernel of J space.

Let $\bar{u}_0 = \bar{v}_0 = [0]$. The two bases for $H_*(SO)$ in 4.2 are hiding behind a change of basis in the following theorem.

Theorem.

$$H_*(J) \cong P\{\sum_{i+j=a} \bar{u}_i * \bar{v}_j \mid a \geq 1\} \otimes E\{\bar{u}'_i \mid i \neq 2^k\}$$

where $\bar{u}'_i \in E\{\bar{u}_i\}$ *and* $\bar{u}_i + \bar{u}'_i$ *is decomposable under* $*$.

Corollary. *No H-map $J \to SG$ can induce a monomorphism on H_2.*

Proof. $H_2(J)$ contains $(\bar{u}_1 + \bar{v}_1)^2 = \bar{v}_1^2$ as an $*$-algebra, and $H_1(SG)$ has basis $\{x_1\}$ with $x_1^2 = 0$ under the $\#$-product, so $\bar{v}_1^2$ maps to 0 under any H-map. □

((More from [May7, §§12–13] ?))

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