Topological Logarithmic Structures

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Outline

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Pre-log rings

Definition

A pre-log ring is a commutative ring A and a commutative monoid M, with a ring homomorphism $\mathbb{Z}[M] \to A$ from the monoid ring of M to A.

- This is the affine form of a notion for schemes introduced by Fontaine, Illusie, Kato.
- Log geometry is useful to relate the special and generic fibers for schemes over e.g. $\mathrm{Spec}(\mathbb{Z}_p)$.

Mild localization

- The pre-log ring (A, M) may be viewed as a less drastic localization of A than $A[M^{-1}]$, giving a geometric object intermediate between $\mathrm{Spec}(A)$ and $\mathrm{Spec}(A[M^{-1}])$:

$$\mathrm{Spec}(A[M^{-1}]) \to \mathrm{Spec}(A, M) \to \mathrm{Spec}(A)$$

- Guiding Question: When does the first map induce an equivalence in de Rham cohomology, algebraic K-theory or other functors?

E_∞ ring spectra and E_∞ spaces

- For the topological theory, we replace commutative rings by some form of E_∞ ring spectra, and commutative monoids by some form of E_∞ spaces.
- In this talk, we model E_∞ ring spectra by commutative symmetric ring spectra [Hovey–Shipley–Smith].
- We model E_∞ spaces by commutative I-space monoids [Bökstedt, Schlichtkrull].
- For brevity, we often say S-algebra for symmetric ring spectrum.

Symmetric spectra and I-spaces

- A symmetric spectrum E in Sp^Σ is a sequence of based Σ_n-spaces E_n for $n \geq 0$, with structure maps $\sigma \colon E_n \wedge S^1 \to E_{n+1}$, such that σ^m is $\Sigma_n \times \Sigma_m$-equivariant for all $n, m \geq 0$.
- The category I has objects the finite sets $n = \{1, \dots, n\}$ for $n \geq 0$, and morphisms the injective functions. An I-space X in S^I is a functor from I to spaces.
- There is an adjunction

$$S[-] \colon S^I \rightleftarrows Sp^\Sigma \colon \Omega^\infty$$

where $S[X] = \{n \mapsto X(n)_+ \wedge S^n\}$, and $\Omega^\infty(E) \colon n \mapsto \Omega^n E_n$.

Commutative monoids

- A commutative S-algebra A in CSp^Σ is a commutative monoid
$$S \xrightarrow{\eta} A \xleftarrow{\mu} A \wedge A$$
with respect to the smash product $\wedge$ of symmetric spectra.
- A commutative I-space monoid M in CS^I is a commutative monoid
$$1 \xrightarrow{\eta} M \xleftarrow{\mu} M \times M$$
with respect to the (convolution) product $\times$ of I-spaces. (NB: Suppressing a comparison $M \boxtimes M \to M \times M$.)
- There is still an adjunction
$$S[-] \colon CS^I \rightleftarrows CSp^\Sigma \colon \Omega^\infty$$

Pre-log S-algebras

Definition

A pre-log S-algebra (A, M) is a commutative S-algebra A and a commutative I-space monoid M, with an S-algebra map $S[M] \to A$ from the spherical monoid ring of M to A.

- When $M = 1$ we get the trivial pre-log S-algebra $(A, 1)$.
- When $A = S[M]$, we get the canonical pre-log S-algebra $(S[M], M)$.

A map $(A, M) \to (B, N)$ consists of an S-algebra map $A \to B$ and an I-space monoid map $M \to N$, so that the two composite maps $S[M] \to B$ agree.

Two building blocks

- Any pre-log S-algebra (A, M) is a pushout of trivial and canonical pre-log S-algebras:

$$\begin{array}{ccc} (S[M], 1) & \longrightarrow & (S[M], M) \\ \downarrow & & \downarrow \\ (A, 1) & \longrightarrow & (A, M) \end{array}$$

- In this sense any pre-log scheme is locally a fiber product of ordinary schemes like $\mathrm{Spec}(A)$ and canonical pre-log schemes like $\mathrm{Spec}(S[M], M)$.

Integral, saturated, Kummer

In deformation theory of pre-log schemes, one works with commutative monoids M satisfying the following conditions:

- M is finitely generated.
- The group completion map $\gamma \colon M \to M^{gp}$ is injective.
- If $x \in M^{gp}$ and $x^n \in M$ for some $n \geq 1$ then $x \in M$.

In the theory of log-étale descent, one works with

- injective monoid homomorphisms $M \to N$ with a constant k such that $x^k \in M$ for all $x \in N$.

These properties have no clear topological analog.

Replete homomorphisms

Their main use is to ensure that certain homomorphisms $\epsilon \colon N \to M$ are replete, in the sense that

- $\epsilon^{gp} \colon N^{gp} \to M^{gp}$ is surjective, and
- $$\begin{tikzcd} N \arrow[r, "\gamma"] \arrow[d, "\epsilon"'] & N^{gp} \arrow[d, "\epsilon^{gp}"] \\ M \arrow[r, "\gamma"] & M^{gp} \end{tikzcd}$$

  is a pullback square.

In other words, $N \to M$ is virtually surjective and exact. We propose to emphasize these properties instead.

Group completion and units

The full embedding

$$(CS^I)^{gp} \longrightarrow CS^I$$

of grouplike objects among all commutative I-space monoids has a left adjoint (group completion)

$$\Gamma = (-)^{gp} \colon CS^I \longrightarrow (CS^I)^{gp}$$

and a right adjoint (homotopy units)

$$F = (-)^* \colon CS^I \longrightarrow (CS^I)^{gp} .$$

Replete maps

Definition

Let $\epsilon \colon N \to M$ be a map of commutative I-space monoids. We say that N is replete over the base M if

- $\pi_0(\Gamma\epsilon) \colon \pi_0(N)^{gp} \to \pi_0(M)^{gp}$ is surjective, and
- the square

$$\begin{array}{ccc} N & \xrightarrow{\gamma} & \Gamma N \\ {\scriptstyle \epsilon}\downarrow & & \downarrow{\scriptstyle \Gamma\epsilon} \\ M & \xrightarrow{\gamma} & \Gamma M \end{array}$$

is a homotopy pullback.

The surjectivity condition is automatic if ϵ has a section $\eta \colon M \to N$, so that N is an object under and over M.

Repletion of commutative I-space monoids

Definition

For a given $\epsilon\colon N \to M$ with $\pi_0(\Gamma\epsilon)$ surjective, the repletion N^{rep} of N over M is defined by the right hand homotopy pullback square:

$$\begin{tikzcd} N \arrow[r] \arrow[d, "\epsilon"'] & N^{rep} \arrow[r] \arrow[d] & \Gamma N \arrow[d, "\Gamma\epsilon"] \\ M \arrow[r, "="] & M \arrow[r, "\gamma"] & \Gamma M \end{tikzcd}$$

Lemma

N^{rep} is replete over M.

One proof uses the Bousfield–Friedlander theorem.

The shear map

Let M be a base commutative I-space monoid, and consider

$$M \xrightarrow{\eta} M \times M \xrightarrow{\mu} M$$

as an object under and over M, where $\mu(x, y) = xy$ is the multiplication map, and $\eta(x) = (x, 1)$ is the left unit.

Lemma

The repletion of $\mu \colon M \times M \to M$ is $pr_1 \colon M \times \Gamma M \to M$, via the *shear map*

$$sh \colon M \times M \longrightarrow M \times \Gamma M$$

with $sh(x, y) = (xy, \gamma(y))$.

Proof

$$
\begin{tikzcd}
M \times M \arrow[rr, "\gamma"] \arrow[dr, "sh"] \arrow[dd, "\mu"'] & & \Gamma M \times \Gamma M \arrow[d, "\simeq"'] \arrow[dd, bend left, "\mu"] & (x, y) \arrow[d, mapsto] \\
& M \times \Gamma M \arrow[r] \arrow[d, "pr_1"'] & \Gamma M \times \Gamma M \arrow[d, "pr_1"'] & (xy, y) \arrow[d, mapsto] \\
M \arrow[r, "="] & M \arrow[r, "\gamma"] & \Gamma M & xy
\end{tikzcd}
$$

so

$$
(M \times M)^{rep} = M \times \Gamma M \,.
$$

Repletion of pre-log S-algebras

Definition

- Let $(B, N) \to (A, M)$ be a map of pre-log S-algebras. We say that (B, N) is replete over the base (A, M) if N is replete over M.
- For a given map $(B, N) \to (A, M)$ with $\pi_0(N)^{gp} \to \pi_0(M)^{gp}$ surjective, the repletion of (B, N) over the base (A, M) is $(B, N)^{rep} = (B^{rep}, N^{rep})$, where

$$B^{rep} = B \wedge_{S[N]} S[N^{rep}] \, .$$

The surjectivity condition is automatic if (B, N) is an object under and over (A, M).

Outline

Reflective subcategories

The full subcategories of replete objects

$$(M/CS^I/M)^{rep} \subset (M/CS^I/M)$$
$$((A,M)/PreLog/(A,M))^{rep} \subset (A,M)/PreLog/(A,M)$$

in commutative I-space monoids under and over M

$$M \xrightarrow{\eta} N \xrightarrow{\epsilon} M\,,$$

resp. in pre-log S-algebras under and over (A,M)

$$(A,M) \xrightarrow{\eta} (B,N) \xrightarrow{\epsilon} (A,M)\,,$$

are reflective, meaning that repletion $(-)^{rep}$ is left adjoint to the forgetful functor.

Topological log geometry

- To do topological log geometry relative to a base (pre-)log S-algebra (A, M), we propose to work in the category

$$((A, M)/PreLog/(A, M))^{rep}$$

  of replete pre-log S-algebras (B, N) under and over (A, M).
- Limits are formed as for pre-log S-algebras over (A, M).
- Colimits are formed by applying repletion to the colimit for pre-log S-algebras under (A, M).
- This category is pointed at (A, M).

Tensored structure

- For (B, N) in $(A, M)/PreLog/(A, M)$ and Y a pointed set,

$$Y \tilde{\otimes}_{(A,M)}(B, N) = (Y \tilde{\otimes}_A B, Y \tilde{\otimes}_M N)$$

- Here

$$Y \tilde{\otimes}_A B = A \wedge_B (B \wedge_A \cdots \wedge_A B)$$
$$Y \tilde{\otimes}_M N = M \times_N (N \times_M \cdots \times_M N)$$

  with one copy of B, resp. N, for each element of Y.

- For $Y_\bullet$ a pointed simplicial set, $Y_\bullet \tilde{\otimes}_{(A,M)}(B, N)$ is the realization of

$$[q] \mapsto Y_q \tilde{\otimes}_{(A,M)}(B, N) \, .$$

Cyclic bar construction, THH

Lemma

The suspension of $M \times M$ in $(M/CS^I/M)$ is the cyclic bar construction

$$B^{cy}(M) = S^1 \tilde{\otimes}_M (M \times M)\,.$$

It equals the unreduced tensor $S^1 \otimes M$ in CS^I.

Lemma

The suspension of $A \wedge A$ in $(A/CSp^\Sigma/A)$ is the topological Hochschild homology

$$THH(A) = S^1 \tilde{\otimes}_A (A \wedge A)\,.$$

It equals the unreduced tensor $S^1 \otimes A$ in CSp^Σ.

Replete bar construction

We are interested in the replete versions.

Definition

The replete bar construction is the repletion $B^{rep}(M) = (B^{cy}(M))^{rep}$ over M of the cyclic bar construction.

It equals the suspension

$$B^{rep}(M) = S^1 \tilde{\otimes}_M (M \times \Gamma M)$$

of $(M \times \Gamma M, pr_1)$ in $(M/CS^I/M)^{rep}$.

The suspended shear map

Lemma

$$B^{rep}(M) \cong M \times B(\Gamma M)$$

where $B(N) = S^1 \tilde{\otimes}_1 N$ *is the usual bar construction.*

- The repletion map $B^{cy}M \to B^{rep}(M)$ is the suspended shear map $S^1 \tilde{\otimes}_M sh$.
- It factors as

$$\begin{tikzcd} B^{cy}(M) \arrow[r, "\Delta"] & B^{cy}(M) \times B^{cy}(M) \arrow[d, "\epsilon \times \pi"] & \\ & M \times B(M) \arrow[r, "id \times \gamma", "\simeq"'] & M \times B(\Gamma M) \end{tikzcd}$$

Log THH

Definition

The log topological Hochschild homology is the repletion

$$(THH(A, M), B^{rep}(M)) = (THH(A), B^{cy}(M))^{rep}$$

over (A, M).

It equals the suspension

$$(S^1 \tilde{\otimes}_{(A,M)} (A \wedge A, M \times M))^{rep}$$

of $(A \wedge A, M \times M)$ in $((A, M)/PreLog/(A, M))^{rep}$, and the unreduced tensor

$$(S^1 \otimes (A, M))^{rep}$$

of (A, M) in $(PreLog/(A, M))^{rep}$.

A pushout square

Theorem

There is a homotopy pushout square

$$\begin{tikzcd} S[B^{cy}(M)] \arrow[r, "\psi"] \arrow[d, "\phi"'] & S[B^{rep}(M)] \arrow[d] \\ THH(A) \arrow[r] & THH(A, M) \end{tikzcd}$$

of commutative S-algebras.

Here ϕ is induced by $S[M] \to A$, and ψ is induced by the suspension of the shear map.

A rewritten pushout square

Corollary

There is a homotopy pushout square

$$\begin{tikzcd} THH(S[M]) \arrow[r, "\psi"] \arrow[d, "\phi"'] & THH(S[M], M) \arrow[d] \\ THH(A) \arrow[r] & THH(A, M) \end{tikzcd}$$

of commutative S-algebras.

Hence log THH is determined by ordinary THH and its value $THH(S[M], M) = S[B^{rep}(M)]$ on canonical pre-log S-algebras.

The Hesselholt–Madsen construction

- Let K by a local field of characteristic 0, with perfect residue field k of characteristic $p \neq 2$, valuation ring A, and uniformizer $\pi \in A$. Let $M = \{\pi^j \mid j \geq 0\}$, with the obvious map $S[M] \to HA$ to the Eilenberg–Mac Lane spectrum.
- In this case, Hesselholt–Madsen define a cyclotomic spectrum $THH(A|K)$ with a homotopy cofiber sequence

$$THH(k) \xrightarrow{i_*} THH(A) \xrightarrow{j^*} THH(A|K)$$

such that e.g. $\pi_1 THH(A|K) = \Omega^1_{(A,M)}$ is the A-module of log Kähler differentials.

A comparison result

Theorem

Suppose that K is wildly ramified, so that the ramification index e of (π) over (p) is divisible by p. Then there is an isomorphism

$$\pi_*(THH(A|K);\mathbb{Z}/p) \cong \pi_*(THH(HA,M);\mathbb{Z}/p)$$

of $\mathbb{Z}/p$-algebras.

Conjecturally,

$$THH(A|K) \simeq THH(A,M)$$

also in the unramified and tamely ramified cases.

Cyclic structure

- The replete bar construction inherits a cyclic structure [Connes] from the pullback square in CS^I

$$\begin{tikzcd} B^{rep}(M) \arrow[r] \arrow[d] & B^{cy}(\Gamma M) \arrow[d, "\epsilon"] \\ M \arrow[r, "\gamma"] & \Gamma M \end{tikzcd}$$

- Log THH gets a cyclic structure from the pushout square

$$\begin{tikzcd} S[B^{cy}(M)] \arrow[r, "\psi"] \arrow[d, "\phi"'] & S[B^{rep}(M)] \arrow[d] \\ THH(A) \arrow[r] & THH(A, M) \end{tikzcd}$$

in CSp^Σ.

Cyclotomic structure (work in progress)

If for each $r \in \mathbb{N}$ the square

$$\begin{tikzcd} M \arrow[r, "\gamma"] \arrow[d, "(-)^r"'] & \Gamma M \arrow[d, "(-)^r"] \\ M \arrow[r, "\gamma"] & \Gamma M \end{tikzcd} \qquad \begin{tikzcd} x \arrow[d, mapsto] \\ x^r \end{tikzcd}$$

is homotopy cartesian, then

$$B^{rep}(M)^{C_r} \simeq B^{rep}(M)$$

and

$$THH(S[M], M) = S[B^{rep}(M)]$$

is a cyclotomic spectrum [Hesselholt–Madsen].

Log topological cyclic homology

- Assuming this, the log topological cyclic homology $TC(S[M], M)$ is defined.
- The cyclotomic structures on $THH(A)$, $THH(S[M]) = S[B^{cy}(M)]$ and $THH(S[M], M) = S[B^{rep}(M)]$ determine a cyclotomic structure on

$$THH(A, M) = THH(A) \wedge_{S[B^{cy}(M)]} S[B^{rep}(M)] \, .$$

- This gives a general definition of $TC(A, M)$. When is it close to $K(A[M^{-1}])$?

An example for log THH

Lemma

Let $M = \{x^j \mid j \geq 0\}$. We have (finite in) S^1-equivariant equivalences

$$B^{cy}(M) \simeq * \sqcup \coprod_{j \geq 1} S^1(j) \simeq * \sqcup \mathcal{L}_{>0} S^1$$

$$B^{rep}(M) \simeq \coprod_{j \geq 0} S^1(j) \simeq \mathcal{L}_{\geq 0} S^1$$

$$B^{cy}(\Gamma M) \simeq \coprod_{j \in \mathbb{Z}} S^1(j) \simeq \mathcal{L} S^1 \,.$$

An example for log TC

Theorem

Let $M = \{x^j \mid j \geq 0\}$. There is a homotopy pullback square

$$\begin{tikzcd} TC(S[M], M) \arrow[r, "\alpha"] \arrow[d, "\beta"'] & \Sigma S[\mathcal{L}_{\geq 0} S^1 \times_{S^1} ES^1] \arrow[d, "trf_{S^1}"] \\ S[\mathcal{L}_{\geq 0} S^1] \arrow[r, "1-\Delta_p"] & S[\mathcal{L}_{\geq 0} S^1] \end{tikzcd}$$

after p-adic completion.

Same proof as in [Bökstedt–Hsiang–Madsen].

Localization sequences

Corollary

Let $M = \{x^j \mid j \geq 0\}$*. There are homotopy cofiber sequences*

$$THH(S) \xrightarrow{i_*} THH(S[M]) \xrightarrow{j^*} THH(S[M], M)$$

and

$$TC(S) \xrightarrow{i_*} TC(S[M]) \xrightarrow{j^*} TC(S[M], M)\,.$$

Should be compared with the fundamental theorem for A-theory [Hüttemann–Klein–Vogell–Waldhausen–Williams].

Outline

1. Replete pre-log S-algebras
2. Log THH and log TC
3. Log TAQ and log modules

Stable commutative augmented A-algebras

- Suspension $B \mapsto E(B) = S^1 \tilde{\otimes}_A B$ defines an endofunctor on $A/CSp^\Sigma/A$.
- By an E-spectrum in $A/CSp^\Sigma/A$ we mean a sequence $\{n \mapsto B_n\}$ of commutative augmented A-algebras, with structure maps $\sigma\colon E(B_n) \to B_{n+1}$.
- The category of E-spectra in $A/CSp^\Sigma/A$ is equivalent to the category of A-modules, taking $\{n \mapsto B_n\}$ to $\operatorname{hocolim}_n \Sigma^{-n}(B_n/A)$. [Basterra–Mandell]

Topological André–Quillen homology

Definition

The topological André–Quillen homology of A is the A-module $TAQ(A)$ that corresponds to the suspension spectrum

$$E^{\infty}(A \wedge A) = \{n \mapsto S^n \tilde{\otimes}_A (A \wedge A)\}$$

of $A \wedge A$ in $A/CSp^{\Sigma}/A$.

Stable repletion

- The n-fold suspension of the shear map

$$S^n \tilde{\otimes}_M sh \colon S^n \tilde{\otimes}_M (M \times M, \mu) \longrightarrow S^n \tilde{\otimes}_M (M \times \Gamma M, pr_1) \cong M \times B^n(\Gamma M)$$

  expresses the repletion of $S^n \tilde{\otimes}_M (M \times M)$ as $M \times B^n(\Gamma M)$.
- The sequence $B^\infty M = \{n \mapsto B^n(\Gamma M)\}$ is the connective spectrum associated to the commutative I-space monoid M, or equivalently, to ΓM.

Log TAQ

Definition

The log topological André–Quillen homology of (A, M) is the A-module $TAQ(A, M)$ that corresponds to the levelwise repletion

$$\{n \mapsto (S^n \tilde{\otimes}_{(A,M)}(A \wedge A, M \times M))^{rep}\}$$

of the suspension spectrum of $(A \wedge A, M \times M)$ in $(A, M)/PreLog/(A, M)$.

In other words, this is the suspension spectrum of $(A \wedge A, M \times M)$ in $((A, M)/PreLog/(A, M))^{rep}$.

A pushout square

Theorem

There is a homotopy pushout square

$$\begin{array}{ccc} A \wedge_{S[M]} TAQ(S[M]) & \xrightarrow{\psi} & A \wedge B^\infty M \\ \phi \downarrow & & \downarrow \\ TAQ(A) & \longrightarrow & TAQ(A, M) \end{array}$$

of A-modules.

Here ϕ is induced by $S[M] \to A$, and ψ is induced from the infinite suspension of the shear map.

Log derivations

Lemma

For each A-module K there is an equivalence

$$(A{-}Mod)(TAQ(A,M),K) \simeq Der((A,M),K)$$

between the space of A-module maps $TAQ(A,M) \to K$ and the space of log derivations of (A,M) with values in K.

In other words, $TAQ(A,M)$ properly plays the role of log Kähler differentials.

Log modules

There does not appear to be a classical notion of modules over a pre-log ring. We propose:

Definition

Let the category

$$(A, M){-}Mod = Sp(((A, M)/PreLog/(A, M))^{rep}, E)$$

of log modules over (A, M) be the stable category of E-spectra in replete pre-log S-algebras under and over (A, M).

Then $TAQ(A, M)$ is a natural example of a log module.

Log K-theory (work in progress)

Definition

Let the log K-theory of (A, M) be the algebraic K-theory

$$K(A, M) = K((A, M){-}Perf)$$

of the category of perfect (= compact) objects in $(A, M){-}Mod$.

- There is a natural localization map

  $$K(A, M) \to K(A[M^{-1}])\,.$$

  When is it an equivalence?
- There seems to be a natural trace map

  $$tr\colon K(A, M) \to THH(A, M)\,.$$

  Does it lift through $TC(A, M)$?

Non-connective E_∞ spaces (work in progress)

- For connective M, the structure map $S[M] \to A$ only touches the connective cover of A.
- There is a category $J = \Sigma^{-1}\Sigma$, extending I, so that commutative J-space monoids have underlying commutative I-spaces monoids [Sagave–Schlichtkrull].
- Just as grouplike commutative I-space monoids are equivalent to connective spectra

$$Sp_{\geq 0} \simeq (CS^I)^{gp} \subset CS^I ,$$

it appears that a full subcategory of commutative J-space monoids is equivalent to all spectra

$$Sp \simeq (CS^J)^{gp} \subset CS^J .$$

Direct image structures

- In the Hesselholt–Madsen example $i \colon A \to K$, the appropriate log structure on A is $M = i_* GL_1(K)$, the direct image of the trivial log structure on K.
- With this log structure they prove that $K(K) \to TC(A|K)$ is a p-adic equivalence except in low degrees.
- For a periodic spectrum E with connective cover $i \colon e \to E$, the appropriate (pre-)log structure on e may be $M = i_* GL_1^J(E)$, where $GL_1^J(E)$ is constructed as a grouplike commutative J-space monoid.
- When will $K(E) \to TC(e, M)$ be a p-adic equivalence with this structure?
- For example, we may consider $E = KU_p$ and $e = ku_p$.