$K_4(\mathbb{Z}) = 0$

John Rognes

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I will talk about the proof that K_4 of the integers is the trivial group.

Recall that the algebraic K-groups $K_i(\mathbb{Z})$ are the homotopy groups of a representing space $K(\mathbb{Z})$, which is defined as a loop space. Let $\mathcal{P}(\mathbb{Z}) = \mathcal{F}(\mathbb{Z})$ denote the category of finitely generated projective (or free) $\mathbb{Z}$-modules, let $Q(\mathbb{Z}) = Q\mathcal{P}(\mathbb{Z})$ be Quillen's Q-construction applied to this category, and let $|Q(\mathbb{Z})| = BQ(\mathbb{Z})$ be the classifying space of the Q-construction. Then $K(\mathbb{Z}) = \Omega|Q(\mathbb{Z})|$, and $|Q(\mathbb{Z})|$ is a model for the first delooping of $K(\mathbb{Z})$.

In 1976 Lee and Sczcarba proved $K_3(\mathbb{Z}) = \mathbb{Z}/48$, and in 1979 Soulé (1979) found $K_4(\mathbb{Z}) \cong$ (two–torsion) $\oplus$ (0 or $\mathbb{Z}/3$). These authors computed the ordinary homology $H_{*+1}(|Q(\mathbb{Z})|)$ of the first delooping of K-theory, in a range, and proceeded by killing homotopy groups à la Serre to extract the desired homotopy groups.

The main new idea is to utilize that the representing space $K(\mathbb{Z})$ is an infinite loop space, *i.e.* can be delooped infinitely often. Instead of working at the level of the first delooping, we will pass to the colimit over higher and higher deloopings, to compute the spectrum homology of the K-theory spectrum, and use the Atiyah–Hirzebruch spectral sequence to extract the homotopy groups of interest.

There is a stabilization map from the singly delooped to the infinitely delooped, or stable, situation, and one point of today's lecture will be to make a comparison of the methods of Lee, Sczcarba and Soulé with mine, and in particular to point out where we are able to overcome the indeterminacy left in Soulé's result on $K_4(\mathbb{Z})$.

1. Filtering singly delooped K-theory

To prove finite generation of the K-theory of the ring of integers in a number field, Quillen introduced a filtration of the Q-construction, by letting $Q_k(\mathbb{Z})$ be the full subcategory of $Q(\mathbb{Z})$ involving only modules of rank at most k. Then there is an increasing filtration $\{Q_k(\mathbb{Z})\}_k$ of subcomplexes of the first delooping of K-theory. By investigating the internal design of the Q-construction we find

$$|Q_k(\mathbb{Z})|/|Q_{k-1}(\mathbb{Z})| \simeq \Sigma^2 B(\mathbb{Z}^k) \wedge_{GL_k(\mathbb{Z})} EGL_k(\mathbb{Z})_+ = \Sigma^2 B(\mathbb{Z}^k)/hGL_k(\mathbb{Z})$$

where $B(\mathbb{Z}^k)$ is the Tits building for $\mathbb{Z}^k$.

The Tits building is usually defined over fields, and $B(\mathbb{Q}^k)$ is the nerve of the partially ordered set of proper nontrivial sub-vectorspaces of $\mathbb{Q}^k$, partially ordered by inclusion. We may extend the definition to $\mathbb{Z}$, letting $B(\mathbb{Z}^k)$ be the nerve of the poset of proper nontrivial direct summands of $\mathbb{Z}^k$, partially ordered by splittable

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inclusions There is a homeomorphism $B(\mathbb{Z}^k) \to B(\mathbb{Q}^k)$ induced by extension of scalars.

We will need explicit models for the suspensions $\Sigma B(\mathbb{Z}^k)$ and $\Sigma^2 B(\mathbb{Z}^k)$. For $\Sigma B(\mathbb{Q}^k)$ consider the simplicial set with q-simplices of the form

$$U_0 \subset U_1 \subset \cdots \subset U_q = \mathbb{Q}^k,$$

and identify any faces with $U_q \neq \mathbb{Q}^k$ to a base point $*$. For $\Sigma^2 B(\mathbb{Q}^k)$ take the simplicial set with q-simplices

$$0 = U_0 \subset U_1 \subset \cdots \subset U_q = \mathbb{Q}^k,$$

and a base point. Similar models may be given over $\mathbb{Z}$ in place of $\mathbb{Q}$, by insisting that every inclusion be splittable.

The following result determines the homotopy type of the Tits buildings.

Theorem (Solomon–Tits). *$\Sigma^2 B(\mathbb{Z}^k) \simeq \bigvee S^k$.*

There is an obvious action of $GL_k(\mathbb{Z})$ on $\Sigma^2 B(\mathbb{Z}^k)$, and the resulting homology representation $\widetilde{H}_k(\Sigma^2 B(\mathbb{Z}^k))$ is the Steinberg representation $St_k = St(\mathbb{Z}^k)$.

Then Lee and Sczcarba, as well as Soulé, proceed by computing

$$H_{*+1}(|Q_k(\mathbb{Z})|, |Q_{k-1}|) \cong H_{*+1-k}(GL_k(\mathbb{Z}); St_k)$$

for $k \leq 3$, and partially for $k = 4$. They then resolve extension problems and find $H_{*+1}(|Q(\mathbb{Z})|)$ for $*$ small. Finally they apply the method of killing homotopy groups à la Serre to find $K_*(\mathbb{Z}) = \pi_{*+1}(|Q(\mathbb{Z})|)$ when $* \leq 3$ or 4.

These calculations may be summarized in the spectral sequence associated to the filtration $\{|Q_k(\mathbb{Z})|\}_k$ of $|Q(\mathbb{Z})|$.

$$(*) \qquad E^1_{s,t} = H_t(GL_{s+1}(\mathbb{Z}); St_{s+1}) \Longrightarrow H_{s+t+1}(|Q(\mathbb{Z})|)$$

which receives a map from the groups of interest $K_{s+t}(\mathbb{Z}) \cong \pi_{s+t+1}(|Q(\mathbb{Z})|)$.

5	$\mathbb{Z}/2$	?	?	?	?	?
4	0	$(\mathbb{Z}/2)^2$	$\mathbb{Z}/2$	?	?	?
3	$\mathbb{Z}/2$	$(\mathbb{Z}/2)^2$	$\mathbb{Z} \oplus (\mathbb{Z}/2)^2$	?	?	?
2	0	$\#4 \oplus \mathbb{Z}/3$	$\#4 \oplus \mathbb{Z}/3$	$T_2 \oplus \mathbb{Z}/3$	?	?
1	$\mathbb{Z}/2$	$\mathbb{Z}/2$	0	0	?	?
0	$\mathbb{Z}$	0	0	0	0	0
	0	1	2	3	4	5

$H_0(GL_k(\mathbb{Z}); St_k) = 0$ for all $k \geq 0$ by a result of Lee and Sczcarba. $\#n$ denotes a group of order n, and T_2 a possible two–torsion group.

Localized at three, Soulé's indeterminacy in $K_4(\mathbb{Z})$ comes from the $\mathbb{Z}/3$ in $E^1_{2,2}$ of $(*)$, which might (or might not) be a boundary, say from the $\mathbb{Z}/3$ in $E^1_{3,2} = H_2(GL_4(\mathbb{Z}); St_4)$.

How are the groups $H_*(GL_k(\mathbb{Z}); St_k)$ in the columns of $(*)$ determined ? One approach is to use the skeleton filtration on $\Sigma B(\mathbb{Z}^k)$. Its nondegenerate q-simplices divide into $GL_k(\mathbb{Z})$-orbits with one representative

$$0 \neq \mathbb{Z}^{i_0} \subset \cdots \subset \mathbb{Z}^{i_q} = \mathbb{Z}^k$$

in each orbit, for every increasing sequence $I = (0 < i_0 < \cdots < i_q = k)$. The stabilizer of this orbit is a parabolic subgroup $P_I \subseteq GL_k(\mathbb{Z})$.

((Sketch some parabolics, *e.g.* $P_{1,12} = P_I$ associated to $I = (0 < 1 < 2 < 3 = 3)$ for $k = 3$, $q = 2$.))

Hence there is a spectral sequence

$$(\dagger) \quad E^1_{s,t} = \bigoplus_I H_t(P_I) \Longrightarrow \widetilde{H}_{s+t}(\Sigma B(\mathbb{Z}^k)/hGL_k(\mathbb{Z})) \cong H_{s+t+1-k}(GL_k(\mathbb{Z}); St_k)$$

with the sum running over the sequences I increasing to k with $(s+1)$ terms.

Examples. In the case $k = 2$ we have the following E^1-term:

$$H_*(GL_2(\mathbb{Z})) \stackrel{d^1}{\longleftarrow} H_*(P_1)$$

The d^1-differentials are induced by group inclusions.

Similarly for $k = 3$, there is a spectral sequence with E^1-term

$$H_*(GL_3(\mathbb{Z})) \stackrel{d^1}{\longleftarrow} H_*(P_1) \oplus H_*(P_{12}) \stackrel{d^1}{\longleftarrow} H_*(P_{1,12})$$

Again the d^1-differentials are induced by group inclusions.

((Sketch shapes of parabolic groups used here.))

2. Filtering the K-theory spectrum

We wish to replace these constructions on the singly–delooped spaces $|Q(\mathbb{Z})| = BK(\mathbb{Z})$ with their infinitely delooped, or stable analogs. We prefer to work with iterated applications of Waldhausen's $S_\bullet$-construction to describe models for the higher deloopings of $K(\mathbb{Z})$. This $S_\bullet$-construction applies to a category with cofibrations and weak equivalences, like $\mathcal{F}(\mathbb{Z})$, and returns a simplicial category of the same kind. Repeating the process n times, $S^{(n)}_\bullet\mathcal{F}(\mathbb{Z})$ is an n-multisimplicial category, and the classifying space of its isomorphism subcategory is an n-fold delooping of $K(\mathbb{Z})$, denoted $K(\mathbb{Z})_n$:

$$K(\mathbb{Z})_n = |iS^{(n)}_\bullet\mathcal{F}(\mathbb{Z})|$$

Then $\Omega K(\mathbb{Z})_{n+1} \simeq K(\mathbb{Z})_n$ for $n \geq 1$.

$S^{(n)}_\bullet\mathcal{F}(\mathbb{Z})$ is a category of diagrams in $\mathcal{F}(\mathbb{Z})$ and we will specify various sub–prespectra of $K(\mathbb{Z})$ by restricting the objects in these diagrams to suitable subclasses of modules.

There is a prespectrum level rank filtration

$$* \simeq F_0K(\mathbb{Z}) \to F_1K(\mathbb{Z}) \to \cdots \to F_kK(\mathbb{Z}) \to \cdots \to K(\mathbb{Z}).$$

where $F_kK(\mathbb{Z})$ is the prespectrum with nth space $F_kK(\mathbb{Z})_n$ defined as the subcomplex of $K(\mathbb{Z})_n = |iS^{(n)}_\bullet\mathcal{F}(\mathbb{Z})|$ realizing the full subcategory of diagrams of modules of rank at most k. The filtration above arises by admitting diagrams involving larger and larger modules.

Proposition. *There are homotopy equivalences of prespectra*

$$F_k K(\mathbb{Z})/F_{k-1}K(\mathbb{Z}) \simeq D(\mathbb{Z}^k) \wedge_{GL_k(\mathbb{Z})} EGL_k(\mathbb{Z})_+ = D(\mathbb{Z}^k)/hGL_k(\mathbb{Z})$$

where $D(\mathbb{Z}^k)$ is a spectrum called the stable building of rank k.

For every $n \geq 1$ the nth space $D(\mathbb{Z}^k)_n$ of the stable building realizes an explicit n-multisimplicial category of diagrams of submodules of $\mathbb{Z}^k$. We omit the explicit description, to avoid getting too technical, but we note that ew can specify sub-prespectra of $D(\mathbb{Z}^k)$ by restricting which submodules of $\mathbb{Z}^k$ may appear in these diagrams.

We have a spectral sequence analogous to $(*)$ abutting to the spectrum homology of $K(\mathbb{Z})$:

$$(**) \qquad E^1_{s,t} = H^{spec}_t(D(\mathbb{Z}^{s+1})/hGL_{s+1}(\mathbb{Z})) \Longrightarrow H^{spec}_{s+t}(K(\mathbb{Z}))$$

There is a stabilization map from $(*)$ to $(**)$, and we list the beginnings of the latter spectral sequence now for comparison.

5	$\mathbb{Z}/2 \leftarrow$	$(\mathbb{Z}/2)^3$	?	?	?	?
4	0	$(\mathbb{Z}/2)^2$	$\mathbb{Z}/2$	?	?	?
3	$\mathbb{Z}/2 \leftarrow$	$(\mathbb{Z}/2)^2 \leftarrow$	$\mathbb{Z}$	?	?	?
2	0	$\mathbb{Z}/3 \oplus \mathbb{Z}/2 \leftarrow$	$\mathbb{Z}/3$	?	?	?
1	$\mathbb{Z}/2 \leftarrow$	$\mathbb{Z}/2$	0	0	?	?
0	$\mathbb{Z}$	0	0	0	0	0
	0	1	2	3	4	5

A theorem like the Solomon–Tits theorem might still hold in the stable situation, that is, $D(\mathbb{Z}^k)$ may be homotopy equivalent to the suspension spectrum on a wedge of $(2k-2)$-spheres, but this is still conjectural in the general case.

We analyze the stable buildings by means of two filtrations. The first, coarser filtration will be used to prove connectivity results for the stable building.

The second, finer filtration gives rise to a spectral sequence for computing the spectrum homology groups $H^{spec}_t(D(\mathbb{Z}^k)/hGL_k(\mathbb{Z}))$, much like the spectral sequence arising from the skeleton filtration on $\Sigma B(\mathbb{Z}^k)$ used above.

First, recall how one using Γ-spaces may construct deloopings by considering categories of sum diagrams. In an analogous fashion, we will filter $D(\mathbb{Z}^k)$ by a collection of sub–prespectra indexed by the various direct sum decompositions of $\mathbb{Z}^k$ as a sum of one or more nontrivial submodules. This is useful because we can describe the filtration subquotients in terms of Tits buildings, corresponding to the singly delooped case.

So let $\mathbb{Z}^k = \oplus_{j=1}^s V_j$ be a sum decomposition of $\mathbb{Z}^k$ in terms of $1 \leq s \leq k$ nontrivial submodules $V_j \subseteq \mathbb{Z}^k$. Let σ denote such a sum decomposition. Then there is a sub–prespectrum $G_\sigma D(\mathbb{Z}^k)$ of $D(\mathbb{Z}^k)$ realizing categories of diagrams involving only submodules of $\mathbb{Z}^k$ which are sums of some of the V_j.

If σ refines a sum decomposition τ, then $G_\tau D(\mathbb{Z}^k)$ will be a sub–prespectrum of $G_\sigma D(\mathbb{Z}^k)$. So $\{G_\sigma D(\mathbb{Z}^k)\}_\sigma$ will be our coarse filtration.

Proposition. *$G_\sigma D(\mathbb{Z}^k)$, modulo the union of the sub–prespectra $G_\tau D(\mathbb{Z}^k)$ for all τ refined by σ, is homotopy equivalent to the suspension spectrum on*

$$\Sigma B(V_1) \wedge \cdots \wedge \Sigma B(V_s) \wedge {}^sA_s$$

where ${}^sA_s$ is homotopy equivalent to the suspension spectrum on a wedge of $(s-1)!$ spheres of dimension $(2s-2)$.

There is a natural action of Σ_s on ${}^sA_s$. Hence $W_s = H^{spec}_{2s-2}({}^sA_s)$ is an integral Σ_s-representation of rank $(s-1)!$. W_1 is trivial of rank one, while W_2 is the rank one sign representation of Σ_2.

So the coarse filtration subquotient of $D(\mathbb{Z}^k)$ has spectrum homology isomorphic to $St(V_1) \otimes \ldots St(V_s) \otimes W_s$, concentrated in degree $\sum_{j=1}^s (\dim(V_j) - 1) + (2s-2) = k + s - 2$. So there is a chain complex (Z_*, d_*) associated to the filtration, with

$$Z_{k+s-2} = \bigoplus_\sigma St(V_1) \otimes \ldots St(V_s) \otimes W_s$$

where the sum runs over sum decompositions σ of $\mathbb{Z}^k = \oplus_{j=1}^s V_j$ into s summands.

The homology of the chain complex (Z_*, d_*) is thus isomorphic to $H^{spec}_*(D(\mathbb{Z}^k))$. The number of summands s runs from 1 to k, so the chain complex is concentrated in degrees $(k-1)$ to $(2k-2)$, and so $D(\mathbb{Z}^k)$ is at least $(k-2)$-connected for all k.

We can analyze the boundary maps in (Z_*, d_*) by making a comparison with the K-theory of the category of finite sets, via the usual functor from sets to free $\mathbb{Z}$-modules. This gives sufficient control of the differentials to prove:

Proposition. *$D(\mathbb{Z}^k)$ is at least $(k-1)$-connected for $k \geq 2$. $D(\mathbb{Z}^4)$ is at least four–connected.*

Hence $D(\mathbb{Z}^k)$ is four–connected for $k \geq 4$, so $F_{k-1}K(\mathbb{Z}) \to F_k K(\mathbb{Z})$ is four–connected in these cases, and $F_3 K(\mathbb{Z}) \to K(\mathbb{Z})$ is four–connected. It might very well be five–connected.

Thus the coarse filtration lets us prove that $F_3K(\mathbb{Z})$ is a reasonably close approximation to $K(\mathbb{Z})$, good enough to give an upper bound on $K_4(\mathbb{Z})$.

The finer filtration may be considered to arise by filtering the suspended Tits buildings $\Sigma B(V_j)$ in the proposition above by their skeleta. It is indexed by tuples of increasing flags of submodules of $\mathbb{Z}^k$ such that the top modules of each of these flags form a sum decomposition of $\mathbb{Z}^k$.

$$\{0 \neq U_0^j \subset \cdots \subset U_{n_j}^j = V_j\}_{j=1}^s \quad \text{with} \quad \oplus_{j=1}^s V_j = \mathbb{Z}^k.$$

We will call such collections of submodules constellations. We are interested in the spectrum homology of $D(\mathbb{Z}^k)/hGL_k(\mathbb{Z})$, so again we can do this by considering the $GL_k(\mathbb{Z})$-orbits of these constellations of submodules in $\mathbb{Z}^k$. Each orbit contains a unique representative of the following type.

Let each of $I^1, \ldots, I^s$ be a sequence of natural numbers increasing to $k_1, \ldots, k_s$, with $\sum_{j=1}^s k_j = k$. We assume $k_1 \geq \cdots \geq k_s$. So for all j we have $I^j = (0 < i_0^j < \cdots < i_{n_j}^j = k_j)$. Then for each j we have a flag $0 \neq \mathbb{Z}^{i_0^j} \subset \cdots \subset \mathbb{Z}^{i_{n_j}^j} = \mathbb{Z}^{k_j}$, which we identify with a flag of submodules in $\mathbb{Z}^k$ via the usual sum decomposition $\mathbb{Z}^k = \oplus_{i=1}^s \mathbb{Z}^{k_i}$.

Let ω be such a constellation of submodules of $\mathbb{Z}^k$, and let $P_\omega \subseteq GL_k(\mathbb{Z})$ be its stabilizer under the $GL_k(\mathbb{Z})$-action. We think of this as a generalized parabolic subgroup. P_ω will be a product of wreath products of symmetric groups and parabolic groups of the form P_I.

Let the *size* of such a constellation ω be the sum

$$(\sum_{j=1}^{s} n_j) + (2s-2) = \sum_{j=1}^{s}(n_j+1) + (s-2).$$

Note that in the case $s = 1$, the constellations ω are precisely simplices in the suspended Tits building $\Sigma B(\mathbb{Z}^k)$, and then the size is one less than the dimension of this simplex, which is the same as the column its group homology appears in in the spectral sequence (†).

We can form the finer filtration $\{G'_\omega D(\mathbb{Z}^k)\}_\omega$ of the stable building indexed by the isomorphism classes of constellations ω, and obtain a spectral sequence (††) abutting to $H_*^{spec}(D(\mathbb{Z}^k)/hGL_k(\mathbb{Z}))$ with input the group homology of the parabolic subgroups P_ω with coefficients in the representations W_s.

$$E^1_{p,q} = \bigoplus_{\omega} H_q(P_\omega; W_s) \Longrightarrow H_*^{spec}(D(\mathbb{Z}^k)/hGL_k(\mathbb{Z})) \tag{††}$$

where ω runs through the constellations of size q, and $s = s(\omega)$ is the number of decomposition summands.

It is again possible to determine the d^1-differential in this spectral sequence via a comparison with the K-theory of finite sets.

Examples. With $k = 2$, the possible constellations ω are generated by the following collections of subsets of $\{1,2\}$:

$\{1,2\}$, $\{1\} \subset \{1,2\}$ and $\{1\},\{2\}$. The corresponding parabolic subgroups are $GL_2(\mathbb{Z})$ itself, P_1 as in the singly delooped case, and $\Sigma_2 \wr \mathbb{Z}/2 = D_2$, the dihedral group of order eight. The E^1-term abutting to $H_*^{spec}(D(\mathbb{Z}^2)/hGL_2(\mathbb{Z}))$ has three columns:

$$H_*(GL_2(\mathbb{Z})) \xleftarrow{d^1} H_*(P_1) \xleftarrow{d^1} H_*(D_2; W_2)$$

The stabilization map $\widetilde{H}_{*+1}(BK(\mathbb{Z})) \to H_*^{spec}(K(\mathbb{Z}))$ induces the map

$$\widetilde{H}_{*+1}(\Sigma^2 B(\mathbb{Z}^2)/hGL_2(\mathbb{Z})) \to H_*^{spec}(D(\mathbb{Z}^2)/hGL_2(\mathbb{Z}))$$

induced by the inclusion of the first two columns into the three–column spectral sequence.

In the case $k = 3$, the representing constellations are induced by the following chains of subsets partitioning $\{1,2,3\}$:

$\{1,2,3\}$ of size zero, $\{1\} \subset \{1,2,3\}$ and $\{1,2\} \subset \{1,2,3\}$ of size one, $\{1\} \subset \{1,2\} \subset \{1,2,3\}$ and $\{1,2\},\{3\}$ of size two, $\{1\} \subset \{1,2\},\{3\}$ of size three, and $\{1\},\{2\},\{3\}$ of size four. The corresponding parabolic subgroups are $GL_3(\mathbb{Z})$, P_1,

P_{12}, $P_{1,12}$, $P_{12,3}$, $P_{1,12,3}$ and $\Sigma_3 \wr \mathbb{Z}/2 = D_3$. The spectral sequence abutting to $H_*^{spec}(D(\mathbb{Z}^3)/hGL_3(\mathbb{Z}))$ has E^1-term:

$$H_*GL_3(\mathbb{Z}) \leftarrow H_*P_1 \oplus H_*P_{12} \leftarrow H_*P_{1,12} \oplus H_*P_{12,3} \leftarrow H_*P_{1,12,3} \leftarrow H_*(D_3; W_3)$$

Again the stabilization map from singly delooped homology maps into the three first columns here. The extra differentials in the stable setting are typically not induced by group inclusions, but we can still find expressions for them by means of the case of the K-theory of finite sets.

3. Extracting K-groups

We can compute the behaviour of these spectral sequences in low degrees, giving the information in $(**)$. The known values of $K_*(\mathbb{Z})$ imply that $H_*^{spec}(K(\mathbb{Z}))$ begins $(\mathbb{Z}, 0, 0, \mathbb{Z}/2, \dots)$ as a graded group. This implies the first few differentials in $(**)$, while the others require careful bookkeeping of represening classes and the nature of the boundary map associated to the spectrum level rank filtration.

The conclusion is $H_*^{spec}(K(\mathbb{Z})) \cong (\mathbb{Z}, 0, 0, \mathbb{Z}/2, 0, \mathbb{Z} \oplus (\text{two–torsion}), \dots)$, and then a spectral sequence argument with the Atiyah–Hirzebruch spectral sequence for stable homotopy theory implies $K_4(\mathbb{Z}) = 0$. This uses Bökstedt's two–complete splitting of $\Omega K^{et}(\mathbb{Z})_2^\wedge$ off from $\Omega K(\mathbb{Z})_2^\wedge$, where $K^{et}(\mathbb{Z}) = JK(\mathbb{Z})$ is a model for étale K-theory of the integers. Hence we can conclude:

Theorem. $K_4(\mathbb{Z}) = 0$.

The three–torsion $\mathbb{Z}/3$ in $E^1_{2,2}$ of $(*)$ comes from a class surviving in $(\dagger)$ from $H_3(P_1) \oplus H_3(P_{12}) \cong (\mathbb{Z}/3)^4$ modulo two–torsion. This class dies when $(\dagger)$ is mapped to its stable analog $(\dagger\dagger)$, by a d^1-differential from $H_3(P_{12,3})$. So this differential represents a reason why the three–torsion found by Soulé in $H_2(GL_3(\mathbb{Z}); St_3)$ turns out to be a boundary, and does not contribute to $K_4(\mathbb{Z})$.

In the stable setting this cancellation takes place entirely within the domain of three–by–three matrices, which is why it is accessible to calculation, in spite of our weak understanding of the homology of $GL_4(\mathbb{Z})$.

Department of Mathematics, University of Oslo, Postboks 1053, Blindern, 0316 Oslo, Norway.