INTRODUCTION TO REDSHIFT

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Abstract. These are notes for the author's talk at Oberwolfach in September 2011. We give examples of redshift from the homological point of view, and develop tools for further calculations, generalizing those used to prove Segal's Burnside ring conjecture, with an eye to applications to the algebraic K-theory of S-algebras.

1. Homotopy fixed points of smash powers

Consider a compact Lie group G and a commutative S-algebra B. The tensor product

$$G \otimes B = \bigwedge_G B$$

is a commutative B-algebra with G-action. Consider the G-homotopy fixed points

$$(G \otimes B)^{hG} = F(EG_+, G \otimes B)^G .$$

Experience has shown that if $\pi_*(B)$ contains v_n-periodic families then $\pi_*(G \otimes B)^{hG}$ often contains v_{n+k}-periodic families, where k is the rank of G. Since the v_{n+k}-periodic families have longer periods, or longer wavelength, than the v_n-periodic families, we refer to this as a *redshift* phenomenon.

The case $G = \mathbb{T} = S^1$ was investigated in the context of topological cyclic homology by Madsen and coauthors [BM94], [HM97]. The tensor product

$$\mathbb{T} \otimes B = THH(B)$$

is called the topological Hochschild homology of B, and $THH(B)^{h\mathbb{T}}$ is closely related to the topological Frobenius homology

$$TF(B;p) = \operatorname{holim}_{n,F} THH(B)^{C_{p^n}}$$

via canonical maps

$$\Gamma_n \colon THH(B)^{C_{p^n}} \longrightarrow THH(B)^{hC_{p^n}}$$

and a p-adic equivalence

$$THH(B)^{h\mathbb{T}} \xrightarrow{\simeq_p} \operatorname{holim}_{n,F} THH(B)^{hC_{p^n}} .$$

The higher abelian cases, with $G = \mathbb{T}^k$ for $k \geq 2$, have been emphasized by Carlsson, Dundas and coauthors [BCD10], [CDD11]. The tensor products $\mathbb{T}^k \otimes B$ are known as higher topological Hochschild homology.

We use the abbreviations $H_*(X) = H_*(X; \mathbb{F}_p)$, $H^*(X) = H^*(X; \mathbb{F}_p)$ and $H = H\mathbb{F}_p$, and write $THH(\mathbb{F}_p)$ instead of $THH(H)$. We shall focus on (co-)homological methods, appealing to the Adams spectral sequence

$$E_2^{s,t} = \operatorname{Ext}^{s,t}_{\mathscr{A}}(H^*(X), \mathbb{F}_p) = \operatorname{Ext}^{s,t}_{\mathscr{A}_*}(\mathbb{F}_p, H_*(X)) \Longrightarrow \pi_{t-s}(X^\wedge_p)$$

(and its generalization to towers in [CMP87]) to pass to information about homotopy groups. In the following examples we shall concentrate on the prime $p = 2$, but all definitions and results have also been extended to odd primes.

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2. Examples of redshift

2.1. The case $G = \mathbb{T}$ and $B = H$. In this case $H_*(H) = \mathscr{A}_*$ is the dual Steenrod algebra

$$\mathscr{A}_* = P(\bar\xi_1, \bar\xi_2, \dots)$$

(the polynomial algebra over $\mathbb{F}_2$ on the listed generators) where the k-th conjugated generator $\bar\xi_k = \chi\xi_k$ has degree $(2^k - 1)$. There is a Bökstedt spectral sequence

$$E^2_{**} = HH_*(H_*(B)) \Longrightarrow H_*(THH(B))$$

and in this case the E^2-term is

$$HH_*(\mathscr{A}_*) = \mathscr{A}_* \otimes E(\sigma\bar\xi_1, \sigma\bar\xi_2, \dots)$$

(exterior algebra) where $\mathscr{A}_*$ lives in filtration 0 and $\sigma\bar\xi_k$ sits in bidegree $(1, 2^k-1)$. The *suspension operator* σ is induced by multiplication by the generator of $H_1(\mathbb{T})$. It is a differential and a derivation. For bidegree reasons there are no differentials, so $E^2_{**} = E^\infty_{**}$. In the abutment there are multiplicative extensions

$$(\sigma\bar\xi_k)^2 = \sigma\bar\xi_{k+1}$$

as can be proved using Dyer–Lashof operations. Hence

$$H_*(THH(\mathbb{F}_2)) = \mathscr{A}_* \otimes P(x)$$

where x is represented by $\sigma\bar\xi_1$. The $\mathscr{A}_*$-coaction on x is trivial, so the Adams spectral sequence collapses to the isomorphism

$$\pi_* THH(\mathbb{F}_2) = P(x) = \mathbb{F}_2[x]$$

with $\deg(x) = 2$.

The odd spheres filtration on $E\mathbb{T} = S(\mathbb{C}^\infty)$ induces a tower of fibrations with limit

$$THH(B)^{h\mathbb{T}} = F(E\mathbb{T}_+, THH(B))^{\mathbb{T}}$$

and an associated homological homotopy fixed point spectral sequence

$$E^2_{*,*} = H^{-*}(\mathbb{T}; H_*(THH(B))) \Longrightarrow H^c_*(THH(B)^{h\mathbb{T}}) .$$

The abutment is the *continuous homology* of $THH(B)^{h\mathbb{T}}$, which by definition is the limit of the homology groups in the tower. Dually, the *continuous cohomology*

$$H^*_c(THH(B)^{h\mathbb{T}})$$

is the colimit of the cohomology groups in the tower.

In this case, the E^2-term is

$$E^2_{*,*} = P(t) \otimes \mathscr{A}_* \otimes P(x)$$

with t in bidegree $(-2, 0)$ and x in bidegree $(0, 2)$. The d^2-differentials are determined by the formula

$$d^2(\alpha) = t \cdot \sigma\alpha$$

for $\alpha \in H_*(THH(B))$, and the fact that t is an infinite cycle. In low degrees we get

$\deg(\alpha)$	0	1	2	2	3	3	3
α	1	$\bar\xi_1$	$\bar\xi_1^2$	x	$\bar\xi_1^3$	$\bar\xi_2$	$\bar\xi_1 x$
$\sigma\alpha$	0	x	0	0	$\bar\xi_1^2 x$	x^2	x^2

so that the cycles on the vertical axis are

$$E^3_{0,*} = P(\bar\xi_1^2, \bar\xi_2 + \bar\xi_1 x, \dots) \otimes P(x)$$

To the left of the vertical axis only the part

$$E^3_{-2i,*} = P(\bar\xi_1^2, \bar\xi_2 + \bar\xi_1 x, \dots)\{t^i\}$$

survives, for $i > 0$. Here

$$P(\bar\xi_1^2, \bar\xi_2 + \bar\xi_1 x, \dots) \cong P(\bar\xi_1^2, \bar\xi_2, \dots) = H_*(H\mathbb{Z})$$

as an $\mathscr{A}_*$-comodule. Hence the E^3-term in the spectral sequence is

$$E^3_{*,*} \cong H_*(H\mathbb{Z})\{\dots, t^2, t, 1, x, x^2, \dots\}$$

In view of the $\mathscr{A}_*$-comodule structure, the spectral sequence collapses at this stage, and there is no room for comodule extensions, so $H^c_*(THH(\mathbb{F}_2)^{h\mathbb{T}})$ is a suitable completion of

$$\bigoplus_{j\in\mathbb{Z}} \Sigma^{2j} H_*(H\mathbb{Z})$$

In cohomology

$$H^*_c(THH(\mathbb{F}_2)^{h\mathbb{T}}) \cong \bigoplus_{j\in\mathbb{Z}} \Sigma^{2j} H^*(H\mathbb{Z}),$$

so by the Adams spectral sequence

$$\pi_* THH(\mathbb{F}_2)^{h\mathbb{T}} \cong \bigoplus_{j\in\mathbb{Z}} \Sigma^{2j} \mathbb{Z}_2 .$$

This is a first example of redshift, where v_0-periodicity arises in $\pi_* THH(B)^{h\mathbb{T}}$, even if $\pi_*(B)$ is v_0-torsion.

2.2. **The case $G = \mathbb{T}^2$ and $B = H$.** There is a Bökstedt spectral sequence

$$E^2_{**} = HH_*(H_*(THH(B))) \Longrightarrow H_*(\mathbb{T}^2 \otimes B).$$

In the case $B = H$, the abutment is

$$H_*(\mathbb{T}^2 \otimes H) = \mathscr{A}_* \otimes P(x_1, x_2) \otimes E(y)$$

where x_1, x_2 and y are represented by $\sigma_1\bar{\xi}_1$, $\sigma_2\bar{\xi}_1$ and $\sigma_1\sigma_2\bar{\xi}_1$, in degrees 2, 2 and 3, respectively. Here σ_1 and σ_2 are the differentials and derivations induced by the standard generators of $H_1(\mathbb{T}^2)$.

There is a homological homotopy fixed point spectral sequence

$$E^2_{**} = H^{-*}(\mathbb{T}^2; H_*(\mathbb{T}^2 \otimes B)) \Longrightarrow H^c_*((\mathbb{T}^2 \otimes B)^{h\mathbb{T}^2}).$$

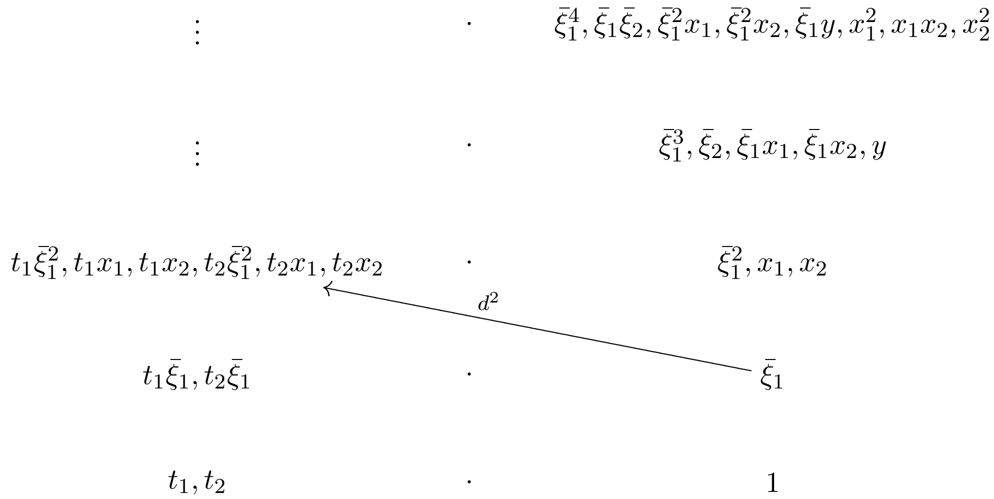
In our case the E^2-term is

$$E^2_{**} = P(t_1, t_2) \otimes \mathscr{A}_* \otimes P(x_1, x_2) \otimes E(y)$$

with t_1 and t_2 in bidegree $(-2, 0)$, x_1 and x_2 in bidegree $(0, 2)$ and y in bidegree $(0, 3)$. The d^2-differentials are given by

$$d^2(\alpha) = t_1 \cdot \sigma_1\alpha + t_2 \cdot \sigma_2\alpha$$

and the fact that t_1 and t_2 are infinite cycles. For instance, $d^2(\bar{\xi}_1) = t_1x_1 + t_2x_2$:



The d^2-cycles on the vertical axis are generated by $P(x_1^2, x_2^2)$ tensored with

$\deg(\alpha)$	0	1	2	3	4	5	6	7	8	9
α	1		$\bar\xi_1^2$		$\bar\xi_1^4$		$\bar\xi_1^6$ $\bar\xi_2^2$	$\hat\xi_3$	$\bar\xi_1^8$ $\bar\xi_1^2\bar\xi_2^2$	$\bar\xi_1^2\hat\xi_3$
				y	$\hat\xi_1 y$	$\bar\xi_1^2 y$	$\bar\xi_1^2(\hat\xi_1 y)$ $\hat\xi_2 y$	$\bar\xi_1^4 y$	$\bar\xi_1^4(\hat\xi_1 y)$ $\bar\xi_1^2(\hat\xi_2 y)$	$\bar\xi_1^6 y$ $\bar\xi_2^2 y$
						$x_1 y$		$\bar\xi_1^2 x_1 y$		$\bar\xi_1^4 x_1 y$
						$x_2 y$		$\bar\xi_1^2 x_2 y$		$\bar\xi_1^4 x_2 y$
								$x_1 x_2 y$		$\bar\xi_1^2 x_1 x_2 y$

where

$$\begin{aligned}\hat\xi_1 y &= \bar\xi_1 y + x_1 x_2\\ \hat\xi_2 y &= \bar\xi_2 y + x_1^2 x_2 + x_1 x_2^2\\ \hat\xi_3 &= \bar\xi_3 + \bar\xi_1\bar\xi_2 y + \bar\xi_2(x_1^2 + x_1x_2 + x_2^2) + \bar\xi_1(x_1^2x_2 + x_1x_2^2)\,.\end{aligned}$$

This is isomorphic to $P(x_1^2, x_2^2)$ tensored with the subspace of

$$H_*(ku)\{1, x_1y, x_2y, x_1x_2y\} \oplus \mathscr{A}_*\{y\}$$

where a copy of $\Sigma^7 H_*(ku) \subset \mathscr{A}_*\{y\}$ is "coidentified" with a copy of $\Sigma^7 H_*(ku) \subset H_*(ku)\{1\}$. Dually, in cohomology, the relation appears as

$$Q_2(1^*) = Q_0 Q_1(y^*)\,.$$

By a generalization, from circle actions to torus actions, of a vanishing result by Bruner and Rognes [BR05], these cycles are also infinite cycles. (Caveat: There might be a d^4-differential on $\bar\xi_1^2$, which would replace $H_*(ku)$ by $H_*(ko)$ in the following, but the qualitative conclusion remains the same.) Hence the image of the edge homomorphism to $H_*(\mathbb{T}^2 \otimes H)$ contains many copies of $H_*(ku)$, which produce copies of $\pi_*(ku_2^\wedge) = \mathbb{Z}_2[v_1]$ in homotopy. This is an example of higher redshift, where v_1-periodicity arises in $\pi_*(\mathbb{T}^2 \otimes B)^{h\mathbb{T}^2}$, even if $\pi_*(B)$ is v_0-torsion.

2.3. **The case $G = \mathbb{T}$, $B = tmf$.** Inspired by the redshift phenomenon, Bruner and Rognes are investigating $THH(tmf)^{h\mathbb{T}}$ as a possible example of a v_3-periodic theory with interesting maps

$$S \to K(tmf) \to THH(tmf)^{h\mathbb{T}}\,.$$

This has the potential of detecting γ-family elements in $\pi_*(S)$, which until now have not been observed at $p = 2$.

2.4. **The case $G = \mathbb{T}$, $B = MU$.** We are also interested in $K(MU)$ and $THH(MU)^{h\mathbb{T}}$, as a half-way house between $K(S) = A(\star)$ and $K(\mathbb{Z})$. Is $K(MU)$ simpler to describe then $K(S)$, in the way that $\pi_*(MU)$ is simpler than $\pi_*(S)$? Can $K(S)$ be recovered from $K(MU)$ by descent along the Hopf–Galois extension $S \to MU$?

3. Algebraic and topological Singer constructions

The prime order case $G = C_p \subset \mathbb{T}$ is quite well understood. In the elementary abelian cases $G = (C_p)^k \subset \mathbb{T}^k$ for $k \geq 2$, there are some open questions.

3.1. **The algebraic Singer construction.** Suppose B is bounded below and of finite type. In the case $G = C_2$, the cyclic group with two elements, the tensor product $C_2 \otimes B = B \wedge B$ is the smash square of B, and the homotopy orbit spectrum

$$D_2(B) = (B \wedge B)_{hC_2} = EC_{2+} \wedge_{C_2} B \wedge B\,,$$

where $EC_2 = S(\mathbb{R}^\infty)$, is known as the *quadratic construction* on B. Its cohomology

$$H^*(D_2(B)) \cong H^*(C_2; H^*(B) \otimes H^*(B))$$

(group cohomology) is determined as an $\mathscr{A}$-module by the Nishida relations [Nis68], which in homology take the form

$$Sq_*^s Q^t = \sum_j \binom{t-s}{s-2j} Q^{t-s+j} Sq_*^j \,.$$

The resulting $\mathscr{A}$-module is complicated, but there is an algebraic localization of it, known as the Singer construction [Sin80], which has much better structural properties.

The diagonal map $\Sigma(B \wedge B) \to \Sigma B \wedge \Sigma B$ induces a tower of quadratic constructions

$$\dots \to \Sigma^n D_2(\Sigma^{-n} B) \to \dots \to D_2(B)$$

and the dual tower in cohomology

$$H^*(D_2(B)) \to \dots \to H^*(\Sigma^n D_2(\Sigma^{-n} B)) \to \dots$$

has a colimit

$$\begin{aligned} \operatorname{colim}_n H^*(\Sigma^n D_2(\Sigma^{-n} B)) &\cong \hat{H}^*(C_2; H^*(B) \otimes H^*(B)) \\ &\cong P(x^{\pm 1}) \otimes \mathbb{F}_2\{a \otimes a\} \end{aligned}$$

(Tate cohomology), where a ranges over a basis for $H^*(B)$.

Miller observed that this colimit is isomorphic to the *algebraic Singer construction* $R_+(M)$ on the $\mathscr{A}$-module $M = H^*(B)$, up to a single desuspension. There is an explicit formula

$$R_+(M) = \Sigma P(x^{\pm 1}) \otimes M$$

where $P(x^{\pm 1}) = \mathbb{F}_2[x, x^{-1}]$ and

$$Sq^s(\Sigma x^r \otimes a) = \sum_j \binom{r-j}{s-2j} \Sigma x^{r+s-j} \otimes Sq^j(a) \,.$$

The $\mathscr{A}$-module isomorphism takes $\Sigma x^r \otimes a \otimes a$ in the continuous cohomology to $\Sigma x^{r+q} \otimes a$ in the Singer construction, where $q = \deg(a)$.

Singer noted that there is a natural $\mathscr{A}$-module homomorphism

$$\begin{aligned} \epsilon \colon R_+(M) &\longrightarrow M \\ \Sigma x^r \otimes a &\longmapsto Sq^{r+1}(a) \end{aligned}$$

for all $\mathscr{A}$-modules M. The really interesting fact about ϵ, which is due to Gunawardena and Miller [AGM85], is that it is an Ext-*equivalence*, in the sense that the induced homomorphism

$$\epsilon^\# \colon \operatorname{Ext}_{\mathscr{A}}^{s,t}(M, \mathbb{F}_2) \xrightarrow{\cong} \operatorname{Ext}_{\mathscr{A}}^{s,t}(R_+(M), \mathbb{F}_2)$$

is an isomorphism.

3.2. **The topological Singer construction.** The terms in the tower of quadratic constructions can be rewritten as

$$\Sigma^n D_2(\Sigma^{-n} B) \simeq F(S^{\mathbb{R}^n}, EC_{2+} \wedge B \wedge B)^{C_2} \,,$$

so that the maps in the tower are given by restriction along $S^{\mathbb{R}^n} \subset S^{\mathbb{R}^{n+1}}$. Hence the homotopy limit of the tower is

$$\operatorname{holim}_n \Sigma^n D_2(\Sigma^{-n} B) \simeq F(\widetilde{EC_2}, EC_{2+} \wedge B \wedge B)^{C_2} = \Sigma^{-1}(B \wedge B)^{tC_2}$$

where $\widetilde{EC_2} = S^{\mathbb{R}^\infty}$ and

$$X^{tG} = [\widetilde{EG} \wedge F(EG_+, X)]^G \simeq \Sigma F(\widetilde{EG}, EG_+ \wedge X)^G$$

denotes the *Tate construction* for G acting on X. The latter equivalence is called Warwick duality by Greenlees [Gre95]. It follows that the algebraic Singer construction

$$R_+(H^*(B)) \cong H_c^*((B \wedge B)^{tC_2})$$

can be identified with the continuous cohomology of the Tate construction for C_2 acting on $B \wedge B$.

Definition 3.1 (Lunøe-Nielsen, Rognes [LNR])**.** The *topological Singer construction* on a spectrum B, denoted $R_+(B)$, is the Tate construction on the C_2-equivariant spectrum $B \wedge B$:

$$R_+(B) = (B \wedge B)^{tC_2} \,.$$

Remark 3.2. When B is an S-algebra, $R_+(B)$ inherits a multiplicative structure from $B \wedge B$, which is not apparent in its realization as the suspended homotopy limit of the tower of quadratic constructions. In actual calculations we must work with homology in a category of complete (topological) $\mathscr{A}_*$-comodule algebras instead of cohomology in a category of $\mathscr{A}$-module coalgebras.

The cofiber sequence $EC_{2+} \to S^0 \to \widetilde{EC}_2$ and the canonical map $B \wedge B \to F(EC_{2+}, B \wedge B)$ induce a map of cofiber sequences known as the *norm–restriction* diagram

$$\begin{array}{ccccccc} D_2(B) & \xrightarrow{N} & (B \wedge B)^{C_2} & \xrightarrow{R} & B & \xrightarrow{\partial} & \Sigma D_2(B) \\ {\scriptstyle =}\downarrow & & {\scriptstyle \Gamma_1}\downarrow & & {\scriptstyle \epsilon_B}\downarrow & & {\scriptstyle =}\downarrow \\ D_2(B) & \xrightarrow{N^h} & (B \wedge B)^{hC_2} & \xrightarrow{R^h} & R_+(B) & \xrightarrow{\partial^h} & \Sigma D_2(B) \end{array}$$

with a (co-)cartesian square in the middle.

Implicit here is the assumption that $B \wedge B$ has the (genuine) C_2-equivariant structure provided by the *Bökstedt smash product*, so that $[\widetilde{EC}_2 \wedge B \wedge B]^{C_2} \simeq B$. The reason why we deviate from the usual notation for ϵ_B, which is $\hat{\Gamma}_1$, will be clear in a moment. The following was previously known for finite B.

Theorem 3.3 (Lunøe-Nielsen, Rognes [LNR])**.** *Let B be bounded below with $H_*(B; \mathbb{F}_2)$ of finite type. Then the natural map*

$$\epsilon_B \colon B \longrightarrow R_+(B) = (B \wedge B)^{tC_2}$$

is a 2-adic equivalence.

Sketch of proof. The thing to check is that the map ϵ_B induces the Ext-equivalence ϵ in continuous cohomology:

$$(\epsilon_B)^* = \epsilon \colon H^*_c(R_+(B)) \cong R_+(H^*(B)) \longrightarrow H^*(B) \,.$$

This is made easier by the isomorphism

$$\epsilon^\# \colon \operatorname{Hom}_{\mathscr{A}}(R_+(M), M) \cong \operatorname{Hom}_{\mathscr{A}}(M, M)$$

for $M = H^*(B)$. Hence ϵ_B induces an isomorphism of Adams spectral sequences from the E_2-term and onwards. □

Corollary 3.4. *Let B be bounded below with $H_*(B; \mathbb{F}_2)$ of finite type. Then the natural map*

$$\Gamma_1 \colon (B \wedge B)^{C_2} \longrightarrow (B \wedge B)^{hC_2}$$

is a 2-adic equivalence.

Example 3.5. When $B = S$ is the sphere spectrum, so that $B \wedge B$ is the C_2-equivariant sphere spectrum, this is the Segal conjecture for C_2, also known as Lin's theorem [LDMA80].

3.3. **Higher ranks.** In the special case $M = \mathbb{F}_2$, the Singer construction

$$R_+(\mathbb{F}_2) = \Sigma P(x^{\pm 1}) = \Sigma H^*(\mathbb{R}P^\infty)[x^{-1}]$$

is a localization of the cohomology of $\mathbb{R}P^\infty = BC_2$.

Adams, Gunawardena and Miller [AGM85] also established an Ext-equivalence for localizations of the cohomology of elementary abelian p-groups of higher rank. Let $G = (C_2)^k$, so that $BG = (\mathbb{R}P^\infty)^k$ and

$$H^*(BG) = P(x_1, \dots, x_k) \,.$$

Let

$$H^*(BG)_{\mathrm{loc}} = P(x_1, \dots, x_k)[w^{-1}]$$

where $w = \lambda_1 x_1 + \cdots + \lambda_k x_k \neq 0$ ranges over all the nonzero elements in $H^1(BG)$. Then there is an Ext-equivalence

$$\epsilon \colon \Sigma^k H^*(BG)_{\mathrm{loc}} \longrightarrow \mathrm{St}_k$$

where St_k is the Steinberg representation of $GL_k(\mathbb{Z}/2)$. It is of dimension $2^{k(k+1)/2}$, and is concentrated in degree zero.

There is also an Ext-equivalence out of the less drastic localization

$$H^*(BG)[x_1^{-1}, \dots, x_k^{-1}] = P(x_1^{\pm 1}, \dots, x_k^{\pm 1}),$$

but the target takes longer to describe.

We can realize each of these localizations of $\Sigma^k H^*(BG)$ as the continuous cohomology of a tower of Thom spectra. It seems to be an interesting question how to realize these Ext-equivalences by a spectrum map into the limit of these towers, which will then be a 2-adic equivalence. Presumably this will give information about the natural map

$$\Gamma_G \colon (G \otimes B)^G \to (G \otimes B)^{hG}$$

in the higher rank cases.

4. Cyclic fixed points

The norm–restriction diagram for the $\mathbb{T}$-action on $THH(B)$ is a map of cofiber sequences

$$\begin{array}{ccccccc}
\Sigma THH(B)_{h\mathbb{T}} & \xrightarrow{N} & TF(B;p) & \xrightarrow{R} & TF(B;p) & \xrightarrow{\partial} & \Sigma^2 THH(B)_{h\mathbb{T}} \\
\downarrow = & & \downarrow \Gamma & & \downarrow \hat\Gamma & & \downarrow = \\
\Sigma THH(B)_{h\mathbb{T}} & \xrightarrow{N^h} & THH(B)^{h\mathbb{T}} & \xrightarrow{R^h} & THH(B)^{t\mathbb{T}} & \xrightarrow{\partial^h} & \Sigma^2 THH(B)_{h\mathbb{T}}
\end{array}$$

(after p-adic completion), with a (co-)cartesian square in the middle.

Remark 4.1. There is a Frobenius trace map

$$tr_F \colon K(B) \longrightarrow TF(B;p)$$

that homotopy equalizes the identity and the restriction map R. This leads to the *cyclotomic trace map* [BHM89]

$$trc \colon K(B) \longrightarrow TC(B;p) = \mathrm{hoeq}(1, R \colon TF(B;p) \to TF(B;p))$$

which is a powerful invariant of algebraic K-theory.

Definition 4.2. Let k be an integer or $-\infty$. A map $f \colon X \to Y$ is *k-coconnected* if $f_* \colon \pi_i(X) \to \pi_i(Y)$ is injective for $i = k$ and bijective for $i > k$.

Theorem 4.3 (Tsalidis [Tsa98])**.** *Let B be a connective S-algebra with $H_*(B;\mathbb{F}_p)$ of finite type. If the map $\Gamma_1 \colon THH(B)^{C_p} \to THH(B)^{hC_p}$ becomes k-coconnected after p-adic completion, then so do all of the maps*

$$\hat\Gamma_1 \colon THH(B) \to THH(B)^{tC_p}\ ,\ \Gamma \colon TF(B;p) \to THH(B)^{h\mathbb{T}} \text{ and } \hat\Gamma \colon TF(B;p) \to THH(B)^{t\mathbb{T}}.$$

Remark 4.4. This tells us that when the Segal conjecture for the C_p-action on $THH(B)$ holds, with suitable coefficients and in high degrees, then we are free to replace the homotopy fixed points $THH(B)^{h\mathbb{T}}$ with the Tate construction $THH(B)^{t\mathbb{T}}$, and either one of these is a good approximation to the topological Frobenius theory $TF(B;p)$. Calculations are often easier for the Tate constructions, since the Tate cohomology of C_p is (almost) a graded field, while the group cohomology has a more complicated module theory.

The following definition can be used to make a more general statement.

Definition 4.5. Let W be in the localizing ideal of spectra generated by S^{-1}/p^∞. Let k be an integer or $-\infty$. We say that a map $f\colon X \to Y$ is (W,k)-*coconnected* if the homomorphism

$$f_*\colon \pi_i F(W,X) \longrightarrow \pi_i F(W,Y)$$

is injective for $i = k$ and bijective for $i > k$.

Example 4.6. If $W = S^{-1}/p^\infty$ and $k = -\infty$ we have $F(S^{-1}/p^\infty, X) = X_p^\wedge$, and the condition is that the map of p-adic completions is an equivalence.

Example 4.7. For $B = H\mathbb{F}_p$, the map $\Gamma_1\colon THH(\mathbb{F}_p)^{C_p} \to THH(\mathbb{F}_p)^{hC_p}$ is $(S^{-1}/p^\infty, -2)$-coconnected (Hesselholt–Madsen [HM97]).

For $B = H\mathbb{Z}$, the map $\Gamma_1\colon THH(\mathbb{Z})^{C_p} \to THH(\mathbb{Z})^{hC_p}$ is $(V(0), -2)$-coconnected, where $V(0) = S/p$ (Bökstedt–Madsen [BM94], Rognes [Rog99]).

For $B = \ell$, the Adams summand in p-local connective K-theory, the map $\Gamma_1\colon THH(\mathbb{Z})^{C_p} \to THH(\mathbb{Z})^{hC_p}$ is $(V(1), -2)$-coconnected, where $p \geq 5$ and $V(1) = S/(p, v_1)$ (Ausoni–Rognes [AR02]).

The degree shifts compared to the original papers arise because we are considering function spectra $F(W, -)$ in place of smash products $W \wedge (-)$.

Theorem 4.8 (Bökstedt, Bruner, Lunøe-Nielsen, Rognes [BBLNR])**.** *Let B be a connective S-algebra with $H_*(B;\mathbb{F}_p)$ of finite type. If the map $\Gamma_1\colon THH(B)^{C_p} \to THH(B)^{hC_p}$ is (W,k)-coconnected, then so are all of the maps*

$$\hat\Gamma_1\colon THH(B) \to THH(B)^{tC_p}\ ,\ \Gamma\colon TF(B;p) \to THH(B)^{h\mathbb{T}} \text{ and } \hat\Gamma\colon TF(B;p) \to THH(B)^{t\mathbb{T}}.$$

There is a similar statement for each cyclic subgroup $C_{p^n} \subset \mathbb{T}$, and the proof proceeds by induction on n.

5. Additive approximations

The unit map $\eta\colon B \to THH(B)$ extends to an $\mathbb{T}$-equivariant map

$$\omega\colon \mathbb{T} \ltimes B \longrightarrow THH(B).$$

Given $\alpha \in H_q(B)$ we write $\sigma\alpha \in H_{q+1}(THH(B))$ for the image of $\sigma \ltimes \alpha$, where $\sigma \in H_1(\mathbb{T})$ is the generator.

The inclusion $C_2 \subset \mathbb{T}$ induces a C_2-equivariant map $\eta_2\colon B \wedge B \to THH(B)$, which extends to an $\mathbb{T}$-equivariant map

$$\omega_2\colon \mathbb{T} \ltimes_{C_2} B \wedge B \longrightarrow THH(B).$$

Applying $(-)^{tC_2}$ we get a $\mathbb{T}/C_2$-equivariant map

$$\omega^t\colon \mathbb{T}/C_2 \ltimes R_+(B) \simeq (\mathbb{T} \ltimes_{C_2} B \wedge B)^{tC_2} \longrightarrow THH(B)^{tC_2}.$$

Lemma 5.1 (Lunøe-Nielsen, Rognes [LNR11])**.** *There is a homotopy commutative square*

$$\begin{array}{ccc} \mathbb{T} \ltimes B & \xrightarrow{\ \omega\ } & THH(B) \\ \downarrow{\scriptstyle \rho\wedge\epsilon_B} & & \downarrow{\scriptstyle \hat\Gamma_1} \\ \mathbb{T}/C_2 \ltimes R_+(B) & \xrightarrow{\ \omega^t\ } & THH(B)^{tC_2} \end{array}$$

where $\rho\colon \mathbb{T} \to \mathbb{T}/C_2$ is the square root isomorphism, and $\epsilon_B\colon B \to R_+(B)$ is the map inducing the Ext*-equivalence in cohomology.*

Using this diagram we can compute the effect of $\hat\Gamma_1$ on classes in $H_*(THH(B))$ that are in the image of ω_*, i.e., the classes α and $\sigma\alpha$ for $\alpha \in H_*(B)$. This is made possible by explicit formulas for $(\epsilon_B)_* = \epsilon_*$ and $(\omega^t)_*$ in homology.

Remark 5.2. If we instead apply $(-)^{t\mathbb{T}}$, we get a map

$$\omega'\colon \Sigma R_+(B) \simeq (\mathbb{T} \ltimes_{C_2} B \wedge B)^{t\mathbb{T}} \longrightarrow THH(B)^{t\mathbb{T}},$$

where we have a Wirthmüller equivalence on the left hand side. We can use this map to get a hold on the completed $\mathscr{A}_*$-comodule structure on $H_*^c(THH(B)^{t\mathbb{T}})$, or dually, the $\mathscr{A}$-module structure on $H^*_c(THH(B)^{t\mathbb{T}})$. This program is currently being pursued by Knut Berg.

6. THH of complex bordism

In view of the examples $B = H\mathbb{F}_p$, $H\mathbb{Z}$, ℓ, ku and tmf, where Γ_1 and $\hat{\Gamma}_1$ only become equivalences in high degrees, the following theorem is a little surprising. It asserts that the C_p-equivariant Segal conjecture holds for $THH(MU)$, in just as strong a form as it holds for the C_p-equivariant sphere spectrum $THH(S) = S$.

Theorem 6.1 (Lunøe-Nielsen, Rognes [LNR11]). *The map*

$$\Gamma_1\colon THH(MU)^{C_p} \longrightarrow THH(MU)^{hC_p}$$

is a p-adic equivalence.

Corollary 6.2. *The maps $\hat{\Gamma}_1\colon THH(MU) \to THH(MU)^{tC_p}$, $\Gamma\colon TF(MU;p) \to THH(MU)^{h\mathbb{T}}$ and $\hat{\Gamma}\colon TF(MU;p) \to THH(MU)^{t\mathbb{T}}$ are p-adic equivalences.*

Outline of proof. We prove that $\hat{\Gamma}_1$ is a p-adic equivalence by showing that there is an isomorphism of $\mathscr{A}$-modules

$$\Phi^*\colon H^*_c(THH(MU)^{tC_p}) \xrightarrow{\cong} H^*_c(R_+(THH(MU)))$$

such that $(\hat{\Gamma}_1)^* = \epsilon \circ \Phi^*$. To achieve this we show that there is a bicontinuous isomorphism of complete $\mathscr{A}_*$-comodules

$$\Phi\colon H_*^c(R_+(THH(MU))) \xrightarrow{\cong} H_*^c(THH(MU)^{tC_p})$$

such that $(\hat{\Gamma}_1)_* = \Phi \circ \epsilon_*$.

As before, we concentrate on the case $p = 2$. There is an $\mathscr{A}_*$-comodule algebra isomorphism

$$H_*(MU) = P(m_\ell \mid \ell \geq 1)$$

where $m_\ell = \bar{\xi}_k^2$ for $\ell = 2^k - 1$, and m_ℓ is $\mathscr{A}_*$-comodule primitive for $\ell \neq 2^k - 1$. The Bökstedt spectral sequence gives

$$H_*(THH(MU)) \cong H_*(MU) \otimes E(\sigma m_\ell \mid \ell \geq 1)$$

where each σm_ℓ is $\mathscr{A}_*$-comodule primitive of degree $2\ell + 1$.

There is a homological C_2-Tate spectral sequence for $THH(MU) \wedge THH(MU)$ (with the twist action)

$$'\hat{E}^2_{**} = \hat{H}^{-*}(C_2; H_*(THH(MU)) \otimes H_*(THH(MU))) \Longrightarrow H_*^c(R_+(THH(MU)))$$

with E^∞-term

$$'\hat{E}^\infty_{**} = P(u^{\pm 1}) \otimes P(m_\ell \otimes m_\ell) \otimes E(\sigma m_\ell \otimes \sigma m_\ell)$$

and another homological C_2-Tate spectral sequence for $THH(MU)$ (with the restricted $\mathbb{T}$-action)

$$''\hat{E}^2_{**} = \hat{H}^{-*}(C_2; H_*(THH(MU))) \Longrightarrow H_*^c(THH(MU)^{tC_2})$$

with E^∞-term

$$''\hat{E}^\infty_{**} = P(u^{\pm 1}) \otimes P(m_\ell^2) \otimes E(m_\ell \sigma m_\ell)\,.$$

In both cases u has bidegree $(-1, 0)$ and ℓ ranges over all natural numbers. The desired isomorphism Φ should then map

$$\begin{aligned} u &\longmapsto u \\ m_\ell \otimes m_\ell &\longmapsto m_\ell^2 \\ \sigma m_\ell \otimes \sigma m_\ell &\longmapsto u^{-1} \cdot m_\ell \sigma m_\ell \end{aligned}$$

modulo Tate filtrations. Due to the filtration shift, this is hopeless to establish algebraically, so we need to start with a topological comparison.

In general, we have a commutative diagram

$$\begin{array}{ccccc} THH(MU) & \xleftarrow{\ \eta\ } & MU & \xrightarrow{\ \eta\ } & THH(MU) \\ \downarrow{\scriptstyle \epsilon} & & \downarrow{\scriptstyle \epsilon} & & \downarrow{\scriptstyle \hat{\Gamma}_1} \\ R_+(THH(MU)) & \xleftarrow{\ R_+(\eta)\ } & R_+(MU) & \xrightarrow{\ \eta^t\ } & THH(MU)^{tC_p} \end{array}$$

of commutative ring spectra, where $\eta^t = \eta_2^{tC_2}$. Passing to (continuous) homology, we get the commutative diagram of complete $\mathscr{A}_*$-comodules given by the solid arrows in the following diagram:

$$\begin{array}{ccccc} & & H_*(THH(MU)) & & \\ & \swarrow{\scriptstyle \epsilon_*} & \downarrow & \searrow{\scriptstyle (\hat{\Gamma}_1)_*} & \\ & & H^c_*(R_+(MU)) \otimes_{H_*(MU)} H_*(THH(MU)) & & \\ & \swarrow{\scriptstyle f} & & \searrow{\scriptstyle g} & \\ H^c_*(R_+(THH(MU))) & & \overset{\Phi}{\dashrightarrow} & & H^c_*(THH(MU)^{tC_p}) \end{array}$$

We wish to construct a bicontinuous $\mathscr{A}_*$-comodule isomorphism Φ making the whole diagram commute.

This is done by two careful calculations. The central term can be rewritten as

$$H^c_*(R_+(MU)) \otimes E(\sigma m_\ell \mid \ell \geq 1)\,.$$

The homomorphisms f and g can be computed by using the explicit formulas for ϵ and $\hat{\Gamma}_1$ in homology. This reveals that both f and g are degreewise pro-isomorphisms of inverse systems (with non-strict = filtration shifting pro-inverses), inducing bicontinuous isomorphisms

$$\hat{f} \colon H^c_*(R_+(MU)) \mathbin{\widehat{\otimes}} E(\sigma m_\ell \mid \ell \geq 1) \xrightarrow{\cong} H^c_*(R_+(THH(MU)))$$

and

$$\hat{g} \colon H^c_*(R_+(MU)) \mathbin{\widehat{\otimes}} E(\sigma m_\ell \mid \ell \geq 1) \xrightarrow{\cong} H^c_*(THH(MU)^{tC_p})$$

of completed tensor products of $\mathscr{A}_*$-comodules. The desired isomorphism Φ is then defined as $\Phi = \hat{g} \circ \hat{f}^{-1}$. □

This result tells us that we have a good chance at determining $K(MU)$ by way of $TC(MU; p)$, since $TF(MU; p)$, $THH(MU)^{h\mathbb{T}}$ and $THH(MU)^{t\mathbb{T}}$ are all p-adically equivalent.

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