ALGEBRAIC K-THEORY OF THE FRACTION FIELD OF TOPOLOGICAL K-THEORY

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On joint work with Christian Ausoni.

(1) Let ku be the connective complex K-theory spectrum. There is a pushout square of associative ku-algebras

$$\begin{array}{ccc} ku & \xrightarrow{\;i\;} & ku/p \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\ H\mathbb{Z} & \xrightarrow{\;i\;} & H\mathbb{Z}/p \end{array}$$

where ku/p is not commutative. We imagine localizations $p^{-1}ku$ and $p^{-1}KU$ such that there is a 3×3 square of cofiber sequences

$$\begin{array}{ccccc} K(\mathbb{Z}/p) & \xrightarrow{\;i_*\;} & K(\mathbb{Z}) & \longrightarrow & K(\mathbb{Z}[1/p]) \\ \downarrow{\scriptstyle \pi_*} & & \downarrow{\scriptstyle \pi_*} & & \downarrow{\scriptstyle \pi_*} \\ K(ku/p) & \xrightarrow{\;i_*\;} & K(ku) & \longrightarrow & K(p^{-1}ku) \\ \downarrow & & \downarrow & & \downarrow \\ K(KU/p) & \xrightarrow{\;i_*\;} & K(KU) & \longrightarrow & K(p^{-1}KU) \end{array}$$

where i_* and π_* are the algebraic K-theory transfer maps. We propose that inverting all primes p in KU this way yields the S-algebraic **fraction field** $\mathit{ff}(KU)$ of KU.

(2) To make computations, it is convenient to p-complete, and to restrict attention to the Adams summands ℓ_p and L_p of ku_p and KU_p, respectively. Here $\pi_*\ell_p = \mathbb{Z}_p[v_1]$ and $\pi_*L_p = \mathbb{Z}_p[v_1^{\pm 1}]$. There are similar diagrams as above, replacing ku/p and KU/p by $\ell/p = k(1)$ and $L/p = K(1)$, respectively. Then $p^{-1}L_p = \mathit{ff}(L_p)$ is the fraction field of the Adams summand of p-adic topological K-theory. We view this as a "brave new" 2-local field, with valuation ring $p^{-1}\ell_p$ and residue field $\mathbb{Q}_p$. Let $V(1)$ be the Smith–Toda complex. It is a ring spectrum for $p \geq 5$, and

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represents homotopy with mod (p, v_1) coefficients. Using topological Hochschild homology, topological cyclic homology and the cyclotomic trace map we compute

$$
\begin{aligned}
V(1)_* K(\mathit{ff}(L_p)) \equiv\ & P(v_2) \otimes E(\partial v_2, \delta_1, \delta_2) \\
& \oplus P(v_2) \otimes E(\delta_2) \otimes \mathbb{F}_p\{t^i \lambda_1 \mid 0 < i < p\} \\
& \oplus P(v_2) \otimes E(\delta_2) \otimes \mathbb{F}_p\{t^i v_2 \delta_1 \mid 0 < i < p^2 - p, (i,p) = 1\} \\
& \oplus P(v_2) \otimes E(\delta_1) \otimes \mathbb{F}_p\{t^{ip} \lambda_2 \mid 0 < i < p\}
\end{aligned}
$$

up to a tiny error term. This is a free module over $P(v_2)$ of rank $(2p^2 + 6)$ and zero Euler characteristic. The symbols $\delta_1 = d\log p$ and $\delta_2 = d\log v_1$ play the role of **logarithmic poles**.

(3) We can arrange these groups as the $E^2 = E^\infty$ term of a hypothetical motivic spectral sequence

$$E^2_{s,t} = H^{-s}_{mot}(\mathit{ff}(L_p); \mathbb{F}_{p^2}(t/2)) \Longrightarrow V(1)_{s+t} K(\mathit{ff}(L_p)) .$$

Then the analogue of the **Bloch–Kato conjecture** holds: Letting the Galois cohomology groups

$$H^n_{Gal}(\mathit{ff}(L_p); \mathbb{F}_{p^2}(k)) = v_2^{-1} H^n_{mot}(\mathit{ff}(L_p); \mathbb{F}_{p^2}(k))$$

be the v_2-inverted motivic groups, one recovers the motivic groups by the truncation formula

$$H^n_{mot}(\mathit{ff}(L_p); \mathbb{F}_{p^2}(k)) = \begin{cases} H^n_{Gal}(\mathit{ff}(L_p); \mathbb{F}_{p^2}(k)) & \text{for } n \le k, \\ 0 & \text{for } n > k. \end{cases}$$

(4) Also, the analogue for 2-local fields of **Tate–Poitou local arithmetic duality** holds: There is an isomorphism

$$H^3_{Gal}(\mathit{ff}(L_p); \mathbb{F}_{p^2}(2)) \xrightarrow{inv} \mathbb{Z}/p$$

taking $\partial \delta_1 \delta_2$ to the generator, and the Galois cohomology cup product

$$H^n_{Gal}(\mathit{ff}(L_p); \mathbb{F}_{p^2}(k)) \otimes H^{3-n}_{Gal}(\mathit{ff}(L_p); \mathbb{F}_{p^2}(2-k)) \xrightarrow{\cup} H^3_{Gal}(\mathit{ff}(L_p); \mathbb{F}_{p^2}(2)) \cong \mathbb{Z}/p$$

is visibly a perfect pairing.

(5) For each local number field F containing $\mathbb{Q}_p$, with ring of integers $\mathcal{O}_F$, there exist non-unique associative ℓ_p-algebras $\ell\mathcal{O}_F$ with $\pi_* \ell\mathcal{O}_F = \ell\mathcal{O}_F[v_1]$, by the A_∞ obstruction theory of Robinson. When F is unramified, $\ell\mathcal{O}_F$ is uniquely a commutative ℓ_p-algebra. In the imagined localizations, there are nontrivial Kähler differentials and derived obstruction groups at p, but we hope that $p^{-1}\ell\mathcal{O}_F$ is uniquely a commutative $p^{-1}\ell_p$-algebra, also for ramified F. Let $\mathbb{Z}_p^{nr}$ and $\bar{\mathbb{Z}}_p$ be the ring of integers in the maximal unramified extension of $\mathbb{Q}$ and the separable closure of $\mathbb{Q}_p$, respectively. Let Ω_1 be the $K(1)$-local separable closure of $\mathit{ff}(L_p)$. We would then have a tower of extensions

$$\mathit{ff}(L_p) \xrightarrow{G_{\mathbb{F}_p}} \mathit{ff}(L\mathbb{Z}_p^{nr}) \xrightarrow{I_p} \mathit{ff}(L\bar{\mathbb{Z}}_p) \xrightarrow{I_{v_1}} \Omega_1$$

where $I_p \to G_{\mathbb{Q}_p} \to G_{\mathbb{F}_p} = \hat{\mathbb{Z}}$ and $I_{v_1} \to G_{\mathit{ff}(L_p)} \to G_{\mathbb{Q}_p}$. We then expect the Galois cohomology of $\mathit{ff}(L_p)$ introduced above to be the continuous cohomology of the absolute Galois group $G_{\mathit{ff}(L_p)} = \mathrm{Gal}(\Omega_1/\mathit{ff}(L_p))$. The discrepancy between the Galois cohomology of $\mathbb{Q}_p$ and that computed for $\mathit{ff}(L_p)$ suggests where to find the remaining extensions from $\mathit{ff}(L\bar{\mathbb{Z}}_p)$ to Ω_1.

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