Algebraic K-Theory of Strict Ring Spectra

John Rognes

University of Oslo, Norway

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Outline

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Symmetric Spectra (Smith)

- A *spectrum* is a sequence of based spaces

$$X_0, X_1, X_2, \dots$$

  and maps $\sigma \colon X_n \wedge S^1 \to X_{n+1}$, for $n \geq 0$.
- A symmetric spectrum is a spectrum equipped with a Σ_n-action on each X_n, such that

$$\sigma^k \colon X_n \wedge S^k \to X_{n+k}$$

  is $\Sigma_n \times \Sigma_k$-equivariant for each $n, k \geq 0$.

Symmetric Ring Spectra

- The category Sp^{Σ} of symmetric spectra is closed symmetric monoidal, with unit the *sphere spectrum* S and monoidal pairing the *smash product* $X \wedge Y$.
- Its localization $Ho(Sp^{\Sigma})$ with respect to the stable equivalences is Boardman's stable homotopy category.
- A symmetric ring spectrum is a symmetric spectrum A with associative and unital structure maps $\mu\colon A \wedge A \to A$ and $\eta\colon S \to A$.

Algebraic K-Theory of Symmetric Ring Spectra

- Mandell defined $K(A)$ as the algebraic K-theory of a category $\mathscr{C}_A$ of finite cell A-modules.
- The algebraic K-theory spectrum $K(A)$ exhibits a group completion

$$|h\mathscr{C}_A| \to \Omega^\infty K(A)$$

of the left hand classifying space, turning cofiber sequences into sums.

Algebraic K-Theory of Spaces

- Let $X \simeq BG$ be a space, with loop group $G \simeq \Omega X$.
- Let $S[G]$ be the spherical group ring spectrum.
- Waldhausen first defined

$$A(X) = K(S[G])$$

as the algebraic K-theory of an unstable model for the category of finite cell $S[G]$-modules, the category of *retractive spaces over X*.

h-Cobordism Spaces

- If X is a compact smooth manifold, let $H(X)$ be the space of h-cobordisms $(W; X, Y)$ with X at one end:

$$\partial W = X \cup Y \,, \qquad X \overset{\simeq}{\to} W \overset{\simeq}{\leftarrow} Y$$

- Let $\mathscr{H}(X) = \operatorname{colim}_k H(X \times [0,1]^k)$ be the stable h-cobordism space.

Theorem (Igusa)

$H(X) \to \mathscr{H}(X)$ is about $n/3$-connected, for $n = \dim X$.

The Stable Parametrized h-Cobordism Theorem

- $A(X) = K(S[G])$ splits as

$$A(X) \simeq S[X] \vee Wh(X)\,,$$

defining the *Whitehead spectrum*.

- Let $\Omega Wh(X) = \Omega^{\infty+1} Wh(X)$ be the looped Whitehead space.

Theorem (Waldhausen–Jahren–R.)

There is a natural homotopy equivalence $\mathscr{H}(X) \simeq \Omega Wh(X)$.

Diffeomorphism Groups: Rational

When X is contractible, $A(*) = K(S) \simeq S \vee Wh(*)$.

Theorem (Borel)

$$K_i(S) \otimes \mathbb{Q} \cong K_i(\mathbb{Z}) \otimes \mathbb{Q} \cong \begin{cases} \mathbb{Q} & \textit{for } i = 0 \textit{ or } 4k+1 \neq 1, \\ 0 & \textit{otherwise.} \end{cases}$$

Example (Farrell–Hsiang)

$$\pi_i Diff(D^n \operatorname{rel} \partial D^n) \otimes \mathbb{Q} \cong \begin{cases} \mathbb{Q} & \text{for } i = 4k-1,\ n \text{ odd}, \\ 0 & \text{otherwise}, \end{cases}$$

for i up to about $n/3$.

Outline

Topological Cyclic Homology

- Bökstedt–Hsiang–Madsen constructed a natural *cyclotomic trace map*

$$K(A) \to TC(A; p)$$

  to the topological cyclic homology of A.
- It is a homotopy limit

$$TC(A; p) = \operatorname*{holim}_{n,R,F} THH(A)^{C_{p^n}}$$

  of cyclic fixed points of the topological Hochschild homology of A.

Nilpotent extensions

An integral version satisfies $TC(A)^\wedge_p \simeq TC(A;p)^\wedge_p$.

Theorem (Dundas–Goodwillie–McCarthy)

Let $A \to B$ be a map of connective symmetric ring spectra, with $\pi_0(A) \to \pi_0(B)$ surjective with nilpotent kernel. The square

$$\begin{array}{ccc} K(A) & \longrightarrow & K(B) \\ \downarrow & & \downarrow \\ TC(A) & \longrightarrow & TC(B) \end{array}$$

is homotopy Cartesian.

The Sphere Spectrum and the Integers

Example

Homotopy Cartesian square

$$\begin{array}{ccc} K(S)^\wedge_p & \longrightarrow & K(\mathbb{Z})^\wedge_p \\ \downarrow & & \downarrow \\ TC(S;p)^\wedge_p & \longrightarrow & TC(\mathbb{Z};p)^\wedge_p . \end{array}$$

R. used this to calculate H_* and π_* of

$$K(S)^\wedge_p \simeq S^\wedge_p \vee Wh(*)^\wedge_p$$

for regular primes p.

K-Theory of the Sphere Spectrum: Cohomology

- Let $\mathscr{A}$ be the mod p Steenrod algebra.
- For $p = 2$ let $C \subset \mathscr{A}$ be generated by admissible Sq^I where $I = (i_1, \dots, i_n)$ with $n \geq 2$ or $I = (i)$ with i odd.

Theorem (R.)

The mod 2 cohomology of $Wh(*)$ is the nontrivial extension

$$\Sigma^{-2} C/\mathscr{A}(Sq^1, Sq^3) \to H^* Wh(*) \to \Sigma^3 \mathscr{A}/\mathscr{A}(Sq^1, Sq^2)$$

of $\mathscr{A}$-modules.

K-Theory of the Sphere Spectrum: Homotopy

Example (R.)

The homotopy groups of $Wh(*)$, modulo p-torsion for irregular primes p, begin:

i	0	1	2	3	4	5	6	7	8	9
$\pi_i Wh(*)$	0	0	0	$\mathbb{Z}/2$	0	$\mathbb{Z}$	0	$\mathbb{Z}/2$	0	$\mathbb{Z} \oplus \mathbb{Z}/2$

i	10	11	12	13	14
$\pi_i Wh(*)$	$\mathbb{Z}/8 \oplus (\mathbb{Z}/2)^2$	$\mathbb{Z}/6$	$\mathbb{Z}/4$	$\mathbb{Z}$	$\mathbb{Z}/36 \oplus \mathbb{Z}/3$

i	15	16	17	18
$\pi_i Wh(*)$	$(\mathbb{Z}/2)^2$	$\mathbb{Z}/24 \oplus \mathbb{Z}/2$	$\mathbb{Z} \oplus (\mathbb{Z}/2)^2$	$\mathbb{Z}/480 \oplus (\mathbb{Z}/2)^3$

Localization and Descent for Algebraic K-Theory

- Seek a conceptual understanding of these calculational results on $K(A)_p$ for $A = S$.
- Can we recover $K(A)_p$ from $K(B)_p$ for suitably local symmetric ring spectra B?
- Can we descend to $K(B)_p$ from $K(C)_p$ for appropriate extensions $B \to C$?
- Is there a simple description of $K(\Omega)_p$ for sufficiently large such extensions $B \to \Omega$?

Algebraic K-Theory of Topological K-Theory

- Adams summand $A = \ell_p$ of ku_p, with $\pi_* \ell_p = \mathbb{Z}_p[v_1]$.
- Localization $B = L_p$, with $\pi_* L_p = \mathbb{Z}_p[v_1^{\pm 1}]$.

$$\begin{array}{ccccccc}
 & & L_p & \longrightarrow & KU_p & & \\
 & & \uparrow & & \uparrow & & \\
S_p & \longrightarrow & \ell_p & \xrightarrow{\phi} & ku_p & \longrightarrow & H\mathbb{Z}_p
\end{array}$$

Theorem (Blumberg-Mandell)

Homotopy cofiber sequence

$$K(\ell_p) \to K(L_p) \to \Sigma K(\mathbb{Z}_p)\,.$$

Chromatic Redshift

For $p \geq 5$, the type 2 Smith–Toda complex

$$V(1) = S \cup_p e^1 \cup_{\alpha_1} e^{2p-1} \cup_p e^{2p}$$

is a ring spectrum up to homotopy, with $v_2 \in \pi_{2p^2-2} V(1)$.

Theorem (Ausoni–R.)

$$V(1)_* K(\ell_p) \quad \textit{and} \quad V(1)_* K(L_p)$$

are finitely generated free $\mathbb{F}_p[v_2]$*-modules, each on* $4p + 4$ *generators, up to small error terms.*

Lichtenbaum–Quillen Conjecture

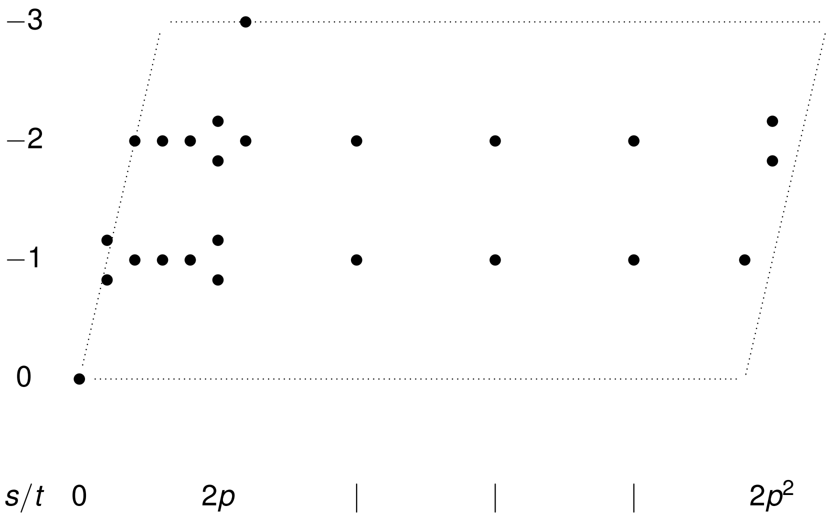
Suggests that $K(\Omega)_p$ is a connective form of the Lubin–Tate spectrum E_2, with $\pi_* E_2 = \mathbb{W}\mathbb{F}_{p^2}[[u_1]][u^{\pm 1}]$ and $V(1)_* E_2 = \mathbb{F}_{p^2}[u^{\pm 1}]$.

Conjecture (R.)

For purely v_1-periodic commutative symmetric ring spectra B there is a spectral sequence

$$E^2_{s,t} = H^{-s}_{\text{mot}}(B; \mathbb{F}_{p^2}(t/2)) \Longrightarrow V(1)_{s+t} K(B)\,.$$

E^2-Term for $V(1)_*K(L_p)$, $p = 5$



Beilinson–Lichtenbaum Conjecture

- Set
$$H^r_{\mathrm{et}}(L_p; \mathbb{F}_{p^2}(*)) = v_2^{-1} H^r_{\mathrm{mot}}(L_p; \mathbb{F}_{p^2}(*)) .$$
- Observe motivic truncation:
$$H^r_{\mathrm{mot}}(L_p; \mathbb{F}_{p^2}(m)) \cong \begin{cases} H^r_{\mathrm{et}}(L_p; \mathbb{F}_{p^2}(m)) & \text{for } 0 \le r \le m, \\ 0 & \text{otherwise.} \end{cases}$$

Tate–Poitou Duality

Symmetry about $(s, t) = (-3/2, p+1)$ similar to arithmetic duality.

Conjecture (R.)

For finite extensions B of L_p there is a perfect pairing

$$H^r_{\mathrm{et}}(B; \mathbb{F}_{p^2}(m)) \otimes H^{3-r}_{\mathrm{et}}(B; \mathbb{F}_{p^2}(p+1-m)) \overset{\cup}{\to} H^3_{\mathrm{et}}(B; \mathbb{F}_{p^2}(p+1)) \cong \mathbb{Z}/p$$

for each r and m.

Outline

Logarithmic Geometry

- Seek to realize more of motivic cohomology as Galois cohomology.
- Difficult to classify/construct *ramified* extensions $B \to C$ by obstruction theory.
- Tamely ramified extensions behave as unramified when rigidified by *logarithmic structures*.

Logarithmic Rings (Fontaine–Illusie, Kato)

- A pre-log ring consists of
  - a commutative ring R;
  - a commutative monoid M;
  - a monoid homomorphism $\alpha\colon M \to (R, \cdot)$.
- *Log ring* if $\alpha^{-1}GL_1(R) \to GL_1(R)$ is an isomorphism.
- *Trivial log structure* on R has $M = GL_1(R)$.
- Localization $R \to R[M^{-1}]$ factors in log rings as

$$R \to (R, M) \to R[M^{-1}] .$$

$\mathcal{J}$-spaces (R., Sagave, Schlichtkrull)

- The "underlying graded space" of a symmetric spectrum A is a $\mathcal{J}$-shaped diagram of spaces

$$\Omega^{\mathcal{J}}(A)\colon (\mathbf{n}_1, \mathbf{n}_2) \mapsto \Omega^{n_2} A_{n_1}$$

- Indexing category $\mathcal{J}$ is isomorphic to Quillen's construction $\Sigma^{-1}\Sigma$, with $B\mathcal{J} \simeq QS^0$.
- Homotopy type of a $\mathcal{J}$-space $X\colon \mathcal{J} \to \mathcal{S}$ is detected by $X_{h\mathcal{J}} = \text{hocolim}_{\mathcal{J}} X$. Positive projective model structure.
- Convolution product $X \boxtimes Y$ maps to smash product under $S^{\mathcal{J}}[-]\colon \mathcal{S}^{\mathcal{J}} \to Sp^{\Sigma}$, Quillen adjoint to $\Omega^{\mathcal{J}}(-)\colon Sp^{\Sigma} \to \mathcal{S}^{\mathcal{J}}$.

Topological logarithmic structures

Definition

A pre-log ring spectrum consists of

- a commutative symmetric ring spectrum A;
- a commutative $\mathcal{J}$-space monoid M;
- a commutative $\mathcal{J}$-space monoid map $\alpha \colon M \to \Omega^{\mathcal{J}}(A)$.

- *Log ring spectrum* if $\alpha^{-1} GL_1^{\mathcal{J}}(A) \to GL_1^{\mathcal{J}}(A)$ is $\mathcal{J}$-equivalence.
- *Trivial log structure* on A has $M = GL_1^{\mathcal{J}}(A) \subset \Omega^{\mathcal{J}}(A)$.
- Localization $A \to A[M^{-1}] = A \wedge_{S^{\mathcal{J}}[M]} S^{\mathcal{J}}[M^{gp}]$ factors as

$$A \to (A, M) \to A[M^{-1}] \,.$$

The replete bar construction

- The *group completion* $\eta\colon M \to M^{gp}$ makes $(M^{gp})_{h\mathcal{J}}$ a group completion of the E_∞ space $M_{h\mathcal{J}}$.
- The *cyclic bar construction* $B^{cy}(M)$ is the usual simplicial object $[q] \mapsto M \boxtimes M \boxtimes \cdots \boxtimes M$.
- The replete bar construction is a homotopy pullback

$$\begin{array}{ccccc} B^{cy}(M) & \xrightarrow{\rho} & B^{rep}(M) & \longrightarrow & B^{cy}(M^{gp}) \\ \downarrow & & \downarrow & & \downarrow \\ M & \xrightarrow{=} & M & \longrightarrow & M^{gp} \end{array}$$

- Repletion in topology plays the role of working with fine and saturated log structures in algebra.

Logarithmic Topological Hochschild Homology

Definition

Log THH of a pre-log ring spectrum (A, M, α) is the pushout

$$\begin{array}{ccc} S^{\mathcal{J}}[B^{cy}(M)] & \xrightarrow{\rho} & S^{\mathcal{J}}[B^{rep}(M)] \\ \downarrow & & \downarrow \\ THH(A) & \xrightarrow{\rho} & THH(A, M) \end{array}$$

of cyclic commutative symmetric ring spectra.

Log Étale Extensions

- $(A, M) \to (B, N)$ is *formally log étale* if $B \wedge_A THH(A, M) \simeq THH(B, N)$.
- The *direct image log structure* of (B, N) along $j \colon A \to B$ is $j_* N = \Omega^{\mathcal{J}}(A) \times_{\Omega^{\mathcal{J}}(B)} N$.

Theorem (R.–Sagave–Schlichtkrull)

$$\phi \colon (\ell_p, j_* GL_1^{\mathcal{J}}(L_p)) \to (ku_p, j_* GL_1^{\mathcal{J}}(KU_p))$$

is log étale.

Localization Sequences

Theorem (R.–Sagave–Schlichtkrull)

Let E be a d-periodic commutative symmetric ring spectrum, with connective cover $j\colon e \to E$. Homotopy cofiber sequence

$$THH(e) \xrightarrow{\rho} THH(e, j_* GL_1^{\mathcal{J}}(E)) \xrightarrow{\partial} \Sigma THH(e[0,d))$$

where $e[0,d)$ is the $(d-1)$-th Postnikov section of e.

Example

Homotopy cofiber sequence

$$THH(\ell_p) \to THH(\ell_p, j_* GL_1^{\mathcal{J}}(L_p)) \to \Sigma THH(\mathbb{Z}_p) .$$

Future Work

- Develop log TC, with a cyclotomic trace map from log K-theory, related to $K(A[M^{-1}])$.
- Develop log obstruction theory to realize tamely ramified extensions $A \to B$ as part of log étale extensions $(A, M) \to (B, N)$.