ALGEBRAIC K-THEORY AND STABLE SMOOTH PSEUDOISOTOPY OF DISCS (SHEFFIELD TALK)

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Pseudoisotopies and h-cobordisms

Let me begin with the motivating geometric topology.

Let M be a compact, connected n-manifold, in the smooth, piecewise-linear or topological category.

An h-cobordism from M to another manifold M' is a compact $(n+1)$-manifold W, with $\partial W = M \cup M'$, such that the inclusions $M \to W$, $M' \to W$ are homotopy equivalences.

To such a pair (W, M) one may associate a Whitehead torsion element $\tau(W, M) \in \mathrm{Wh}(\pi)$, where $\pi = \pi_1(M)$ and $\mathrm{Wh}(\pi) = K_1(\mathbb{Z}[\pi])/\pm\pi$ is the Whitehead group. The h-cobordism is trivializable, i.e., there exists an isomorphism

$$W \cong M \times I$$

relative to $M \cong M \times 0$, if and only if $\tau(W, M) = 0$.

Two such trivializations differ by a pseudoisotopy, alias a concordance, which is an automorphism of $M \times I$ relative to $M \times 0$. These form a topological group

$$P(M) = \mathrm{Aut}(M \times I \quad \mathrm{rel}\, M \times 0)$$

called the pseudoisotopy space of M.

A pseudoisotopy $\psi\colon M \times I \xrightarrow{\cong} M \times I$ that can be written on the form $\psi(x, t) = (\phi_t(x), t)$ amounts to a family ϕ_t of diffeomorphisms of M starting with the identity, i.e., an isotopy. In this sense pseudoisotopies generalize isotopies by not necessarily respecting the projection to the I-factor.

We may also consider the space of h-cobordisms from M. Each h-cobordism (W, M) may be embedded in a cylinder $(M \times I, M \times 0)$, for if (W, M) is composed with an h-cobordism (W', M') with $\tau(W', M') + \tau(W, M) = 0$ then $W \cup W'$ is a trivializable h-cobordism, isomorphic to $M \times I$. Hence we may consider the h-cobordism space $H(M)$ of codimension 0 submanifolds $W \subset M \times I$ that are h-cobordisms from $M = M \times 0$.

In the smooth case, Whitehead torsion identifies the set of path components of $H(M)$ with the Whitehead group. The spaces $P(M)$ and $H(M)$ are closely related; see e.g. [Waldhausen].

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Lemma. *$H(M)$ is a delooping of $P(M)$, so $\Omega H(M) \simeq P(M)$.*

Proof. To see this, recall that a collar of M in $M\times I$ is an embedding $M\times I \to M\times I$ relative to M. Its image $W \subset M \times I$ is a trivial h-cobordism of M, and the collar provides a choice of trivialization. Let $C(M)$ be the space of collars. Then $P(M)$ acts freely on $C(M)$ by reparametrizing the collaring, and there is a principal fibration

$$P(M) \to C(M) \to C(M)/P(M) \subset H(M).$$

The orbit space $C(M)/P(M)$ consists of the collars with their parameterization forgotten, i.e., the space of trivial h-cobordisms. In this way it is a subspace of the h-cobordism space $H(M)$. Now the collar space is contractible ("choosing a collar is a contractible choice"), and the lemma follows. □

The pseudoisotopy space is also related to the automorphism space $\mathrm{Aut}(M)$ of diffeomorphisms or (PL-)homeomorphisms of M. For by restriction a pseudoisotopy $\psi\colon M\times I \to M\times I$ determines a diffeomorphism ψ_1 of the upper end $M\times 1$. There results a fiber sequence

$$\mathrm{Aut}(M\times I) \to P(M) \xrightarrow{r} \mathrm{Aut}(M)$$

with $r(\psi) = \psi_1$; see e.g. [Weiss and Williams] and the Hatcher spectral sequence.

The eventual aim is to understand these spaces of diffeomorphisms, pseudoisotopies and h-cobordisms for smooth manifolds M, and similarly in the topological category. This is the study of symmetries of geometric objects.

Waldhausen's A-theory

For now, we can only do this is a stable range. There are stabilization maps $P(M) \to P(M \times I)$ and $H(M) \to H(M \times I)$, which induce isomorphisms on homotopy groups in a range of degrees that grows linearly with the dimension of M. This is the pseudoisotopy stable range; see [Igusa]. In the limit we form the stable pseudoisotopy space $\mathcal{P}(M) = \mathrm{colim}_m\, P(M\times I^m)$ and the stable h-cobordism space $\mathcal{H}(M) = \mathrm{colim}_m\, H(M\times I^m)$.

Remarkably, these extend to homotopy functors $X \mapsto \mathcal{P}(X)$ and $X \mapsto \mathcal{H}(X)$ that take values in infinite loop spaces and maps. Here X can be any space. (See [Waldhausen].) There is a Whitehead space functor $\mathrm{Wh}(X)$ (in the smooth, PL and topological categories) from spaces to infinite loop spaces, such that for manifolds $\Omega\,\mathrm{Wh}(M) \simeq \mathcal{H}(M)$ and $\Omega^2\,\mathrm{Wh}(M) \simeq \mathcal{P}(M)$. In particular $\mathrm{Wh}(M)$ is connected when M is connected, and $\pi_1\,\mathrm{Wh}(M) \cong \mathrm{Wh}(\pi_1 M)$.

The construction of the Whitehead space $\mathrm{Wh}(X)$ is related to Waldhausen's construction of the space $A(X)$, representing his algebraic K-theory of spaces, alias A-theory. We may use the following definition for X based and path connected:

$$A(X) = \Omega B(\coprod_{k\geq 0} \mathrm{colim}_n\, BH^n_k(G))$$

Here ΩB group completes, $G = G(X)$ is the Kan loop group of X, $H^n_k(G)$ is the monoid of G-equivariant self-homotopy equivalences of $\bigvee^k (G_+ \wedge S^n)$, and B denotes the bar construction. (Think of $X = *$ and $G = 1$ as a simplest case.)

Theorem (Waldhausen). *(a) There is a natural splitting of infinite loop spaces*

$$A(X) \simeq Q(X_+) \times \mathrm{Wh}^{\mathrm{Diff}}(X).$$

(b) There is a natural fiber sequence of infinite loop spaces

$$\Omega^\infty(A(*) \wedge X_+) \to A(X) \to \mathrm{Wh}^{\mathrm{PL}}(X).$$

In (b) we use the spectrum smash product of the spectrum $A(*)$ with the space X_+. An older notation would be $h(X; A(*))$.

Hence, for understanding the smooth or PL Whitehead spaces, and thus the smooth or PL pseudoisotopy and h-cobordism spaces in the stable range, it suffices to study the spectra $A(X)$, with $A(*)$ having a prominent role. Note that $A(*) \simeq Q(S^0) \times \mathrm{Wh}^{\mathrm{Diff}}(*)$ relates to pseudoisotopies and h-cobordisms of discs, in a range of degrees increasing to infinity with the dimension of the disc.

The linearization map

A-theory is related to algebraic K-theory. For a ring R, the K-theory space is

$$K(R) = \Omega B(\coprod_{k \geq 0} BGL_k(R)).$$

The nth homology of $\bigvee^k (G_+ \wedge S^n)$ is R^k with $R = \mathbb{Z}[\pi_0 G]$, so a G-equivariant self-homotopy equivalence of this space induces an invertible R-homomorphism $R^k \to R^k$, with associated matrix in $GL_k(R)$. Hence taking a map to its homology class defines a linearization map

$$L\colon A(X) \to K(R)$$

where $R = \mathbb{Z}[\pi_1 X]$. This map is a rational equivalence, and was used to obtain rational information about the smooth Whitehead space; see e.g. [Farell and Hsiang].

Topological cyclic homology

Developments in the last two years now enable us to obtain 2-torsion information about these spaces.

The topological cyclic homology of Bökstedt, Hsiang and Madsen specializes to define spectra $TC(X)$ and $TC(R)$, for spaces X and rings R. There is a cyclotomic trace map $\mathrm{trc}_X\colon A(X) \to TC(X)$ and $\mathrm{trc}_R\colon K(R) \to TC(R)$. In Waldhausen's first paper on A-theory, he determined the first nonzero homotopy group of the fiber of $L\colon A(X) \to K(R)$ using Hochschild homology. The topological cyclic refinement of Hochschild homology gives a perfect relative invariant.

Theorem (B. Dundas). *Let X be a space and $R = \mathbb{Z}[\pi_1 X]$. The square*

$$\begin{array}{ccc} A(X) & \xrightarrow{L} & K(R) \\ \downarrow{\scriptstyle \mathrm{trc}_X} & & \downarrow{\scriptstyle \mathrm{trc}_R} \\ TC(X) & \xrightarrow{L} & TC(R) \end{array}$$

is homotopy Cartesian.

Hence determining $TC(X)$, $K(R)$ and $TC(R)$ and the maps between them suffices to determine $A(X)$. We will specialize to the case $X = *$, when $\pi = 1$ and $R = \mathbb{Z}$, since determining $A(*)$ is of interest in itself.

The topological cyclic homology $TC(X)$ has a homotopy-theoretical description for each space X. When $X = *$ it specializes as follows:

Theorem (Bökstedt, Hsiang and Madsen).

$$TC(*) \simeq Q(S^0) \times BQ(\mathbb{C}P^\infty_{-1}).$$

Here $\mathbb{C}P^\infty_{-1}$ is the Thom spectrum of the negative of the canonical complex line bundle over $\mathbb{C}P^\infty$, and $Q(\mathbb{C}P^\infty_{-1})$ denotes its underlying space. We can think of $\mathbb{C}P^\infty_{-1}$ as the suspension spectrum on $\mathbb{C}P^\infty_+$ with an extra cell in complex dimension -1, i.e., real dimension -2. There is a fiber sequence

$$BQ(\mathbb{C}P^\infty_{-1}) \to Q(\Sigma(\mathbb{C}P^\infty_+)) \xrightarrow{\mathrm{trf}_{S^1}} Q(S^0)$$

where trf_{S^1} is the S^1-transfer map.

In more detail, recall that the Thom complex of k times the canonical complex line bundle over $\mathbb{C}P^n$ is homeomorphic to the stunted projective space $\mathbb{C}P^{n+k}_k = \mathbb{C}P^{n+k}/\mathbb{C}P^{k-1}$; see [Atiyah]. The stable homotopy type of these spaces is periodic in k, with period depending on n. This is James periodicity. Hence, when working with spectra one may formally desuspend, and consider spectra $\mathbb{C}P^{n+k}_k$ for arbitrary integers k. Then forming unions over n one forms e.g. $\mathbb{C}P^\infty_{-1}$. We shall return to the homotopy of this space below.

Hence we can split off a factor $Q(S^0)$ from both $A(*)$ and $TC(*)$, and obtain the following two homotopy Cartesian squares:

$$\begin{array}{ccccc}
\mathrm{Wh}^{\mathrm{Diff}}(*) & \longrightarrow & A(*) & \xrightarrow{L} & K(\mathbb{Z}) \\
\downarrow & & \downarrow{\scriptstyle \mathrm{trc}_*} & & \downarrow{\scriptstyle \mathrm{trc}_{\mathbb{Z}}} \\
BQ(\mathbb{C}P^\infty_{-1}) & \longrightarrow & TC(*) & \xrightarrow{L} & TC(\mathbb{Z}).
\end{array}$$

We will use the resulting fiber sequence

$$\Omega\,\mathrm{Wh}^{\mathrm{Diff}}(*) \to Q(\mathbb{C}P^\infty_{-1}) \xrightarrow{\Delta} \mathrm{hofib}(\mathrm{trc}_{\mathbb{Z}})$$

to describe the stable smooth h-cobordism space $\mathcal{H}(*) = \Omega\,\mathrm{Wh}^{\mathrm{Diff}}(*)$, which deloops the stable smooth pseudoisotopy space $\mathcal{P}(*)$. These are approximated by $H(D^k)$ and $P(D^k)$.

Linear calculations

Now the 2-primary algebraic K-theory of the integers can be determined as a consequence of Voevodsky's proof of the Milnor conjecture. (See [Weibel], and the fix in [Rognes and Weibel].) One way to formulate the result follows:

Theorem (Voevodsky, Weibel, etc.). *There is a fiber sequence*

$$K(\mathbb{Z}) \to \mathbb{Z} \times BO \xrightarrow{c(\psi^3-1)} BSU$$

after 2-adic completion. Here $\psi^3 - 1\colon \mathbb{Z} \times BO \to BSpin$ *is the Adams operation, and* $c\colon BSpin \to BSU$ *is complexification. Hence* $K_0(\mathbb{Z}) \cong \mathbb{Z}$*,* $K_1(\mathbb{Z}) \cong \mathbb{Z}/2$*, and for* $n \geq 2$

$$K_n(\mathbb{Z}) \cong \begin{cases} 0 & \textit{for } n \equiv 0, 4, 6 \bmod 8, \\ \mathbb{Z}/2 \oplus \mathbb{Z} & \textit{for } n \equiv 1 \bmod 8, \\ \mathbb{Z}/2 & \textit{for } n \equiv 2 \bmod 8, \\ \mathbb{Z}/16 & \textit{for } n \equiv 3 \bmod 8, \\ \mathbb{Z} & \textit{for } n \equiv 5 \bmod 8, \\ \mathbb{Z}/2^{v_2(k)+4} & \textit{for } n = 8k-1 \end{cases}$$

modulo odd finite groups.

Here $v_2(k)$ is the 2-adic valuation of k, i.e., the exponent of the exact power of 2 dividing k. No corresponding theorem about the odd torsion in $K(\mathbb{Z})$ is available, but there are detailed conjectures.

The 2-completed topological cyclic homology of the integers, $TC(\mathbb{Z}; 2)$, was determined by the author, using the methods of [Bökstedt and Madsen]. (In fact, parts of that calculation are essentially used in the computation of $K(\mathbb{Z})$ reviewed above.)

Theorem (Rognes). *The homotopy type of* $TC(\mathbb{Z}; 2)$ *is known. Its homotopy groups are*

$$TC_n(\mathbb{Z}; 2) \cong \begin{cases} \mathbb{Z} & \textit{for } n = -1, 0, \\ \mathbb{Z}/2 \oplus \mathbb{Z} & \textit{for } n \equiv 1, 5 \bmod 8, \\ \mathbb{Z}/2 & \textit{for } n \equiv 2, 6 \bmod 8, \\ \mathbb{Z}/8 \oplus \mathbb{Z} & \textit{for } n \equiv 3 \bmod 8, \\ \mathbb{Z}/8 & \textit{for } n \equiv 4 \bmod 8, \\ \mathbb{Z}/2^{v_2(k)+4} \oplus \mathbb{Z} & \textit{for } n = 8k-1, \\ \mathbb{Z}/2^{v_2(k)+4} & \textit{for } n = 8k. \end{cases}$$

after 2-adic completion.

In nonnegative degrees this agrees with the algebraic K-theory of the 2-adic integers $\mathbb{Z}_2$.

A key result is that both $K(\mathbb{Z})$ and $TC(\mathbb{Z}; 2)$ agree with their Bousfield K-localizations above degree 1. Hence the Adams v_1^4-action on their mod 2 homotopy is periodic above degree 1. Thus this also applies to the homotopy fiber $\mathrm{hofib}(\mathrm{trc}_{\mathbb{Z}})$ of the cyclotomic trace map linking these spectra.

To summarize, we have homotopical descriptions of 3 of the 4 corners of Dundas' homotopy Cartesian square, and hence we have a description of $A(*)$ and $\Omega\, \mathrm{Wh}^{\mathrm{Diff}}(*)$. It remains to make this identification explicit, and to interpret the results.

Stable homotopy of Thom spectra

We implicitly complete all calculations at 2.

The stable homotopy of $\mathbb{C}P^\infty$ was studied by R. Mosher, and later by J. Mukai, who also studied the S^1-transfer map. By combining and extending their calculations I have obtained the homotopy groups $\pi_n Q(\mathbb{C}P^\infty_{-1})$ for $n \le 20$, with some unresolved extension for $15 \le n \le 20$.

Theorem. *The homotopy groups of the Thom spectrum* $\mathbb{C}P^\infty_{-1}$ *begin as follows:*

n	-2	-1	0	1	2	3	4	5	6	7	8
$\pi_n Q(\mathbb{C}P^\infty_{-1})$	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	$\mathbb{Z}/8$	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2 \oplus \mathbb{Z}$	$\mathbb{Z}/16$	$\mathbb{Z}/2 \oplus \mathbb{Z}$

n	9	10	11	12	13	14
$\pi_n Q(\mathbb{C}P^\infty_{-1})$	$(\mathbb{Z}/2)^2 \oplus \mathbb{Z}/8$	$\mathbb{Z}$	$\mathbb{Z}/8 \oplus \mathbb{Z}/2$	$\mathbb{Z}$	$\mathbb{Z}/2 \oplus \mathbb{Z}/4$	$(\mathbb{Z}/2)^2 \oplus \mathbb{Z}$

The proof uses the Atiyah–Hirzebruch spectral sequence

$$E^2_{s,t} = H_s(\mathbb{C}P^\infty_{-1}; \pi_t Q(S^0)) \cong \begin{cases} \pi_t Q(S^0) & \text{for } s \ge -2 \text{ even,} \\ 0 & \text{otherwise} \end{cases}$$
$$\Longrightarrow \pi_{s+t} Q(\mathbb{C}P^\infty_{-1})$$

for stable homotopy viewed as a generalized homology theory, applied to the spectrum $\mathbb{C}P^\infty_{-1}$. This extends the spectral sequence for the stable homotopy of $\mathbb{C}P^\infty$ considered by Mosher, by adjoining the two columns $s = -2$ and $s = 0$. In Mosher's case, the nonzero d^2-, d^4- and d^8-differentials are given in terms of multiplication by η, ν or σ in the stable stems, while the d^6-differentials are expressed in terms of Toda brackets. By James periodicity his formulas for these differentials also extend into nonpositive homological degrees as in the spectral sequence above. The higher differentials are found by comparing with Mukai's calculations using composition methods, a la Toda.

The fiber of the cyclotomic trace map

We can compute the cyclotomic trace map $\mathrm{trc}_\mathbb{Z} \colon K(\mathbb{Z}) \to TC(\mathbb{Z})$ at 2. It is mostly injective on homotopy, except in degrees $8k+3$ where the image of $\eta^2 \mu_{8k+1} \in \pi_* Q(S^0)$ maps to zero.

Theorem. *The homotopy groups of the homotopy fiber of* $\mathrm{trc}_\mathbb{Z}$ *begin as follows:*

n	-2	-1	0	1	2	3	4	5	6	7	8
$\pi_n \operatorname{hofib}(\mathrm{trc}_\mathbb{Z})$	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	$\mathbb{Z}/16$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}$	$\mathbb{Z}/16$	0

n	9	10	11	12	13	14
$\pi_n \operatorname{hofib}(\mathrm{trc}_\mathbb{Z})$	0	$\mathbb{Z}$	$\mathbb{Z}/16$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}$

We can also describe $\operatorname{hofib}(\mathrm{trc}_\mathbb{Z})$ as a spectrum, with a proviso. There is a reduction map $K(\mathbb{Z}_2) \to K(\mathbb{F}_3)$ and a Brauer lift $K(\mathbb{F}_3) \to K(\mathbb{C})$, such that the composite is homotopic to the map induced by a complex embedding of $\mathbb{Z}_2$. Let us assume that this reduction map can be chosen so that the composite $K(\mathbb{Z}) \to K(\mathbb{Z}_2) \to K(\mathbb{F}_3)$ is homotopic to the map induced by the ring homomorphism $\mathbb{Z} \to \mathbb{F}_3$. (The maps are known to agree on the level of homotopy groups.)

Then there is a fiber sequence of infinite loop spaces

$$B(\mathbb{Z} \times BO) \to K(\mathbb{F}_3) \to \mathrm{hofib}(\mathrm{trc}_{\mathbb{Z}})$$

where the left map is the composite

$$B(\mathbb{Z} \times BO) \xrightarrow{c} B(\mathbb{Z} \times BU) \underset{\simeq}{\xrightarrow{\beta}} U \xrightarrow{\partial} K(\mathbb{F}_3)$$

of complexification, Bott periodicity and the connecting map from Quillen's identification of $K(\mathbb{F}_3)$ with the homotopy fiber of $\psi^3 - 1\colon \mathbb{Z} \times BU \to BU$.

Homotopy of the stable smooth h-cobordism space

We know that $A(*)$ and $K(\mathbb{Z})$ are connective, and that the linearization map $L\colon A(*) \to K(\mathbb{Z})$ is (at least) 2-connected. Hence $\mathrm{Wh}^{\mathrm{Diff}}(*)$ is 1-connected, and so the map Δ is also 1-connected.

$$\Omega\, \mathrm{Wh}^{\mathrm{Diff}}(*) \to Q(\mathbb{C}P^\infty_{-1}) \xrightarrow{\Delta} \mathrm{hofib}(\mathrm{trc}_{\mathbb{Z}})$$

Thus $\pi_*(\Delta)$ induces an isomorphism in degrees -2 and 0. By keeping track of explicit generators, and using the action of the stable stems $\pi_* Q(S^0)$ on the homotopy of infinite loop spaces, I have computed the map $\pi_*(\Delta)$ for $* \le 12$, with some ambiguity in degree 13. This yields:

Theorem. *The homotopy groups of $\mathcal{H}(*) = \Omega\, \mathrm{Wh}^{\mathrm{Diff}}(*)$ begin as follows:*

n	0	1	2	3	4	5	6	7	8
$\pi_n\mathcal{H}(*)$	0	0	$\mathbb{Z}/2$	0	$\mathbb{Z}$	0	$\mathbb{Z}/2$	0	$\mathbb{Z}/2 \oplus \mathbb{Z}$
gen.			$\partial(\eta^3)$		$2\zeta_2$		$\nu^2\zeta_0$		$\nu^2\zeta_1, \zeta_4$

n	9	10	11	12
$\pi_n\mathcal{H}(*)$	$(\mathbb{Z}/2)^2 \oplus \mathbb{Z}/8$	$\mathbb{Z}/2$	$\mathbb{Z}/4$	$\mathbb{Z}$
gen.	$\eta^2\sigma\zeta_0, \eta\bar{\nu}\zeta_0, \sigma\zeta_1$	$\partial(\eta^2\nu)$	$\sigma\zeta_2$	$2\zeta_6$

The classes labelled $\partial(x)$ are in the image of the connecting map from $\mathrm{hofib}(\mathrm{trc}_{\mathbb{Z}})$. The composite map

$$\mathcal{H}(*) = \Omega\, \mathrm{Wh}^{\mathrm{Diff}}(*) \to Q(\mathbb{C}P^\infty_{-1}) \to Q(\mathbb{C}P^\infty_{+}) \xrightarrow{\bar{R}} \mathbb{Z} \times BU$$

takes the classes labelled $\alpha\zeta_n$ to $\alpha \in \pi_* Q(S^0)$ times the generator $\zeta_n \in \pi_{2n} Q(\mathbb{C}P^\infty_+)$ that maps to a generator of $\pi_{2n}(\mathbb{Z} \times BU)$, for $n \ge 0$. Here the rotation map $\bar{R}$ is adjoint to the embedding $R\colon \Sigma\mathbb{C}P^\infty_+ \to U$ taking (L, z) to rotation by $z \in S^1$ about the line $L \subset \mathbb{C}^\infty$.

Space level structure of $\Omega\,\mathrm{Wh}^{\mathrm{Diff}}(*)$

It remains to determine the space or spectrum level structure of $\Omega\,\mathrm{Wh}^{\mathrm{Diff}}(*)$. There is a Hatcher–Waldhausen map

$$hw\colon G/O \to \Omega\,\mathrm{Wh}^{\mathrm{Diff}}_{\otimes}(*)$$

that is a π_2-isomorphism and a rational equivalence. There is also a spectrum map

$$\Omega\,\mathrm{Wh}^{\mathrm{Diff}}_{\otimes}(*) \to BSO_{\otimes}$$

induced by the linearization $L\colon A(*) \to K(\mathbb{Z})$, and the composite $G/O \to BSO_{\otimes}$ factors as the natural map $G/O \to BSO$ followed by the 2-primary cannibalistic equivalence $\rho^3\colon BSO \to BSO_{\otimes}$, followed by some self-map of $BSO_{\otimes}$. I expect the latter self map to be an equivalence, but this remains to be proven. We have the following diagram of horizontal fiber sequences:

$$\begin{array}{ccccccc}
SG & \longrightarrow & G/O & \longrightarrow & BSO & \xrightarrow{Bj} & BSG \\
\downarrow & & \downarrow{\scriptstyle hw} & & \downarrow{\scriptstyle j\eta} & & \downarrow{\scriptstyle w} \\
\Omega A(*)_1 & \longrightarrow & \Omega\,\mathrm{Wh}^{\mathrm{Diff}}_{\otimes}(*) & \xrightarrow{*} & SG & \longrightarrow & A(*)_1 \\
\downarrow{\scriptstyle \Omega L} & & \downarrow & & \downarrow{\scriptstyle e_{\otimes}} & & \downarrow{\scriptstyle L} \\
\Omega K(\mathbb{Z})_1 & \longrightarrow & BSO_{\otimes} & \xrightarrow{\zeta_{\otimes}\eta} & SJ_{\otimes} & \longrightarrow & K(\mathbb{Z})_1
\end{array}$$

All that needs to be done is to relate the map $\Omega\,\mathrm{Wh}^{\mathrm{Diff}}(*) \to BSO$ to the cyclotomic trace map to $Q(\mathbb{C}P^{\infty}_{-1}) \to \mathbb{Z} \times BU$, by which the calculation above was expressed.

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