ELLIPTIC COHOMOLOGY AND ALGEBRAIC K-THEORY OF 2-VECTOR SPACES

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§1. Chromatic notation

Let p be a prime à la Euclid.

Connective p-completed Johnson–Wilson spectrum $BP\langle n\rangle_p$ with

$$\pi_* BP\langle n\rangle_p = \mathbb{Z}_p[v_m \mid m \geq 1]/(v_m \mid m > n)\,.$$

Examples: $BP\langle -1\rangle_p = H\mathbb{F}_p$, $BP\langle 0\rangle_p = H\mathbb{Z}_p$, $BP\langle 1\rangle_p = \ell_p$ Adams summand of connective p-complete K-theory ku_p. These three are commutative S-algebras.

v_n-periodic Johnson–Wilson spectrum $E(n) = v_n^{-1}BP\langle n\rangle$ detects chromatic filtration $\leq n$ phenomena.

Smith–Toda spectrum $V(n)$ with

$$BP_* V(n) = BP_*/(v_m \mid m \leq n)\,.$$

Examples: $V(-1) = S$, $V(0) = S/p$, and $V(1) = C_\alpha$ mapping cone of Adams map $\alpha\colon \Sigma^{2p-2} S/p \to S/p$, for $p \neq 2$.

§2. Topological cyclic homology

A a connective S-algebra, $TC(A)_p$ its p-completed topological cyclic homology.

Theorem (Hesselholt–Madsen). *$\pi_* TC(\mathbb{F}_p)_p$ is a free $\mathbb{Z}_p$-module on 2 generators.*

Theorem (Bökstedt–Madsen). *$V(0)_* TC(\mathbb{Z}_p)$ is a free $\mathbb{F}_p[v_1]$-module on $p+3$ generators, for $p \neq 2$.*

Theorem (Ausoni–Rognes). *$V(1)_* TC(\ell_p)$ is a free $\mathbb{F}_p[v_2]$-module on $4p+4$ generators, for $p \neq 2, 3$.*

If p and n are such that $BP\langle n\rangle_p$ is a commutative S-algebra and $V(n)$ is a ring spectrum, then $V(n)_* TC(BP\langle n\rangle_p)$ is a free $\mathbb{F}_p[v_{n+1}]$-module on

$$2^{n+2} + 2^n(n+1)(p-1)$$

generators.

Note that $V(n)_* BP\langle n+1\rangle_p = \mathbb{F}_p[v_{n+1}]$. This suggests that $TC(BP\langle n\rangle_p)_p$ is built as an extension of the above number of copies of $BP\langle n+1\rangle_p$.

Chromatic red-shift problem. *In what generality does TC take chromatic filtration n connective S-algebras to chromatic filtration $(n+1)$ S-modules ?*

§3. Algebraic K-theory

A an S-algebra, $K(A)$ its algebraic K-theory.

Cofiber sequence (Hesselholt and Madsen)

$$K(A)_p \to TC(A)_p \to \Sigma^{-1} H\mathbb{Z}_p$$

e.g. for A connective with $\pi_0(A) = \mathbb{F}_p$ or $\mathbb{Z}_p$.

$$K(\mathbb{F}_p)_p \simeq H\mathbb{Z}_p$$

and

$$K(\mathbb{Z}_p)_p \simeq j_p \vee \Sigma j_p \vee \Sigma^3 ku_p$$

for $p \neq 2$.

Homotopy type of $K(\ell_p)_p$ is not known, but $V(1)_* K(\ell_p)$ agrees in high degrees with a free $\mathbb{F}_p[v_2]$-module on $4p+4$ generators.

Composite

$$K(\ell_p) \xrightarrow{a_*} K(ku_p) \xrightarrow{a^!} K(\ell_p)$$

is multiplication by $(p-1)$, so $K(\ell_p)_p$ is a homotopy retract of $K(ku_p)_p$.

Suggests $V(1)_*K(ku_p)$ agrees in high degrees with a free finitely generated $\mathbb{F}_p[v_2]$-module.

§4. Topological K-theory

Homotopy Cartesian square

$$\begin{array}{ccc} K(ku)_p & \longrightarrow & K(ku_p)_p \\ \downarrow & & \downarrow \\ K(\mathbb{Z})_p & \longrightarrow & K(\mathbb{Z}_p)_p \end{array}$$

by Dundas.

$V(1)_*K(\mathbb{Z})$ and $V(1)_*K(\mathbb{Z}_p)$ are finite, assuming the Lichtenbaum–Quillen conjecture for $K(\mathbb{Z})$. Thus $V(1)_*K(ku)$ and $V(1)_*K(ku_p)$ agree in high degrees.

Suggests $K(ku)$ has p-completion built from finitely many copies of $BP\langle 2\rangle_p$, for each prime p.

Elliptic cohomology spectrum $\mathcal{E}\ell\ell$ of Landweber, Ravenel and Stong, with $\mathcal{E}\ell\ell_* = \mathbb{Z}[1/2, \delta, \epsilon][\Delta^{-1}]$ ($\Delta = \epsilon^2(\delta^2 - \epsilon)$) is built from finitely many copies of $E(2) = v_2^{-1}BP\langle 2\rangle$ for each $p \neq 2$ (with some adjustment for $p = 3$).

Think of $K(ku)$ as (a candidate for) an integrally defined, connective, commutative S-algebra model for an elliptic cohomology.

§5. 2-VECTOR SPACES

Obtain $K(ku)$ from the 2-category of 2-vector spaces of Kapranov and Voevodsky by group completing two distinct semi-group structures: form the 2-K-theory of 2-vector spaces.

Definition. Let $(\mathcal{V}, \oplus, \otimes)$ be the bipermutative category of finite dimensional complex vector spaces, under direct sum and tensor product.

$\mathcal{V}$ has objects the $\mathbb{C}^n$ for integers $n \geq 0$, and morphisms $\mathbb{C}^m \to \mathbb{C}^n$ the $n \times m$ complex matrices $M_{n,m}(\mathbb{C})$. The direct sum and tensor product are both strictly unital and associative, and coherently commutative. One distributive law holds strictly, the other up to coherent isomorphism.

Replacing complex numbers, sums and products in the definition of the complex vector space $\mathbb{C}^n$, by complex vector spaces, direct sums and tensor products, respectively, we obtain a definition of the 2-vector space $\mathcal{V}^n$. This is the product category of

n-tuples of complex vector spaces, with some extra structure.

Definition. Let $2-\mathcal{V}$ be the 2-category with objects the $\mathcal{V}^n$ for integers $n \geq 0$, 1-morphisms

$$A\colon \mathcal{V}^m \to \mathcal{V}^n$$

the $n \times m$ matrices $A = (V_{ij})$ with complex vector spaces (objects in $\mathcal{V}$) as entries, and 2-morphisms

$$\Phi\colon A \Rightarrow B$$

for $A = (V_{ij})$, $B = (W_{ij})$ the $n \times m$ matrices $\Phi = (\phi_{ij})$ of linear maps $\phi_{ij}\colon V_{ij} \to W_{ij}$.

§6. Isomorphism 2-groupoids

$i(2-\mathcal{V}) \subset 2-\mathcal{V}$ isomorphism 2-groupoid.

1-isomorphisms $A\colon \mathcal{V}^n \to \mathcal{V}^n$: square matrices with one entry $\mathbb{C}^1$ in each row and column, all other entries $\mathbb{C}^0$.

2-isomorphisms $\Phi\colon A \Rightarrow A$: square matrices with one linear isomorphism $\mathbb{C}^1 \to \mathbb{C}^1$ in each row and column, zero maps elsewhere.

$\mathcal{V}_1 \subset \mathcal{V}$ full subcategory of the 1-dimensional vector spaces. $i\mathcal{V}_1 \subset \mathcal{V}_1$ its isomorphism groupoid.

Automorphism groupoid of $\mathcal{V}^n$ is $GL_n(i\mathcal{V}) \cong \Sigma_n \wr i\mathcal{V}_1$, with nerve

$$|GL_n(i\mathcal{V})| = \Sigma_n \wr B\mathbb{C}^* \,.$$

The 2-nerve of the full 2-groupoid of $i(2-\mathcal{V})$ generated by $\mathcal{V} = \mathcal{V}^1$ is

$$B|GL_1(\mathcal{V})| = B|i\mathcal{V}_1| = BB\mathbb{C}^* \simeq K(\mathbb{Z}, 3) \,.$$

The 2-nerve of the whole 2-groupoid $i(2-\mathcal{V})$ is

$$|i(2-\mathcal{V})| = \coprod_{n\ge 0} B|GL_n(i\mathcal{V})| = \coprod_{n\ge 0} B(\Sigma_n \wr B\mathbb{C}^*) \,.$$

§7. Virtual vector spaces

The scarcity of 1-isomorphisms in $2-\mathcal{V}$ can be relieved by extending the coefficient category $\mathcal{V}$ of vector spaces to include virtual (negative-dimensional) vector spaces. Technically we group complete $\mathcal{V}$ with respect to the direct sum $\oplus$ of vector spaces by forming a Γ-category $K\mathcal{V}$ as defined by Segal or Shimada and Shimakawa.

Definition. Let Γ be the category of finite sets $p_+ = \{0, 1, \dots, p\}$ pointed at 0, and base-point preserving functions. A Γ-object X in a category $\mathcal{C}$ pointed at $*$ is a pointed functor $X \colon \Gamma \to \mathcal{C}$.

Definition. Let $K\mathcal{V}$ be the Γ-category with $K\mathcal{V}(p_+)$ the category of p-dimensional cubical diagrams

$$S \mapsto V_S \in \mathcal{V}$$

for $\emptyset \neq S \subseteq \{1, \dots, p\}$, with coherent isomorphisms $V_S \oplus V_T \cong V_{S \cup T}$ for $S \cap T = \emptyset$, and isomorphisms of such diagrams. Underlying category is $K\mathcal{V}(1_+) \cong i\mathcal{V}$.

The nerve of $K\mathcal{V}$ is a Γ-space $|K\mathcal{V}|$, whose associated spectrum $\{k \mapsto |K\mathcal{V}|(S^k)\}$ is a model for ku, i.e., connective topological K-theory.

The tensor product $\otimes$ on $\mathcal{V}$ lets us choose a transformation

$$K\mathcal{V}(p_+) \wedge K\mathcal{V}(q_+) \xrightarrow{\mu} K\mathcal{V}(p_+ \wedge q_+)$$

natural in p_+ and q_+. This defines a map

$$|K\mathcal{V}| \wedge |K\mathcal{V}| \xrightarrow{\mu} |K\mathcal{V}|$$

representing the ring spectrum product on ku. This model for the product is coherently, not strictly, associative.

The underlying infinite loop space of $|K\mathcal{V}|$ inherits a coherently associative product when defined as

$$\Omega^\infty_I |K\mathcal{V}| = \operatorname*{hocolim}_{k \in I} \Omega^k |K\mathcal{V}|(S^k) \simeq \mathbb{Z} \times BU$$

with I the category of finite sets and injective functions (Bökstedt).

§8. Virtual 2-vector spaces

Let

$$M'_n(|K\mathcal{V}|) = F(n_+, n_+ \wedge |K\mathcal{V}|) \subset M_n(|K\mathcal{V}|)$$

be the restricted $n \times n$ matrices, with at most one nonzero entry in each column. The inclusion is a stable equivalence:

$$\Omega^\infty_I M'_n(|K\mathcal{V}|) \simeq \Omega^\infty_I M_n(|K\mathcal{V}|) \simeq M_n(\mathbb{Z} \times BU)\,,$$

and the left hand term has a coherently associative product. Let

$$\widehat{GL}_n(|K\mathcal{V}|) \subset \Omega^\infty_I M'_n(|K\mathcal{V}|)$$

be the coherently associative grouplike monoid of homotopy invertible components.

The zeroth space inclusion

$$\coprod_{m\ge 0} BU(m) \simeq |i\mathcal{V}| \to \Omega^\infty_I |K\mathcal{V}| \simeq \mathbb{Z} \times BU$$

induces a map

$$\Sigma_n \wr B\mathbb{C}^* = |GL_n(i\mathcal{V})| \to \widehat{GL}_n(|K\mathcal{V}|)\,.$$

We view these as the nerves of the automorphism groupoids of $\mathcal{V}^n$ viewed as a 2-vector space, and of $(K\mathcal{V})^n$ viewed as a "virtual 2-vector space", respectively. The inclusion corresponds to relaxing the notion of a vector space (in $\mathcal{V}$) to a virtual vector space (in $K\mathcal{V}$) in the definition of a 2-vector space.

Taking classifying spaces for the (coherently associative) matrix products, we have the inclusion

$$\coprod_{n\geq 0} B(\Sigma_n \wr B\mathbb{C}^*) = |i(2-\mathcal{V})| \to \coprod_{n\geq 0} B\widehat{GL}_n(|K\mathcal{V}|)\,.$$

Applying group completion with respect to block sum of matrices we obtain the inclusion

$$\begin{aligned} Q(BB\mathbb{C}^*_+) &\simeq \Omega B(\coprod_{n\geq 0} B(\Sigma_n \wr B\mathbb{C}^*)) \\ &\to \Omega B(\coprod_{n\geq 0} B\widehat{GL}_n(|K\mathcal{V}|)) =: (2-K)(2-\mathcal{V})\,. \end{aligned}$$

The right hand side defines the 2-K-theory of the 2-category of 2-vector spaces. It is obtained from the 2-nerve of the 2-groupoid of 2-vector spaces by group completing both the direct sum of vector spaces

$$\oplus \colon \mathcal{V} \times \mathcal{V} \to \mathcal{V}$$

to obtain $K\mathcal{V}$, and group completing the block sum of matrices

$$B\widehat{GL}_n(-) \times B\widehat{GL}_m(-) \to B\widehat{GL}_{n+m}(-)\,.$$

The definition (Bökstedt, Hsiang and Madsen) of $K(ku)$ involves choosing an FSP model for ku, i.e., a stably equivalent Γ-space $X \simeq |K\mathcal{V}|$ with a strictly associative product. Then $\widehat{GL}_n(X) \simeq \widehat{GL}_n(K\mathcal{V})$ and clearly

$$K(ku) \simeq (2-K)(2-\mathcal{V})\,.$$

§9. Stacks

We interpret this 2-K-theory as the receptacle for an additive invariant of certain sheaves of Γ-categories that we call $(K\mathcal{V})^n$-bundles, providing a "geometric" interpretation of 2-K-theory parallel to the interpretation of complex K-theory as the receptacle for an additive invariant of certain sheaves of sets, namely complex vector bundles.

A stack is a sheaf of categories, which has stalks that are categories rather than sets. A $\mathbb{C}^n$-bundle $E \downarrow X$ is in particular a locally trivial sheaf of sets with stalks isomorphic to $\mathbb{C}^n$. A $\mathcal{V}^n$-bundle $\mathcal{E} \downarrow X$ is in particular a locally trivial sheaf of categories, with stalks isomorphic to $\mathcal{V}^n$.

There is a tautological $\mathcal{V}^n$-bundle over $B|GL_n(i\mathcal{V})|$, which can be pulled back along a classifying map $X \to B|GL_n(i\mathcal{V})|$ to give a $\mathcal{V}^n$-bundle over X.

Expect there is a notion of $(K\mathcal{V})^n$-bundle, or locally trivial sheaf of Γ-categories with stalks isomorphic to $(K\mathcal{V})^n$, and a tautological such $(K\mathcal{V})^n$-

bundle over $B\widehat{GL}_n(|K\mathcal{V}|)$.

Maps $X \to B\widehat{GL}_n(|K\mathcal{V}|)$ then "represent" such $(K\mathcal{V})^n$-bundles over X. The semi-group of $(K\mathcal{V})^n$-bundles over X, for $n \geq 0$, can then be group completed (with respect to the block sum of matrices) and "represented" by maps

$$X \mapsto (2 - K)(2 - \mathcal{V}) \simeq K(ku)\,.$$

These are elements in $K(ku)^0(X)$.

§10. Gerbes

The special case of $\mathbb{C}^*$-gerbes plays the rôle among $K\mathcal{V}$-bundles that the special case of line bundles plays among vector bundles.

A $\mathbb{C}^*$-gerbe $\mathcal{G} \downarrow X$ is a $\mathcal{V}_1$-bundle over X, i.e., a locally trivial sheaf of categories with stalks isomorphic to $\mathcal{V}_1 \subset \mathcal{V}$. They are classified by maps $X \to B|GL_1(i\mathcal{V})| = BB\mathbb{C}^* \simeq K(\mathbb{Z}, 3)$, and thus give a "geometric" interpretation of elements in $H^3(X; \mathbb{Z})$.

§11. 1-DIMENSIONAL ELLIPTIC OBJECTS

Brylinski shows how a $\mathbb{C}^*$-gerbe $\mathcal{G}$ over a smooth manifold M (together with a contractible choice of connecting and curving structures) determines a complex line bundle L over the free loop space ΛM. Thus to each oriented circle C over M there is associated a 1-dimensional vector space, namely the fiber L_C over $C \in \Lambda M$. Furthermore, for any compact oriented surface Σ over M viewed as an oriented cobordism from some boundary circles $C_1, \ldots, C_m$ to the remaining boundary circles $C'_1, \ldots, C'_n$ there is a parallel transport isomorphism

$$S(\Sigma)\colon L_{C_1} \otimes \cdots \otimes L_{C_m} \xrightarrow{\cong} L_{C'_1} \otimes \cdots \otimes L_{C'_n} .$$

This provides a particular 2-dimensional field theory, analogous to the elliptic objects described by Segal, except that the vector spaces L_C are 1-dimensional rather than infinite-dimensional Hilbert spaces.

§12. Virtual elliptic objects

We think that the differential geometry of $\mathbb{C}^*$-gerbes can be extended to $K\mathcal{V}$-bundles, so as to associate to each $K\mathcal{V}$-bundle $\mathcal{E}$ over M (together with connecting and curving structures) a 2-dimensional field theory that to each oriented circle C over M associates a virtual vector space V_C, and to each oriented surface Σ as above associates an isomorphism

$$S(\Sigma)\colon V_{C_1} \otimes \cdots \otimes V_{C_m} \xrightarrow{\cong} V_{C'_1} \otimes \cdots \otimes V_{C'_n} .$$

This may then serve the purposes desired of the elliptic objects of Segal.

§13. Conclusion

Algebraic K-theory of topological K-theory viewed of as a version of elliptic cohomology and interpreted as the 2-K-theory of 2-vector spaces, is a spectrum representing a receptacle for the geometric structures ($(K\mathcal{V})^n$-bundles) that give rise to elliptic objects and 2-dimensional field theories.