THE STABLE PARAMETRIZED h-COBORDISM THEOREM

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These are the author's notes for lectures on the paper: "Spaces of PL manifolds and categories of simple maps (the manifold part)" by F. Waldhausen and W. Vogell.

Lecture I

1. Whitehead torsion and simple homotopy equivalence.

We first recall some of J.H.C. Whitehead's simple homotopy theory.

A finite polyhedron is the geometric realization of a finite simplicial complex. Let K be a fixed finite polyhedron and consider finite polyhedral pairs (L, K).

There is an elementary collapse $\operatorname{rel} K$ from (L_0, K) to (L_1, K) if L_0 is obtained by attaching a simplex Δ to L_1 along a horn in its boundary, i.e., the complement Λ in $\partial\Delta$ of the interior of a face $\partial_i\Delta$ of Δ:

$$L_0 = L_1 \cup_\Lambda \Delta$$

We write $L_0 \searrow L_1 \operatorname{rel} K$. The equivalence relation generated by elementary collapses $\operatorname{rel} K$ is called simple homotopy equivalence $\operatorname{rel} K$.

The space of deformation retractions of L_0 onto L_1 equals the space of deformation retractions of Δ onto Λ, which is contractible. In particular it is non-empty, so simple homotopy equivalence $\operatorname{rel} K$ implies homotopy equivalence $\operatorname{rel} K$. A chain of elementary collapses beginning with (L_0, K) and ending with (L_1, K) thus determines a homotopy equivalence $L_0 \to L_1 \operatorname{rel} K$ up to contractible choice.

Definition. The Whitehead group $Wh_1(K)$ is the set of simple homotopy classes $\operatorname{rel} K$ of finite polyhedral pairs (L, K) such that the inclusion $K \subseteq L$ is a homotopy equivalence. The sum operation is given by union along K. The equivalence class $\tau(L, K) \in Wh_1(K)$ is called its Whitehead torsion.

The condition that $K \subseteq L$ is a homotopy equivalence is equivalent to asserting that L contains K as a deformation retract.

Suppose K is connected. Whitehead's main theorem (1952?) is that $Wh_1(K)$ only depends on the fundamental group $\pi_1(K)$, in that there is an algebraically defined functor

$$Wh_1(\pi) = K_1(\mathbb{Z}[\pi])/\{\pm\pi\}$$

and a natural equivalence $Wh_1(K) \cong Wh_1(\pi_1(K))$. There is an algebraic construction of the Whitehead torsion class $\tau(L, K) \in Wh_1(\pi_1(K))$ in terms of a model for (L, K) as a pair of simplicial complexes.

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So for $\pi_1(K) = 1$ or $\pi_1(K) = \mathbb{Z}$, when $Wh_1(\pi) = 0$, the notions of homotopy equivalence and simple homotopy equivalence rel K agree. Generally they do not, as the calculation $Wh_1(C_5) \cong \mathbb{Z}$ indicates. (If $\pi = C_5 = \langle T \mid T^5 \rangle$, a generator is the image of $1 - T + T^2$ under $GL_1(\mathbb{Z}[\pi]) \to K_1(\mathbb{Z}[\pi]) \to Wh_1(\pi)$.)

2. Parametrized h-cobordism theory.

An h-cobordism (W, M) is a compact PL manifold W^{n+1} with boundary $\partial W \cong M \cup (\partial M \times I) \cup M'$, such that the inclusions $M \subset W$ and $M' \subset W$ are homotopy equivalences. The trivial h-cobordism is $(M \times I, M)$, where we identify $M \cong M \times \{0\}$.

The s-cobordism theorem of Barden, Mazur and Stallings (1963) says that for a compact connected PL manifold M^n, $n \geq 5$, the isomorphism classes of h-cobordisms (W, M) are in one-to-one correspondence with the elements of $Wh_1(M)$, via their Whitehead torsions. The trivial h-cobordism corresponds to zero in $Wh_1(M)$.

In particular there exists an isomorphism $(W, M) \cong (M \times I, M)$ if and only if $\tau(W, M) = 0 \in Wh_1(M)$. Such h-cobordisms are called s-cobordisms, and such an isomorphism is called a trivialization of (W, M). It provides (W, M) with a product structure.

Such product structures have applications in the classification of manifolds, such as in the proof of the generalized Poincaré conjecture in dimensions ≥ 5 by Smale in the smooth case (1960), and Stallings in the PL case. Likewise for the classification of acyclic manifolds such as lens spaces into PL homeomorphism vs. homotopy equivalence classes.

Parametrized h-cobordism theory is concerned with the existence of product structures in parametrized families of h-cobordisms, i.e., the comparison of families of product structures on $(M \times I, M)$.

Definition. There is a space $H(M)$ of h-cobordisms on M, or more precisely a simplicial set $[q] \mapsto H(M)_q$. When M is closed, a zero-simplex is a PL submanifold $W \subseteq M \times I$ such that (W, M) is an h-cobordism. A q-simplex is a q-parameter family of such h-cobordisms, meaning a PL fiber bundle over Δ^q that (i) is contained as a sub-bundle in the trivial bundle $M \times I \times \Delta^q \to \Delta^q$ and (ii) is a fiberwise h-cobordism.

When M has non-empty boundary we also assume that $W \cap (U \times I) = U \times [0, t]$ for some neighborhood U of $\partial M \subset M$ and $0 < t < 1$. The boundary neighborhood U and height t may vary in a PL fashion within a parametrized family.

Then the path components of $H(M)$ are in bijective correspondence with the isomorphism classes of h-cobordisms on M, and to elements of $Wh_1(M)$ by means of their Whitehead torsion.

Let P be a finite polyhedron. A PL fiber bundle over P of h-cobordisms $\{(W_p, M) \in H(M) \mid p \in P\}$ is classified by a PL map $P \to H(M)$, and vice versa. A fiber (W_p, M) admits a product structure $(W_p, M) \cong (M \times I, M)$ if and only if p maps to the component of $H(M)$ containing the trivial h-cobordism $(M \times I, M)$, which we view as the base point in $H(M)$. The fiber bundle admits a global product structure, i.e., an isomorphism of PL fiber bundles over P of $\{(W_p, M)\}_p$ with the constant family $\{(M \times I, M)\}_p$, if and only if the map $P \to H(M)$ is homotopic to the constant map at the base point.

Hence parametrized h-cobordism theory is concerned with the homotopy classes of maps $P \to H(M)$, and in the universal case, with the homotopy type of the representing space of h-cobordisms $H(M)$. We know $\pi_0 H(M) \cong Wh_1(M)$, but desire a combinatorial, computable model for the full homotopy type of $H(M)$.

3. Categories of simple maps.

The composite of two elementary collapses is typically not an elementary collapse, so the arrows $L_0 \searrow L_1 \operatorname{rel} K$ do not provide the morphisms in a category. Instead there is a weaker notion, called a simple map, that generates the same equivalence relation as simple homotopy equivalence.

Definition. A PL map of polyhedra is simple if it has contractible point inverses. In particular the point inverses are non-empty, so the map is surjective.

A simplicial map of simplicial sets is simple if its geometric realization has contractible point inverses. Again such a map is surjective.

An elementary collapse $L_0 \searrow L_1 \operatorname{rel} K$ provides a simple map $f\colon L_0 \to L_1$, with point inverses single points or closed intervals. Conversely a simple map is a simple homotopy equivalence. Hence the existence of a chain of simple maps $\operatorname{rel} K$ is equivalent to the existence of a chain of elementary collapses $\operatorname{rel} K$, so simple maps generate the same equivalence relation as simple homotopy equivalence.

Lemma [W4]**.** *The composite of two simple maps of polyhedra is a simple map.*

((Look at this proof later.))

Definition. Let K be a finite polyhedron. Let $s\mathcal{E}^h_0(K)$ be the category of finite polyhedra L that contain K as a deformation retract, and simple maps $\operatorname{rel} K$.

More generally let $s\mathcal{E}^h_q(K)$ be the category with objects the q-parameter families of such finite polyhedra, meaning PL fiber bundles over Δ^q with finite polyhedral fibers, containing the trivial bundle $K \times \Delta^q \to \Delta^q$ as a fiberwise deformation retract. The morphisms are maps of PL fiber bundles which are simple maps $\operatorname{rel} K$ in each fiber.

The functor $[q] \mapsto s\mathcal{E}^h_q(K)$ defines a simplicial category, briefly denoted $s\mathcal{E}^h_\bullet(K)$, with geometric realization $s\mathcal{E}^h(K)$.

Let X be a compact PL manifold. There is a forgetful functor $f_\bullet\colon H(X)_\bullet \to s\mathcal{E}^h_\bullet(X)$ taking an h-cobordism (W, X) on X to the underlying polyhedral pair (W, X), and likewise for parametrized families. Here we view the simplicial set $H(X)$ as a discrete simplicial category, with only identity morphisms. It induces a map

$$f\colon H(X) \to s\mathcal{E}^h(X)$$

which in turn induces an isomorphism on π_0:

$$\pi_0 H(X) \cong \pi_0 s\mathcal{E}^h(X) \cong Wh_1(X)$$

According to Hatcher (1975) the map $f\colon H(X) \to s\mathcal{E}^h(X)$ is k-connected whenever $n = \dim X$ is large with respect to k, and $n \geq 3k + 5$ will do. This is the parametrized h-cobordism theorem of Hatcher. If this is correct, the space $s\mathcal{E}^h(X)$ is a combinatorial model for the geometric space $H(X)$ up to dimension k, providing the desired model for parametrized h-cobordism theory in up to k parameters.

Unfortunately "the published proof is not satisfactory." Hatcher's argument involves forming mapping cylinders of general PL maps, which is not a functorially meaningful construction. Instead, Waldhausen and Vogell provide a proof of a stabilized version of this result, passing to a colimit over maps taking X to $X \times I$. This is the stable parametrized h-cobordism theorem of Waldhausen, or rather its manifold part.

4. Stable parametrized h-cobordism theory.

There is a (lower) stabilization map

$$\sigma\colon H(X) \to H(X \times I)$$

essentially taking an h-cobordism (W, X) to $(W \times I, X \times I)$, but making adjustments near $\partial(X \times I)$. Let

$$\mathcal{H}(X) = \operatorname*{colim}_n H(X \times I^n)$$

denote the homotopy colimit of the resulting direct system. This is the stable h-cobordism space of X.

Theorem (Waldhausen, Vogell). *Let X be a compact PL manifold. There is a homotopy equivalence*

$$\mathcal{H}(X) \simeq s\mathcal{E}^h(X)\,.$$

This effects the translation from compact manifolds to finite polyhedra. The non-manifold part of the theory concerns the translation from finite polyhedra to finite simplicial sets.

Let $sC^h_f(X)$ be the category of finite simplicial sets containing X as a deformation retract, and simple maps. It remains to produce a homotopy equivalence $sC^h_f(X) \to s\mathcal{E}^h(X)$ taking a simplicial set to an associated polyhedron. This is the subject of [W4].

Union along X makes $sC^h_f(X)$ a category with sums, and a Γ-category in the sense of Segal. It admits a delooping $Wh^{PL}(X) = sN_\bullet C^h_f(X)$, where $N_\bullet$ refers to Segal's simplicial category of sum diagrams. Then $Wh_1(X) \cong \pi_0 \mathcal{H}(X) \cong \pi_0 \Omega Wh^{PL}(X)$ and so $Wh^{PL}(X)$ is a connected space carrying the Whitehead group $Wh_1(X)$ as its fundamental group.

In [W2], Waldhausen establishes a homotopy equivalence

$$sN_\bullet C^h_f(X) \simeq sS_\bullet R^h_f(X^{\Delta^\bullet})$$

and a fiber sequence

$$\Omega sS_\bullet R^h_f(X^{\Delta^\bullet}) \to \Omega sS_\bullet R_f(X^{\Delta^\bullet}) \xrightarrow{\alpha} \Omega hS_\bullet R_f(X^{\Delta^\bullet})\,.$$

The right hand term is a model for $A(X)$, the middle term is the homology theory in X with coefficient spectrum $A(*)$, and the map α is the assembly map. The left hand term is homotopy equivalent to $\Omega Wh^{PL}(X) \simeq sC^h_f(X)$.

Given all this, the full stable parametrized h-cobordism theorem asserts that for compact PL manifolds X there is a homotopy equivalence

$$\mathcal{H}(X) \simeq \Omega Wh^{PL}(X)\,.$$

Here the left hand side is geometrically defined, while the right hand side is defined in terms of algebraic K-theory, and is homotopy equivalent to the fiber of the assembly map for the functor $X \mapsto A(X)$.

LECTURE II

5. Categories of PL manifolds.

Let M be an n-dimensional PL manifold, i.e., a polyhedron that is locally PL homeomorphic to the ball $B^n = [-1,1]^n \subset \mathbb{R}^n$. Equivalently, M is a combinatorial manifold, so it admits a triangulation with all links PL-homeomorphic to either $\partial B^n \cong S^{n-1}$ or $B^{n-1} \cong D^{n-1}$.

Let $\tau_M \downarrow M$ be the tangent PL microbundle of M, which is a germ of PL neighborhoods of the diagonal $\Delta(M) \subset M \times M$.

A stable framing of M is an equivalence class of isomorphisms $\phi\colon \tau_M \oplus \epsilon^k \cong \epsilon^{n+k}$ of PL microbundles over M. Here $\epsilon^k \downarrow M$ is the trivial k-dimensional microbundle over M. The trivializing isomorphism ϕ is equivalent to its stabilization

$$\tau_M \oplus \epsilon^k \oplus \epsilon^l = \tau_M \oplus \epsilon^{k+l} \xrightarrow[\cong]{\phi \oplus id} \epsilon^{n+k} \oplus \epsilon^l = \epsilon^{n+k+l} .$$

For stably framed manifolds M and N, a framed map $f\colon M \to N$ is one such that the composite map of bundles over M

$$\epsilon^{n+k} \cong \tau_M \oplus \epsilon^k \xrightarrow{f_* \oplus id} f^*\tau_N \oplus \epsilon^k \cong \epsilon^{n+k}$$

is the identity.

Consider the category $\mathcal{M}_0^n$ with objects all stably framed n-dimensional compact PL manifolds, and morphisms all framed PL homeomorphisms of such manifolds. This category is in fact a groupoid, i.e., all morphisms are invertible.

Let $h\mathcal{M}_0^n$ be the category with the same objects as $\mathcal{M}_0^n$, i.e., stably framed n-dimensional compact PL manifolds, and morphisms all framed homotopy equivalences of such manifolds. There is a forgetful functor $\mathcal{M}_0^n \to h\mathcal{M}_0^n$ regarding PL homeomorphisms as special cases of homotopy equivalences.

We shall need parametrized versions of these categories.

Let P be a polyhedron. A family over P of compact n-dimensional PL manifolds is a PL fiber bundle $\pi\colon E \to P$ with fibers $M_p = \pi^{-1}(p)$ which are compact n-dimensional PL manifolds.

A stable framing of such a family of manifolds is a family of stable framings $\phi_p\colon \tau_{M_p} \oplus \epsilon^k \cong \epsilon^{n+k}$ which vary in a PL manner with $p \in P$. This means that there is an isomorphism $\Phi\colon \tau_E^v \oplus \epsilon^k \cong \epsilon^{n+k}$ of PL microbundles over E, restricting to ϕ_p over $M_p \subset E$. Here $\tau_E^v = \bigcup_{p \in P} \tau_{M_p} \to E$.

Let $\mathcal{M}_q^n$ be the groupoid with objects all families over Δ^q of stably framed n-dimensional compact PL manifolds, and morphisms all framed PL homeomorphisms of such. Let $h\mathcal{M}_q^n$ be the category with the same objects, and morphisms the framed PL bundle maps which are fiberwise homotopy equivalences.

The now obvious functor $[q] \mapsto \mathcal{M}_q^n$ is a simplicial groupoid, denoted $\mathcal{M}_\bullet^n$. Likewise there is a simplicial category $h\mathcal{M}_\bullet^n$, and a forgetful functor $\mathcal{M}_\bullet^n \to h\mathcal{M}_\bullet^n$.

We shall also need to stabilize these categories with respect to the manifold dimension. The stabilization map

$$\sigma\colon \mathcal{M}_\bullet^n \to \mathcal{M}_\bullet^{n+1}$$

takes a PL manifold M to $M \times I$, and likewise for parametrized families, etc. There is an induced forgetful functor $\operatorname{colim}_n \mathcal{M}_\bullet^n \to \operatorname{colim}_n h\mathcal{M}_\bullet^n$.

6. Categories of simple maps.

Let $s\mathcal{E}_0$ be the category of finite polyhedra and simple maps, and let $h\mathcal{E}_0$ be the category of finite polyhedra and homotopy equivalences. The forgetful functor $s\mathcal{E}_0 \to h\mathcal{E}_0$ regards a simple map as a homotopy equivalence.

More generally let $s\mathcal{E}_q$ be the category of PL fiber bundles over Δ^q of finite polyhedra, and fiberwise simple PL bundle maps. Then $[q] \mapsto s\mathcal{E}_q$ is a simplicial category, denoted $s\mathcal{E}_\bullet$. Likewise let $h\mathcal{E}_q$ be the category of PL fiber bundles over Δ^q of finite polyhedra, and PL bundle maps that are fiberwise homotopy equivalences. Then $[q] \mapsto h\mathcal{E}_q$ is a simplicial category, denoted $h\mathcal{E}_\bullet$, and there is a forgetful functor $s\mathcal{E}_\bullet \to h\mathcal{E}_\bullet$.

There are stabilization maps $\sigma\colon s\mathcal{E}_\bullet \to s\mathcal{E}_\bullet$ and $\sigma\colon h\mathcal{E}_\bullet \to h\mathcal{E}_\bullet$ taking a finite polyhedron K to $K \times I$, and similarly for parametrized families etc.

7. The main theorem.

There is a functor $f\colon \mathcal{M}^n_\bullet \to s\mathcal{E}_\bullet$ that takes a compact PL manifold M to its underlying finite polyhedron, and likewise for parametrized families. It is well defined, since a PL homeomorphism is a simple map. Likewise there is a functor $g\colon h\mathcal{M}^n_\bullet \to h\mathcal{E}_\bullet$, and the two are compatible under the forgetful functors. The functors commute with stabilization, and induce maps

$$\operatorname*{colim}_n f\colon \operatorname*{colim}_n \mathcal{M}^n_\bullet \to \operatorname*{colim}_n s\mathcal{E}_\bullet$$

and

$$\operatorname*{colim}_n g\colon \operatorname*{colim}_n h\mathcal{M}^n_\bullet \to \operatorname*{colim}_n h\mathcal{E}_\bullet\,.$$

Hence we have the following commutative diagram:

$$\begin{array}{ccccc}
\operatorname{colim}_n \mathcal{M}^n_\bullet & \xrightarrow{\operatorname{colim}_n f} & \operatorname{colim}_n s\mathcal{E}_\bullet & \xleftarrow{i_0} & s\mathcal{E}_\bullet \\
\downarrow & & \downarrow & & \downarrow \\
\operatorname{colim}_n h\mathcal{M}^n_\bullet & \xrightarrow{\operatorname{colim}_n g} & \operatorname{colim}_n h\mathcal{E}_\bullet & \xleftarrow{i_0} & h\mathcal{E}_\bullet
\end{array}$$

Lemma [WV, 1.1]**.** *The canonical maps $i_0\colon s\mathcal{E}_\bullet \to \operatorname{colim}_n s\mathcal{E}_\bullet$ and $i_0\colon h\mathcal{E}_\bullet \to \operatorname{colim}_n h\mathcal{E}_\bullet$ are homotopy equivalences.*

Proof. The stabilization map $\sigma\colon s\mathcal{E}_\bullet \to s\mathcal{E}_\bullet$ is homotopic to the identity via the natural transformation $\sigma \to id$ given on a polyhedron K by the projection $K \times I \to K$ away from I. Its point inverses are intervals, which are contractible, so the projection is a simple map. The proof in the h-case is the same. □

The stable parametrized h-cobordism theorem will follow from the following four claims:

Theorem. *Let X be a stably framed compact PL manifold.*

(a) The homotopy fiber of $\operatorname{colim}_n \mathcal{M}^n_\bullet \to \operatorname{colim}_n h\mathcal{M}^n_\bullet$ at X is homotopy equivalent to the stable h-cobordism space $\mathcal{H}(X)$.

(b) The homotopy fiber of $s\mathcal{E}_\bullet \to h\mathcal{E}_\bullet$ at X is homotopy equivalent to the space $s\mathcal{E}^h(X)$.

(c) The stabilized map

$$\operatorname*{colim}_n f\colon \operatorname*{colim}_n \mathcal{M}^n_\bullet \to \operatorname*{colim}_n s\mathcal{E}_\bullet$$

is a homotopy equivalence.

(d) The stabilized map

$$\operatorname*{colim}_n g\colon\ \operatorname*{colim}_n h\mathcal{M}^n_\bullet \to \operatorname*{colim}_n h\mathcal{E}_\bullet$$

is a homotopy equivalence.

The proofs of (a) and (d) are left as exercises, to which we shall return. We shall also return to why we can assume X is stably framed. The proof of (b) appears in the non-manifold part [W4]. We now turn to the proof of (c), which is [WV, 1.2].

8. Simple manifolds over polyhedra.

We aim to prove that the map

$$\operatorname*{colim}_n f\colon\ \operatorname*{colim}_n \mathcal{M}^n_\bullet \to \operatorname*{colim}_n s\mathcal{E}_\bullet$$

is a homotopy equivalence. As a first step, we shall construct a simplicial category $\mathcal{ME}^n_\bullet$ and functors $\alpha\colon \mathcal{ME}^n_\bullet \to \mathcal{M}^n_\bullet$ and $\beta\colon \mathcal{ME}^n_\bullet \to s\mathcal{E}_\bullet$ such that α is a homotopy equivalence and $f\alpha$ is homotopic to β. Upon stabilization we get a map

$$\operatorname*{colim}_n \beta\colon\ \operatorname*{colim}_n \mathcal{ME}^n_\bullet \to s\mathcal{E}_\bullet$$

such that $\operatorname{colim}_n f \operatorname{colim}_n \alpha \simeq i_0 \operatorname{colim}_n \beta$. Hence $\operatorname{colim}_n f$ is a homotopy equivalence if and only if $\operatorname{colim}_n \beta$ is one. This replaces $\operatorname{colim}_n f$ by a map where the target category is not stabilized.

Let K be a finite polyhedron. A simple n-manifold over K is an n-dimensional compact PL manifold M together with a PL simple map $M \to K$. If M is also stably framed, this is a stably framed simple n-manifold over K.

Let $\mathcal{ME}^n_0$ be the category with objects $u\colon M \to K$ where K is a finite polyhedron and M is a stably framed simple n-manifold over K. The morphisms from $u\colon M \to K$ to $u'\colon M' \to K'$ are commutative squares

$$\begin{array}{ccc} M & \longrightarrow & M' \\ \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle u'} \\ K & \longrightarrow & K' \end{array}$$

where $M \to M'$ is a framed PL homeomorphism, and $K \to K'$ is a simple map.

Let $\mathcal{ME}^n_q$ be the q-parameter version of this category, with objects the simple maps $M \to K \times \Delta^q$ over Δ^q where $M \to \Delta^q$ is a PL fiber bundle of stably framed n-dimensional compact PL manifolds, as usual, and K is a finite polyhedron. The morphisms are the obvious commutative squares of PL fiber bundles over Δ^q. Let $\mathcal{ME}^n_\bullet$ be the resulting simplicial category.

Let the functors $\alpha\colon \mathcal{ME}^n_\bullet \to \mathcal{M}^n_\bullet$ and $\beta\colon \mathcal{ME}^n_\bullet \to s\mathcal{E}_\bullet$ take the object $u\colon M \to K$ to M and K, respectively.

Lemma. *α is a homotopy equivalence.*

Proof. Let $\iota\colon \mathcal{M}^n_\bullet \to \mathcal{ME}^n_\bullet$ be the functor taking M to the identity map $M \xrightarrow{=} M$, viewing the target M as a finite polyhedron. Then $\alpha\iota = id$. There is a natural transformation $\iota\alpha \to id$ of endo-functors on $\mathcal{M}\mathcal{E}^n_\bullet$, taking a simple map $u\colon M \to K$ to the commutative square:

$$\begin{array}{ccc} M & \xrightarrow{=} & M \\ \downarrow{\scriptstyle =} & & \downarrow{\scriptstyle u} \\ M & \xrightarrow{u} & K \end{array}$$

Hence α is a homotopy equivalence. □

Lemma. *The maps $f\alpha$ and β are homotopic.*

Proof. The structural map $M \to K$ of an object of $\mathcal{M}\mathcal{E}^n_\bullet$ defines a natural transformation from $f\alpha$ to β of functors $\mathcal{M}\mathcal{E}^n_\bullet \to s\mathcal{E}_\bullet$. □

There is a stabilization map

$$\sigma\colon \mathcal{M}\mathcal{E}^n_\bullet \to \mathcal{M}\mathcal{E}^{n+1}_\bullet$$

taking the object $u\colon M \to K$ to $M \times I \to K$, i.e., the product map $u \times id\colon M \times I \to K \times I$ followed by projection $pr\colon K \times I \to K$ away from the interval. This is compatible under α with the dimension-increasing stabilization $\sigma\colon \mathcal{M}^n_\bullet \to \mathcal{M}^{n+1}_\bullet$, and under β with the trivial stabilization $id\colon s\mathcal{E}^n_\bullet \to s\mathcal{E}^n_\bullet$.

Hence we obtain a (non-commutative) diagram:

$$\begin{array}{ccc} \operatorname{colim}_n \mathcal{M}\mathcal{E}^n_\bullet & \xrightarrow{\operatorname{colim}_n \beta} & s\mathcal{E}_\bullet \\ \simeq \downarrow{\scriptstyle \operatorname{colim}_n \alpha} & & \simeq \downarrow{\scriptstyle i_0} \\ \operatorname{colim}_n \mathcal{M}^n_\bullet & \xrightarrow{\operatorname{colim}_n f} & \operatorname{colim}_n s\mathcal{E}_\bullet \end{array}$$

The lower left composite takes $u\colon M \to K$ to M, and the upper right composite takes $u\colon M \to K$ to K. Restricted to a fixed $\mathcal{M}\mathcal{E}^n_\bullet$, these maps are homotopic since the simple map $u\colon M \to K$ defines a homotopy from $f\alpha$ to $i_0\beta$. However, this homotopy is not compatible with stabilization, so we have to work a little harder.

There is a corresponding diagram of mapping telescopes:

$$\begin{array}{ccc} \operatorname{Tel}_n \mathcal{M}\mathcal{E}^n_\bullet & \xrightarrow{\operatorname{Tel}_n \beta} & s\mathcal{E}_\bullet \\ \simeq \downarrow{\scriptstyle \operatorname{Tel}_n \alpha} & & \simeq \downarrow{\scriptstyle i_0} \\ \operatorname{Tel}_n \mathcal{M}^n_\bullet & \xrightarrow{\operatorname{Tel}_n f} & \operatorname{Tel}_n s\mathcal{E}_\bullet \end{array}$$

The composite maps

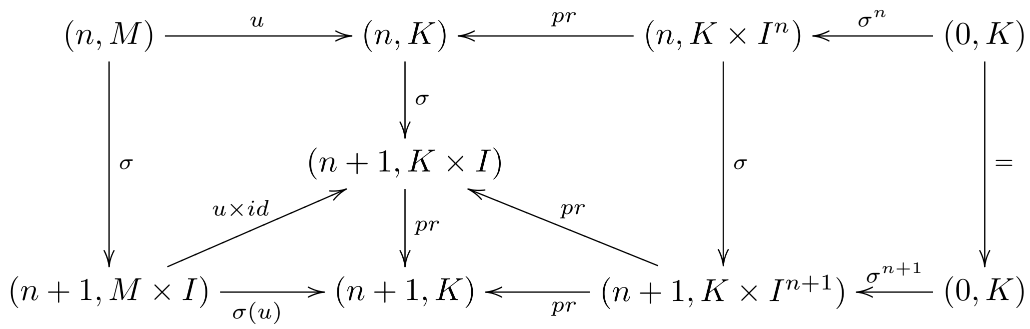
$$\operatorname{Tel}_n \mathcal{M}\mathcal{E}^n_\bullet \underset{i_0 \operatorname{Tel}_n \beta}{\overset{\operatorname{Tel}_n f\alpha}{\rightrightarrows}} \operatorname{Tel}_n s\mathcal{E}_\bullet$$

take $(n, u\colon M \to K)$ to (n, M) and $(0, K)$, respectively. The edge in the source telescope from $(n, u\colon M \to K)$ to $(n+1, \sigma(u)\colon M \times I \to K)$ maps to the edge from (n, M) to $(n+1, M \times I)$ and to the degenerate edge at $(0, K)$ in the target telescope, respectively.

There is a path in $\mathrm{Tel}_n\, s\mathcal{E}_\bullet$ from (n, M) to $(0, K)$, along the edges:

$$(n, M) \xrightarrow{u} (n, K) \xleftarrow{pr} (n, K \times I^n) \xleftarrow{\sigma^n} (0, K)\,.$$

The following diagram in $\mathrm{Tel}_n\, s\mathcal{E}_\bullet$ describes how to extend this path to a homotopy from $\mathrm{Tel}_n\, f\alpha$ to $i_0\,\mathrm{Tel}_n\,\beta$.



The canonical maps

$$\kappa\colon \operatorname*{Tel}_n \mathcal{M}\mathcal{E}^n_\bullet \to \operatorname*{colim}_n \mathcal{M}\mathcal{E}^n_\bullet$$

and

$$\kappa\colon \operatorname*{Tel}_n s\mathcal{E}_\bullet \to \operatorname*{colim}_n s\mathcal{E}_\bullet$$

are homotopy equivalences since the stabilization maps are cofibrations.

Now suppose we can show that $\mathrm{colim}_n\,\beta$ is a homotopy equivalence. Then so is $i_0\,\mathrm{colim}_n\,\beta$, and $i_0\,\mathrm{Tel}_n\,\beta$. This is homotopic to $\mathrm{Tel}_n\, f\alpha$, so also $\mathrm{colim}_n\, f\alpha$ is a homotopy equivalence. Thus $\mathrm{colim}_n\, f$ is a homotopy equivalence, which is what we want to prove.

Lecture III

9. Spaces of thickenings.

Let $\mathcal{S}^n_0(K)$ be the set of stably framed simple n-manifolds over K. More generally let $\mathcal{S}^n_q(K)$ the the set of PL fiber bundles over Δ^q of stably framed n-dimensional compact PL manifolds together with a PL simple map over Δ^q to $K \times \Delta^q$. The simplicial set $[q] \mapsto \mathcal{S}^n_q(K)$ is denoted $\mathcal{S}^n_\bullet(K)$.