THE SPHERE SPECTRUM

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October 28th 2004

Following J.H.C. Whitehead [14] and Lima [7], a sequential spectrum $E$ is a sequence of based topological spaces (or simplicial sets) $E_n$ and structure maps $\sigma_n : \Sigma E_n \to E_{n+1}$, for all $n \geq 0$. Here the suspension $\Sigma E_n = S^1 \wedge E_n$ equals the smash product with the based topological (or simplicial) circle $S^1$. The sphere spectrum $S$ is the most basic example: its $n$-th space is the $n$-sphere $S^n = S^1 \wedge \cdots \wedge S^1$ and its structure maps are the resulting homeomorphisms $\sigma_n : \Sigma S^n \to S^{n+1}$.

The $k$-th homotopy group $\pi_k(E)$ is the colimit over $n$ of the (unstable) homotopy groups $\pi_k(E_n)$, i.e., of the homotopy classes of based maps $S^k \to S^n$. By the Freudenthal suspension theorem [5] the homomorphisms in this colimit are isomorphisms for $n \geq k + 2$, and the common limiting value $\pi_k(S)$ is known as the $k$-th stable homotopy group of spheres.

The generalized homology theory $\pi_*^S(X) = \pi_*(S \wedge X)$ and the generalized cohomology theory $\pi_*^S(X) = \pi_* \text{Map}(X, S)$ associated to the sphere spectrum are called stable homotopy and stable cohomotopy, respectively. As a consequence of the proven Segal conjecture [3], stable cohomotopy has the exceptional property that $\pi_*^S(BG_+)$ vanishes for all $* < 0$. Here $BG$ is the classifying space of an arbitrary finite group.

A map of sequential spectra $f : E \to F$ is a sequence of based maps $f_n : E_n \to F_n$ commuting with the structure maps. It induces a homomorphism $f_* : \pi_*(E) \to \pi_*(F)$ of homotopy groups, and is called a stable equivalence if $f_*$ is an isomorphism in each degree. The stable homotopy category is the category obtained from

\[ E_{s,t}^2 = H^{-s}(BG; \pi_t(S)) \Rightarrow \pi_{-s-t}^S(BG_+) \]

there is a complicated differential interplay between group cohomology and the stable homotopy groups of spheres, making $E_{s,t}^\infty = 0$ for $s + t < 0$. For $G = \mathbb{Z}/p$ the Segal conjecture also provides a copy of $\pi_*(S)_p^\wedge$ as a direct summand in the abutment, so each class $x \in \pi_k(S)_p^\wedge$ is represented at $E_{s,t}^\infty$ by some coset $M(x) \subset \pi_t(S)_p^\wedge$, with $t \geq k$, called the Mahowald root invariant of $x$. Empirically, when $x$ is part of a periodic family in $\pi_*(S)_p^\wedge$ detected by the $n$-th Morava K-theory $K(n)$, then $M(x)$ is part of a family detected by the next Morava K-theory $K(n+1)$ [9].
the category of spectra by inverting the stable equivalences. Let $Sp$ denote the category of sequential spectra, and let $B$ (for Boardman) denote its associated stable homotopy category. The sphere spectrum $S$ generates $B$ in the sense that for each spectrum $E$ there exists a cell spectrum $E^c$, which has been assembled from integer suspensions of $S$ in the same way that a cell complex is built from non-negative suspensions of $S^0$, and a stable equivalence $E^c \to E$.

The smash product of two maps $S^{k+m} \to S^m$ and $S^{\ell+n} \to E_n$ composes with the iterated structure map $S^m \wedge E_n \to E_{n+m}$ to produce a map $S^{k+m+\ell+n} \to E_{n+m+n}$. With some care, especially about the ordering of the various circle factors in these smash products, this rule induces a pairing $E = \text{iterated structure map}$ and module structures are realized by morphisms spectra [1], well-defined up to stable equivalence, so that these commutative ring categories of spectra by inverting the stable equivalences. Let $E$ be a graded commutative ring, and in general $\pi_*(E)$ is a graded module over $\pi_*(S)$.

In the stable homotopy category $B$ there is a functorial smash product $E \wedge F$ of spectra [1], well-defined up to stable equivalence, so that these commutative ring and module structures are realized by morphisms $S \wedge S \to S$ and $S \wedge E \to E$. However, in the category $Sp$ of sequential spectra there is no definition of a smash product $E \wedge F$ such that the product on $S$ is commutative. It is at best associative, and sequential spectra $E$ and $F$ may be regarded as left (or right) $S$-modules, but no natural $S$-module structure remains on their smash product $E \wedge F$. The situation is reminiscent of that of modules over a non-commutative ring.

To overcome this defect, modern stable homotopy theory takes place in one of several possible modified categories $Sp'$ of spectra, three of which are reviewed below. In each of these there is a smash product $E \wedge F$ defined within the category of spectra, that is so well-behaved that the sphere spectrum $S$ admits a commutative product $S \wedge S \to S$ in $Sp'$. More precisely, the smash product is a symmetric monoidal pairing (= coherently unital, associative and commutative) with $S$ as the unit object. The spectra $E$, $F$ are naturally modules over $S$ with this product, i.e., $S$-modules, and the smash product $E \wedge F$ over $S$ of two $S$-modules is again an $S$-module, because $S$ is commutative. Furthermore, there is a notion of stable equivalence on $Sp'$, so chosen that the associated homotopy category is equivalent to $B$.

This makes the sphere spectrum $S$ the initial ground “ring” for stable homotopy theory, much like the integers $\mathbb{Z}$ is the initial ground ring for algebra. The categories of $S$-modules, resp. associative or commutative $S$-algebras, can be thought of as enriched versions of the categories of $\mathbb{Z}$-modules (= abelian groups), resp. associative or commutative $\mathbb{Z}$-algebras (= rings). This is a fruitful point of view for promoting ideas from algebra, algebraic geometry or number theory to the algebraic-topological context. The Eilenberg–MacLane functor embeds algebra into topology, and the enrichment amounts to a change of ground ring along the Hurewicz map $h: S \to \mathbb{Z}$. The earlier theories of $A_\infty$ and $E_\infty$ ring spectra [11] provide many more examples of associative and commutative $S$-algebras in topology, beyond those coming from algebra. These are therefore “brave new rings,” a term coined by Waldhausen.

Several modern reinterpretations $Sp'$ of the category of spectra appeared shortly after 1994. The principal three are (a) the $S$-modules $M_S$ of Elmendorf, Kriz, Mandell and May [4], (b) the symmetric spectra $Sp^\Sigma$ of Hovey, Shipley and Smith...
The $S$-modules of May et al. were introduced in [4]. To start, a coordinate-free spectrum $E$ is a rule that assigns a based space $EV$ to each finite-dimensional vector subspace $V \subseteq \mathbb{R}^\infty$, together with a compatible system of homeomorphisms $EV \cong \Omega^{W-V} EW$ whenever $V \subseteq W$. Here $W-V$ is the orthogonal complement of $V$ in $W$, $S^{W-V}$ is its one-point compactification, and $\Omega^{W-V} X = F(S^{W-V}, X)$ is the mapping space. The coordinate-free sphere spectrum $S$ is the rule with $SV = \mathrm{colim}_{V \subseteq W} \Omega^{W-V} SW$. Its 0th space $S0$ is also known as $Q(S^0)$.

An $L$-spectrum is a coordinate-free spectrum equipped with a suitable action by the space $L(1)$ of linear isometries $\mathbb{R}^\infty \to \mathbb{R}^\infty$, which is part of the linear isometries operad $L$. The sphere spectrum $S$ is canonically an $L$-spectrum, and there is an operadic smash product $E \wedge_L F$ of $L$-spectra. Finally, the $S$-modules are the $L$-spectra $E$ such that a natural map $S \wedge E \to E$ is an isomorphism. The sphere spectrum is an $S$-module, and the operadic smash product of $L$-spectra $E \wedge_L F$ restricts to the desired smash product $E \wedge F$ on the full subcategory of $S$-modules.

(b) The symmetric spectra of J. Smith et al. were introduced in [6]. First, a symmetric sequence $E$ is a sequence of based simplicial sets $E_n$ with an action by the symmetric group $\Sigma_n$, for each $n \geq 0$. The sphere symmetric sequence $S$ has the $n$-fold smash product $S^n = S^1 \wedge \cdots \wedge S^1$ as its $n$-th space, with $\Sigma_n$ permuting the factors. There is a symmetric monoidal pairing $E \otimes F$ of symmetric sequences, so defined that a map $E \otimes F \to G$ corresponds to a set of $(\Sigma_m \times \Sigma_n)$-equivariant maps $E_m \wedge F_n \to G_{m+n}$. Then $S$ is a commutative monoid with product $S \otimes S \to S$ corresponding to the equivariant isomorphisms $S^m \wedge S^n \cong S^{m+n}$.

A symmetric spectrum $E$ is defined to be an $S$-module in symmetric sequences, i.e., a symmetric sequence with a unital and associative action $S \otimes E \to E$. Explicitly, the module action amounts to a set of $(\Sigma_m \times \Sigma_n)$-equivariant maps $S^m \wedge E_n \to E_{m+n}$. The sphere spectrum $S$ is then a symmetric spectrum, and the desired smash product $E \wedge F$ of two symmetric spectra is defined as the coequalizer of two obvious maps $E \otimes S \otimes F \to E \otimes F$. There is a notion of a stable equivalence $f: E \to F$ of symmetric spectra, strictly more restrictive than asking that $\pi_\ast(f)$ is an isomorphism, so that the associated homotopy category is equivalent to $B$.

A variant of symmetric spectra, called orthogonal spectra [10], is obtained by replacing the symmetric group actions by orthogonal group actions. Then the $\pi_\ast$-isomorphisms are the correct weak equivalences to invert, in order to obtain a homotopy category equivalent to $B$.

(c) Let $\Gamma$ be the category of finite sets $n_+ = \{0, 1, \ldots, n\}$ based at 0, for $n \geq 0$, and base-point preserving functions. Segal [13] defined a $\Gamma$-space $E$ to be a functor from $\Gamma$ to based simplicial sets, such that $E(0_+)$ is a point. Each $\Gamma$-space can be prolonged (degreewise) to an endofunctor of based simplicial sets, and there is an associated sequential spectrum with $n$-th space $E(S^n)$. Bousfield and Friedlander [2] show that the homotopy category of $\Gamma$-spaces under stable equivalences is equivalent to the stable homotopy category of connective spectra, i.e., spectra with $\pi_k(E) = 0$ for $k < 0$.

The sphere $\Gamma$-space $S$ is the functor that takes $n_+$ to itself, considered as a based simplicial set. Its prolongation is the identity endofunctor, and the associated
sequential spectrum is the sphere spectrum $\mathbb{S}$. The smash product $E \wedge F$ of two $\Gamma$-spaces is defined so that a map $E \wedge F \to G$ of $\Gamma$-spaces amounts to a natural transformation $E(k_+) \wedge F(\ell_+) \to G(k_+ \wedge \ell_+)$, for $k_+$ and $\ell_+$ in $\Gamma$. This defines a symmetric monoidal pairing on $\Gamma$-spaces, with the sphere as the unit object. Lydakis [8] realized that this categorical construction also has good homotopical properties, in particular that it really models the smash product of spectra.

References


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