TWO-PRIMARY ALGEBRAIC K-THEORY OF POINTED SPACES

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0.1. These are notes for the author's talk at the 1999 British Topology Meeting in Swansea, April 6th to 8th.

0.2. I aim to talk about a calculation of the two-primary homotopy type of the algebraic K-theory space, $A(*)$, of the category of finite pointed CW-complexes, as constructed by Waldhausen.

(1) How can this homotopy type be defined and interpreted ?
(2) How does the calculation proceed, and what is the result ?
(3) How can the result be explained, or understood ?

1.1. In one interpretation, $A(*)$ is an infinite loop space acting as the target for a generalized Euler characteristic.

The ordinary Euler characteristic can be defined e.g. for finite CW-complexes, and takes values in the integers $\mathbb{Z}$ viewed as an abelian group. It will be convenient to work with pointed CW-complexes and the reduced Euler characteristic $\tilde{\chi}$. It satisfies two characterizing properties: homotopy invariance and additivity. For each pointed CW-complex Y the reduced Euler characteristic $\tilde{\chi}(Y)$ only depends on the homotopy type of Y, and for every cofiber sequence $Y' \to Y \to Y''$ we have the addition formula

$$\tilde{\chi}(Y) = \tilde{\chi}(Y') + \tilde{\chi}(Y'') .$$

In fact the reduced Euler characteristic is universal among additive homotopy invariants of finite CW-complexes taking values in abelian groups.

We now generalize this to an Euler characteristic of diagrams of finite CW-complexes and homotopy equivalences, taking values in an infinite loop space.

Let $hR_f(*)$ be the category of finite pointed CW-complexes and homotopy equivalences. We view a small category C and a functor

$$Y : C \to hR_f(*)$$

as a diagram of finite pointed CW-complexes and pointed homotopy equivalences, shaped like the nerve $|C|$ of the category C. The infinite loop space $A(*)$ comes equipped with a map $e\colon |hR_f(*)| \to A(*)$. Thus to the diagram $Y : C \to hR_f(*)$ we can associate a map

$$|C| \xrightarrow{|Y|} |hR_f(*)| \xrightarrow{e} A(*)$$

with the following homotopy invariance and additivity properties: Given a map $Y \to Y'$ of diagrams, i.e., a natural transformation from Y to Y', the composite maps $e \circ |Y|$ and $e \circ |Y'|$ are homotopic. Given a cofiber sequence

$$Y' \to Y \to Y''$$

of diagrams, i.e., natural transformations yielding a cofiber sequence $Y'(c) \to Y(c) \to Y''(c)$ for every object c in C, the composite map $e \circ |Y|$ is homotopic to the sum of the composite maps $e \circ |Y'|$ and $e \circ |Y''|$ with respect to the (homotopy commutative) loop sum in $A(*)$.

The homotopy class $\tilde{\chi}(Y) \in [|C|, A(*)]$ of the composite map $e \circ |Y|$ is thus an additive homotopy invariant for small diagrams of finite pointed CW-complexes and homotopy equivalences, taking values in an infinite loop space. In fact $A(*)$ and the map e are universal with respect to these properties, and we can think of $A(*)$ as the universal receptacle for a generalized Euler characteristic, defined for diagrams of spaces.

1.2. This generalizes the classical example. When C is the trivial category with only one object and one morphism, the diagram Y is a single finite pointed CW-complex, $|C| = *$ is a point, and the homotopy class $\tilde{\chi}(Y) \in [|C|, A(*)] \cong \pi_0 A(*) \cong \mathbb{Z}$ is the ordinary reduced Euler characteristic of Y.

More generally, if C has a single object $*$ and a group G of morphisms $* \to *$, then a diagram $Y\colon C \to hR_f(*)$ is equivalent to the finite pointed CW-complex $Y(*)$, equipped with a G-action. The nerve $|C|$ is the usual model for the classifying space BG for principal G-bundles, so the generalized Euler characteristic of such a G-space $Y(*)$ is a homotopy class of maps $BG \to A(*)$. Already this may carry more information than just the reduced Euler characteristic of the space $Y(*)$.

For example, if $G = \mathbb{Z}$, so $Y(*)$ is a space equipped with an automorphism, then $BG \simeq S^1$ and $\tilde{\chi}(Y) \in [|C|, A(*)] \cong \pi_0 A(*) \oplus \pi_1 A(*) \cong \mathbb{Z} \oplus \mathbb{Z}/2$ detects not only the reduced Euler characteristic of $Y(*)$, but also a $\mathbb{Z}/2$-valued invariant of the automorphism. The automorphism induces an automorphism on the rational homology group $H_i(Y(*); \mathbb{Q})$, in each dimension $i \geq 0$. The $\mathbb{Z}/2$-valued invariants detects whether an even or odd number of these have negative determinant.

In general, determining the homotopy classes of maps $|C| \to A(*)$ can be considered as an Adams spectral sequence problem, starting with the mod p cohomology of the nerve $|C|$ and the mod p spectrum cohomology of the spectrum with underlying infinite loop space $A(*)$:

$$E_2^{s,t} = \mathrm{Ext}_A(H^*_{spec}(A(*); \mathbb{F}_p), H^*(|C|; \mathbb{F}_p)) \Longrightarrow [|C|, A(*)^\wedge_p]_{t-s} \,.$$

The Adams spectral sequence will be conditionally convergent to the p-adically completed homotopy classes of maps, but a separate argument must be made to check strong convergence in individual cases.

1.3. The infinite loop space $A(*)$ is constructed by an infinite loop space machine, considered by Waldhausen. It takes as input the category $R_f(*)$ of finite pointed CW-complexes and pointed maps, equipped with a subcategory of cofibrations, being the cellular embeddings, and a subcategory of weak equivalences, being the homotopy equivalences. It yields as output the infinite loop space $A(*)$, together with the universal map $e\colon |hR_f(*)| \to A(*)$. We say that $A(*)$ is the algebraic

K-theory space of the category of finite pointed CW-complexes. Alternatively we can think of it as a connective spectrum.

Just as the reduced Euler characteristic is invariant under double suspension, $\tilde{\chi}(\Sigma^2 Y) = \tilde{\chi}(Y)$, the double suspension functor $\Sigma^2 \colon R_f(*) \to R_f(*)$ induces a map $\Sigma^2 \colon A(*) \to A(*)$ which is homotopic to the identity. From this it follows that the input category for Waldhausen's infinite loop space machine can be stabilized with respect to suspensions, i.e., that the category of finite pointed CW-complexes can be replaced by the category of finite CW-spectra.

Hence $A(*)$ can also be constructed, or described, as the algebraic K-theory spectrum of the category of finite CW-spectra, with respect to suitable notions of cofibrations and weak equivalences between such spectra.

There are now several largely equivalent categories of spectra that come equipped with a symmetric monoidal smash product. The main examples are the $\mathbb{S}$-modules of Elmendorf, Kriz, Mandell and May, the symmetric spectra of J. Smith, and the Γ-spaces or simplicial functors of G. Segal and M. Lydakis. In each case, the sphere spectrum $\mathbb{S}$ is the neutral element for the smash product, so the respective categories of spectra can all be interpreted as categories of module spectra over $\mathbb{S}$, viewed as a ring spectrum. The category of finite CW-spectra then plays the role of the category of finitely generated projective modules over $\mathbb{S}$, as in ordinary algebraic K-theory, and so we can also think of $A(*)$ as the algebraic K-theory spectrum of the ring spectrum $\mathbb{S}$, writing $A(*) = K(\mathbb{S})$.

Since the sphere spectrum $\mathbb{S}$ is commutative, its algebraic K-theory $A(*) = K(\mathbb{S})$ is in turn a ring spectrum.

Finally, any algebraic K-theory spectrum is a module spectrum over $A(*) = K(\mathbb{S})$. For example, when R is a ring its Eilenberg–Mac Lane spectrum HR is an algebra spectrum over $\mathbb{S}$. When $X = BG$ is a space, with G a simplicial group, the suspension spectrum $\Sigma^\infty G_+$ is an algebra spectrum over $\mathbb{S}$. In general, any A_∞ ring spectrum admits a model as an algebra spectrum over $\mathbb{S}$, in the spectrum categories considered above. So the algebraic K-theory of a ring, in the sense of Quillen, or of a space, in the sense of Waldhausen, or of any A_∞-ring spectrum, are all examples of the algebraic K-theory of an $\mathbb{S}$-algebra, and come equipped with a module action by the ring spectrum $K(\mathbb{S}) = A(*)$.

1.4. The original motivation for studying Waldhausen's algebraic K-theory $A(*) = K(\mathbb{S})$, or more generally the algebraic K-theory $A(X) = K(\Sigma^\infty G_+)$ of a space $X = BG$, stems from geometric topology. When X is a manifold, the spectrum $A(X)$ provides the link between algebraic K-theory and the geometric topology concerning the spaces of concordances (= pseudoisotopies), h-cobordisms and automorphisms of X.

Working in one of the three geometric categories DIFF, PL or TOP of smooth, piecewise linear or topological manifolds, respectively, Waldhausen constructed natural cofiber sequences

$$\Sigma^\infty X_+ \xrightarrow{i} A(X) \to \mathrm{Wh}^{DIFF}(X)$$

and

$$A(*) \wedge X_+ \xrightarrow{\alpha} A(X) \to \mathrm{Wh}^{PL}(X)\,.$$

There is a natural homotopy equivalence $\mathrm{Wh}^{PL}(X) \simeq \mathrm{Wh}^{TOP}(X)$. Furthermore, he constructed a trace map $tr_X \colon A(X) \to \Sigma^\infty \Lambda X_+$, which shows that the DIFF cofiber sequence splits. Here $\mathrm{Wh}^{CAT}(X)$ is called the CAT Whitehead spectrum. Its underlying infinite loop space is more commonly referred to, and is called the CAT Whitehead space.

Let $C^{CAT}(X) = CAT(X \times I, X \times 1)$ be the CAT concordance space of a CAT manifold X. It is the space of CAT automorphisms of the cylinder $X \times I$, fixing a neighborhood of the part $X \times 0 \cup \partial X \times I$ of the boundary. There are stabilization maps $C^{CAT}(X) \to C^{CAT}(X \times I)$, which in the direct limit induce a stabilization map

$$\Sigma_X^{CAT} \colon C^{CAT}(X) \to \operatorname*{hocolim}_k C^{CAT}(X \times I^k) \simeq \Omega^\infty \Omega^2 \,\mathrm{Wh}^{CAT}(X)\,.$$

Thus the double loop space of the CAT Whitehead space is the stabilized CAT concordance space. K. Igusa proved, in the DIFF category, that the stabilization map Σ_X^{CAT} is approximately $n/3$-connected, when $n = \dim(X)$ is the dimension of X, and similar estimates apply in the PL and TOP categories.

A. Hatcher showed how to recover information about the space $CAT(X)$ of CAT automorphisms of a manifold X (diffeomorphisms, PL homeomorphisms or homeomorphisms, respectively), from the concordance space $C^{CAT}(X)$ together with an involution on $C^{CAT}(X)$ thought of as 'turning a concordance upside-down.' His spectral sequence was given a space level interpretation by M. Weiss and B. Williams, who constructed a map

$$\Phi_X^{CAT} \colon \widetilde{CAT}(X)/CAT(X) \to \Omega^\infty(EC_{2+} \wedge_{C_2} \Omega \,\mathrm{Wh}^{CAT}(X))$$

which is also roughly $n/3$-connected. Here $\widetilde{CAT}(X)$ is the block automorphism space of X, which is a simplicial group accessible through surgery theory. The involution on concordance spaces stabilizes to a C_2-action on $\Omega\,\mathrm{Wh}^{CAT}(X)$, and the target of Φ_X^{CAT} is the underlying space of the spectrum level homotopy orbits of this action.

Thus from an understanding of $A(*) \simeq \Sigma^\infty S^0 \vee \mathrm{Wh}^{DIFF}(*)$ we obtain an understanding of $\mathrm{Wh}^{DIFF}(X)$ up to dimension $n/3$ for smooth, n-dimensional, $n/3$-connected manifolds X. By Igusa's stability theorem this amounts to knowing the DIFF concordance space $C^{DIFF}(X)$ in roughly the same range. Assuming the involution on $\Omega\,\mathrm{Wh}^{DIFF}(X)$ or $A(*)$ can be pinned down, this leads to knowledge of the DIFF automorphisms space $DIFF(X)$, again roughly up to dimension $n/3$. Already with $X = D^n$ a disc, this space of diffeomorphisms has a rich and interesting homotopy type.