1. Topological cyclotomy

For a commutative (symmetric or orthogonal) ring spectrum $A$ we initially defined its topological Hochschild homology

$$THH(A) = |A \otimes S^1|$$

as the topological realization of the cyclic bar construction $[q] \mapsto A \otimes S^1_q = A \wedge A \wedge \cdots \wedge A$. This is a commutative ring spectrum with $S^1$-action. Bökstedt, Hsiang and Madsen (1993) gave a modified definition that makes $THH(A)$ a “genuine” $S^1$-equivariant orthogonal ring spectrum. In particular, for each finite subgroup $C_r \subset S^1$ one can form the $C_r$-fixed point spectrum $THH(A)^{C_r}$. For $q|r$ we have $C_q \subset C_r$, and there is a natural (Frobenius) map

$$F: THH(A)^{C_r} \to THH(A)^{C_q}$$

that forgets part of the invariance. In the stable homotopy category, there is also a transfer (Verschiebung) map

$$V: THH(A)^{C_q} \to THH(A)^{C_r}.$$ 

Let $ES^1 = S(C\infty)$ be a free, contractible $S^1$-CW complex. It is also a free, contractible $C_r$-CW space for each $C_r \subset S^1$. Let $\hat{ES}^1 = S^{C\infty}$ be the mapping cone of the collapse map $ES^1_+ \to S^0$. Let $p$ be a prime. The geometric $C_p$-fixed point spectrum of $THH(A)$ can (provisionally) be defined as

$$THH(A)^{C_p} = \Phi^{C_p}THH(A) = [\hat{ES}^1 \wedge THH(A)]^{C_p}.$$ 

This has a residual $S^1/C_p$-action. There is a non-obvious, “cyclotomic”, equivalence

$$THH(A)^{C_p} \simeq THH(A),$$

which is equivariant with respect to the $p$-th root isomorphism $S^1/C_p \cong S^1$. The inclusion $S^0 \to \hat{ES}^1$ induces an $S^1/C_p$-equivariant map

$$THH(A)^{C_p} \to THH(A)^{C_p}.$$ 

Passing to $C_p$-fixed points, with $pq = r$, we obtain a natural restriction map

$$R: THH(A)^{C_r} \to THH(A)^{C_q}.$$ 

The $F$- and $R$-maps commute. Restricting attention to prime powers $r = p^n$ for $n \geq 0$, we obtain a (non-commutative) diagram

$$\cdots \xrightarrow{F} THH(A)^{C_{p^{n+1}}} \xrightarrow{F} THH(A)^{C_{p^n}} \xrightarrow{F} THH(A)^{C_{p^{n-1}}} \xrightarrow{F} \cdots$$

There is a trace map $\text{tr}: K(A) \to THH(A)$ from the algebraic $K$-theory of $A$, which admits lifts

$$\text{tr}_{p^n}: K(A) \to THH(A)^{C_{p^n}}$$

for all $n \geq 0$. These are compatible (up to homotopy) with the $R$- and $F$-maps, hence induce a map to the homotopy limit of the diagram above. This homotopy limit

$$TC(A; p) = \text{holim}_{n,R,F} THH(A)^{C_{p^n}}$$

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is the \((p\text{-typical})\) topological cyclic homology of \(A\). The lifted map
\[
\text{trc}: K(A) \longrightarrow TC(A; p)
\]
is the \((p\text{-typical})\) cyclotomic trace map. It is a sensitive invariant of algebraic \(K\)-theory, especially for connective \(A\) with \(\pi_0(A)\) close to \(\mathbb{Z}/p, \mathbb{Z}_p\) or similar rings. See work by Bökstedt (ca. 1985, unpublished), Goodwillie (1986), Bökstedt-Madsen (1993 and 1994), McCarthy (1997), Dundas (1997), Hesselholt-Madsen (1997 and 2003), Rognes (1998 and 1999), Ausoni-Rognes (2002), etc.

2. **Topological Frobenius and restriction homology**

Concentrating on the sequence of \(F\)-maps, let
\[
TF(A; p) = \varinjlim_{n,F} THH(A)^{C_p^n}.
\]
The \(R\)-maps then induce a self-map
\[
R: TF(A; p) \longrightarrow TF(A; p).
\]
Concentrating on the sequence of \(R\)-maps, let
\[
TR(A; p) = \varinjlim_{n,R} THH(A)^{C_p^n}.
\]
The \(F\)-maps then induce a self-map
\[
F: TR(A; p) \longrightarrow TF(R; p).
\]
There are homotopy equivalences of ring spectra
\[
\operatorname{hoeq}(\text{id}, R) \simeq TC(A; p) \simeq \operatorname{hoeq}(\text{id}, F).
\]
Additively,
\[
\operatorname{hoeq}(\text{id}, R: TF(A; p) \to TF(A; p)) \simeq \operatorname{hofib}(\text{id} - R: TF(A; p) \to TF(A; p))
\]
and
\[
\operatorname{hoeq}(\text{id}, F: TR(A; p) \to TR(A; p)) \simeq \operatorname{hofib}(\text{id} - F: TR(A; p) \to TR(A; p)).
\]
In particular, there is a short exact sequence
\[
0 \to \operatorname{Rlim}_{n,F} \pi_{+1} THH(A)^{C_p^n} \longrightarrow \pi_\ast TF(A; p) \longrightarrow \operatorname{lim}_{n,F} \pi_\ast THH(A)^{C_p^n} \to 0
\]
and a long exact sequence
\[
\cdots \longrightarrow \pi_{+1} TF(A; p) \longrightarrow \pi_\ast TC(A; p) \longrightarrow \pi_\ast TF(A; p) \longrightarrow \pi_{+1} TC(A; p) \longrightarrow \cdots
\]
which we can use to calculate \(\pi_\ast TF(A; p)\) and \(\pi_\ast TC(A; p)\) in some cases.

3. **Homotopy fixed points**

Let
\[
THH(A)^{hC_r} = F(ES^1_{+}, THH(A))^{C_r}
\]
be the \(C_r\)-homotopy fixed points of \(THH(A)\). The collapse map \(ES^1_{+} \to S^0\) induces a comparison map
\[
THH(A)^{C_r} \longrightarrow THH(A)^{hC_r}.
\]
When \(r = p^n\) we denote this map by \(\Gamma_n\). Let
\[
THH(A)^{hS^1} = F(ES^1_{+}, THH(A))^{S^1}
\]
be the \(S^1\)-homotopy fixed points of \(THH(A)\). The natural map
\[
THH(A)^{hS^1} \longrightarrow \varinjlim_{n,F} THH(A)^{hC_p^n}
\]
becomes an equivalence after \(p\)-completion. Hence there is a natural comparison map
\[
\Gamma: TF(A; p) \longrightarrow THH(A)^{hS^1}_p
\]
making the following diagram commute.

\[
\begin{array}{ccc}
TF(A; p) & \xrightarrow{\sim} & \text{holim}_{n,F} THH(A)^{C_r^n} \\
\Gamma_f & \downarrow & \downarrow \text{holim}_{n,F} \Gamma_n \\
THH(A)^{hS^1}_p & \xrightarrow{\sim} & \text{holim}_{n,F} THH(A)^{hC_r^n}_p
\end{array}
\]

The skeleton filtration of \(ES^1\) leads to an algebra spectral sequence

\[
E^2_{s,t}(C_r) = H^{-s}(C_r; \pi_t THH(A)) \Rightarrow \pi_{s+t} THH(A)^{hC_r}
\]

(group cohomology) called the homotopy fixed point spectral sequence. The \(C_r\)-action on \(\pi_s THH(A)\) extends over the connected group \(S^1\), hence is trivial. There is a similar spectral sequence

\[
E^2_{s,t}(C_r; \mathbb{Z}/p) = H^{-s}(C_r; \pi_t (THH(A); \mathbb{Z}/p)) \Rightarrow \pi_{s+t} (THH(A)^{hC_r}; \mathbb{Z}/p)
\]

for homotopy with \(\mathbb{Z}/p\)-coefficients, which is an algebra spectral sequence for \(p \geq 5\), and sometimes for \(p = 3\) or \(p = 2\). Here \(\pi_*(X) = \pi_*(X \wedge S/p)\), where \(S/p\) is the mod \(p\) Moore spectrum. This is a ring spectrum in the homotopy category for \(p \geq 5\). In many cases \(\pi_*(\Gamma_n)\) or \(\pi_*(\Gamma_n; \mathbb{Z}/p)\) is an isomorphism for \(*\) sufficiently large, in which case the \(C_{p^n}\)-homotopy fixed point spectral sequence gives a way to access the homotopy groups of \(THH(A)^{hC_r}\).

The Frobenius maps \(F: THH(A)^{hC_r} \to THH(A)^{hC_{r+1}}\) induce a map of algebra spectral sequences

\[
F: E^2_{s,*}(C_{p^n}) \to E^2_{s,*}(C_{p^{n-1}})
\]

converging to

\[
F: \pi_*(THH(A)^{hC_r^n}) \to \pi_*(THH(A)^{hC_{r+1}^{n-1}}).
\]

Passing to the limit over \(n\), there is also an algebra spectral sequence

\[
E^2_{s,t}(S^1) = H^{-s}(S^1; \pi_t THH(A)) \Rightarrow \pi_{s+t} THH(A)^{hS^1}
\]

(group cohomology, suitably defined), with \(p\)-complete and mod \(p\) variants. When \(\pi_*(\Gamma)\) is an isomorphism for \(*\) sufficiently large, this gives a way to access the homotopy groups of \(TF(A; p)\), usually after \(p\)-completion or with mod \(p\) coefficients.

The Verschiebung maps \(V: THH(A)^{hC_{r+1}} \to THH(A)^{hC_r^n}\) also induce a map of spectral sequences

\[
V: E^2_{s,*}(C_{p^{n-1}}) \to E^2_{s,*}(C_{p^n})
\]

converging to

\[
V: \pi_*(THH(A)^{hC_{r+1}^{n-1}}) \to \pi_*(THH(A)^{hC_r^n}).
\]

4. Tate constructions

Let

\[
THH(A)^{tC_r} = \hat{H}(C_r, THH(A)) = [\hat{E}S^1 \wedge F(ES^1, THH(A))]^{C_r}
\]

be the \(C_r\)-Tate construction for \(THH(A)\). The collapse map \(ES^1_+ \to S^0\) induces a comparison map

\[
THH(A)^{C_r} \simeq [THH(A)^{\varphi C_r}]^{C_r} = [\hat{E}S^1 \wedge THH(A)]^{C_r} \to THH(A)^{tC_r}
\]

for \(pq = r\). When \(r = p^n\) we denote this map by \(\hat{\Gamma}\). Nikolaus-Scholze call \(\hat{\Gamma}_1: THH(A) \to THH(A)^{tC_r}\) the Frobenius map \(\varphi_{p^n}\). The same map was denoted \(\hat{\gamma}: T(A) \to \hat{T}(A)\), e.g. by Lunøe-Nielsen and Rognes. Let

\[
THH(A)^{tS^1} = \hat{H}(S^1, THH(A)) = [\hat{E}S^1 \wedge F(ES^1, THH(A))]^{S^1}
\]

be the \(S^1\)-Tate construction on \(THH(A)\), which Hesselholt (2016, arXiv) denotes \(TP(A)\). The natural map

\[
THH(A)^{tS^1} \to \text{holim}_{n,F} THH(A)^{tC_r^n}
\]

becomes an equivalence after \(p\)-completion. Hence there is a natural comparison map

\[
\hat{\Gamma}: TF(A; p) \to THH(A)^{tS^1}_{3}
\]
making the following diagram commute.

\[
\begin{array}{ccc}
TF(A; p) & \rightarrow & \text{holim}_{n,F} THH(A)^{C_{\rho^n-1}} \\
\text{\Gamma}_n & \downarrow & \text{holim}_{n,F} \Gamma_n \\
THH(A)^{tS^1}_p & \rightarrow & \text{holim}_{n,F} THH(A)^{tC_{\rho^n}}
\end{array}
\]

The skeleton filtration of $ES^1$ and the induced filtration of $\tilde{E}S^1$ lead to an algebra spectral sequence

\[
\tilde{E}^2_{s,t}(C_r) = \hat{H}^{-s}(C_r; \pi_t THH(A)) \Rightarrow \pi_{s+t} THH(A)^{tC_r}
\]

(Tate cohomology) called the Tate spectral sequence. There is a similar spectral sequence

\[
\hat{E}^2_{s,t}(C_r, \mathbb{Z}/p) = \hat{H}^{-s}(C_r; \pi_t THH(A); \mathbb{Z}/p) \Rightarrow \pi_{s+t} THH(A)^{tC_r; \mathbb{Z}/p}
\]

for homotopy with $\mathbb{Z}/p$-coefficients. In many cases $\pi_*(\hat{\Gamma}_n)$ or $\pi_*(\hat{\Gamma}_n; \mathbb{Z}/p)$ is an isomorphism for $s$ sufficiently large, in which case the $C_{\rho^n}$-Tate spectral sequence gives a way to access the homotopy groups or mod $p$ homotopy groups of the fixed point spectrum $THH(A)^{C_{\rho^n-1}}$.

The Frobenius maps $F: THH(A)^{tC_{\rho^n}} \rightarrow THH(A)^{tC_{\rho^n-1}}$ induce a map of algebra spectral sequences

\[
F: \tilde{E}^2_{s,t}(C_{\rho^n}) \rightarrow \tilde{E}^2_{s,t}(C_{\rho^{n-1}})
\]

converging to

\[
F: \pi_{s}(THH(A)^{tC_{\rho^n}}) \rightarrow \pi_{s}(THH(A)^{tC_{\rho^{n-1}}}).
\]

Passing to the limit over $n$, there is also an algebra spectral sequence

\[
\hat{E}^2_{s,t}(S^1) = \hat{H}^{-s}(S^1; \pi_t THH(A)) \Rightarrow \pi_{s+t} THH(A)^{tS^1}
\]

(Tate cohomology, suitably defined), with $p$-complete and mod $p$ variants. When $\pi_*(\hat{\Gamma})$ is an isomorphism for $s$ sufficiently large, this gives a second way to access the homotopy groups of $TF(A; p)$, usually after $p$-completion and with mod $p$ coefficients.

The Verschiebung maps $V: THH(A)^{tC_{\rho^n-1}} \rightarrow THH(A)^{tC_{\rho^n}}$ also induce a map of spectral sequences

\[
V: \hat{E}^2_{s,t}(C_{\rho^{n-1}}) \rightarrow \hat{E}^2_{s,t}(C_{\rho^n})
\]

converging to

\[
V: \pi_{s}(THH(A)^{tC_{\rho^n-1}}) \rightarrow \pi_{s}(THH(A)^{tC_{\rho^n}}).
\]

5. THE RESTRICTION AND HOMOTOPY RESTRICTION MAPS

By taking the smash product of the $S^1$-equivariant homotopy cofiber sequence

\[
ES^1_+ \rightarrow S^0 \rightarrow \tilde{E}S^1 \rightarrow \Sigma(ES^1_+)
\]

and the natural $S^1$-map

\[
THH(A) \rightarrow F(ES^1_+, THH(A))
\]

and passing to $C_{\rho^n}$-fixed points, we obtain a vertical map of horizontal homotopy cofiber sequences:

\[
\begin{array}{ccc}
THH(A)_{hC_{\rho^n}} & \rightarrow & \text{holim}_{n,F} THH(A)^{C_{\rho^n}} \\
\text{\Gamma}_n & \downarrow & \text{holim}_{n,F} \Gamma_n \\
THH(A)_{hC_{\rho^n}} & \rightarrow & \text{holim}_{n,F} THH(A)^{tC_{\rho^n}}
\end{array}
\]

Note that the middle square is homotopy (co-)cartesian. It follows by induction that each spectrum $THH(A)^{tC_{\rho^n}}$ is connective (assuming that $A$ is connective). There is a map of spectral sequences

\[
R^h: \tilde{E}^2_{s,t}(C_{\rho^n}) = \hat{H}^{-s}(C_{\rho^n}; \pi_t THH(A)) \rightarrow \hat{E}^2_{s,t}(C_{\rho^n}) = \hat{H}^{-s}(C_{\rho^n}; \pi_t THH(A))
\]

converging to

\[
R^h: \pi_{s}THH(A)^{hC_{\rho^n}} \rightarrow \pi_{s}THH(A)^{tC_{\rho^n}}.
\]
Nikolaus-Scholze call $R^h$ the canonical map. In the range where $\pi_*(\Gamma_n)$ and $\pi_*(\hat{\Gamma}_n)$ are isomorphisms, this is identified with

$$R : \pi_*\text{THH}(A)_{p^{\infty}} \longrightarrow \pi_*\text{THH}(A)_{p^{n-1}}.$$ 

Passing to homotopy limits over $n$ using the $F$-maps, we obtain an implicitly $p$-completed vertical map of horizontal homotopy cofiber sequences:

$$\Sigma \text{THH}(A)_{hS^1} \xrightarrow{N} TF(A; p) \xrightarrow{R} TF(A; p) \xrightarrow{\partial} \Sigma^2 \text{THH}(A)_{hS^1}$$

$$\Sigma \text{THH}(A)_{hS^1} \xrightarrow{N^h} \text{THH}(A)^{hS^1} \xrightarrow{R^h} \text{THH}(A)^{hS^1} \xrightarrow{\partial^h} \Sigma^2 \text{THH}(A)_{hS^1}$$

Again there is a map of spectral sequences

$$R^h : E^2_{s,t}(S^1) = H^{-s}(S^1; \pi_*\text{THH}(A)) \longrightarrow E^2_{s,t}(S^1) = \hat{H}^{-s}(S^1; \pi_*\text{THH}(A))$$

converging to

$$R^h : \pi_*\text{THH}(A)^{hS^1} \longrightarrow \pi_*\text{THH}(A)^{hS^1}.$$ 

In the range where $\pi_*(\hat{\Gamma}_1)$ is an isomorphism, this is identified with

$$R : \pi_*TF(A; p) \longrightarrow TF(A; p).$$

6. The homotopy limit property for finite cyclic groups

We say that a map $f : X \rightarrow Y$ is $k$-coconnected if $\pi_*(f) : \pi_*(X) \rightarrow \pi_*(Y)$ is injective for $* = k$ and an isomorphism for $* > k$. Note that $\Gamma_n : \text{THH}(A)_{p^{\infty}} \rightarrow \text{THH}(A)_{p^{n-1}}$ is $k$-coconnected if and only if

$$\hat{\Gamma}_n : \text{THH}(A)_{p^{\infty}} \rightarrow \text{THH}(A)_{p^{n-1}}$$

is $k$-coconnected.

**Theorem 6.1** (Tsalidis). Let $A$ be bounded below with $H_*^s(A; Z/p)$ of finite type. If $\Gamma_1 : \text{THH}(A)_{p^{\infty}} \rightarrow \text{THH}(A)_{p^{n-1}}$ is $k$-coconnected, then $\pi_*\text{THH}(A)_{p^{\infty}} \rightarrow \text{THH}(A)_{p^{n-1}}$ is $k$-coconnected for each $n \geq 1$.

This was proved by Tsalidis (1998?), generalizing earlier work by Ravenel. Later joint work by Bökstedt, Bruner, Lurie-Nielsen and Rognes (2014) generalized the result further.

**Theorem 6.2.** Let $A$ be bounded below with $H_*^s(A; Z/p)$ of finite type. If $\Gamma_1 : \text{THH}(A)_{p^{\infty}} \wedge S/p \rightarrow \text{THH}(A)_{p^{n-1}} \wedge S/p$ is $k$-coconnected, then $\hat{\Gamma}_n : \text{THH}(A)_{p^{\infty}} \wedge S/p \rightarrow \text{THH}(A)_{p^{n-1}} \wedge S/p$ is $k$-coconnected for each $n \geq 1$.

7. Group and Tate cohomology

Let $p$ be an odd prime. The group- and Tate cohomology of $C_{p^n}$ with coefficients in $Z/p$ are given by

$$H^*(C_{p^n}; Z/p) = E(u_n) \otimes P(t)$$

$$\hat{H}^*(C_{p^n}; Z/p) = E(u_n) \otimes P(t, t^{-1})$$

with $u_n \in H^1 \cong \hat{H}^1$ and $t \in H^2 \cong \hat{H}^2$. Here $E(u_n) = Z/p[u_n]/(u_n^2)$ is the exterior algebra, $P(t) = Z/p[t]$ is the polynomial algebra, and $P(t, t^{-1}) = Z/p[t, t^{-1}]$ is the Laurent polynomial algebra. The canonical homomorphism

$$R^h : H^*(C_{p^n}; Z/p) \longrightarrow \hat{H}^*(C_{p^n}; Z/p)$$

inverts t.

The inclusion $C_{p^{n-1}} \rightarrow C_{p^n}$ induces homomorphisms

$$F : H^*(C_{p^n}; Z/p) \longrightarrow H^*(C_{p^{n-1}}; Z/p)$$

$$\hat{F} : \hat{H}^*(C_{p^n}; Z/p) \longrightarrow \hat{H}^*(C_{p^{n-1}}; Z/p)$$

given by $F(u_n) = 0$ and $\hat{F}(t) = t$. In the limiting case

$$H^*(S^1; Z/p) = P(t)$$

$$\hat{H}^*(S^1; Z/p) = P(t, t^{-1})$$

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with \( t \in H^2 \cong \hat{H}^2 \).

There are also Verschiebung (transfer) homomorphisms
\[
V : H^*(C_{p^{n-1}}; \mathbb{Z}/p) \to H^*(C_{p^n}; \mathbb{Z}/p)
\]
\[
\hat{V} : \hat{H}^*(C_{p^{n-1}}; \mathbb{Z}/p) \to \hat{H}^*(C_{p^n}; \mathbb{Z}/p)
\]
given by \( V(u_{n-1}) = u_n \) and \( V(t) = 0 \).

8. Calculations for \( \mathbb{Z}/p \)

Consider the case \( A = H\mathbb{Z}/p \). Recall that \( \pi_\ast\text{THH}(\mathbb{Z}/p) = P(\mu_0) = \mathbb{Z}/p[\mu_0] \), where \( \mu_0 \in \pi_2\text{THH}(\mathbb{Z}/p) \) has mod \( p \) Hurewicz image \( \sigma_\ast \tau_0 \in H_2(\text{THH}(\mathbb{Z}/p)) \).

**Theorem 8.1.** The \( S^1 \)-homotopy fixed point spectral sequence
\[
E^2_{*,*}(S^1) = P(t) \otimes P(\mu_0) \Rightarrow \pi_\ast\text{THH}(\mathbb{Z}/p)^{hS^1}
\]
collapses at the \( E^2 \)-term. There are additive extensions, so that the image of \( p \in \pi_0(S) = \mathbb{Z} \) is represented by (a unit in \( \mathbb{Z}/p \) times)
\[
t_\mu_0 \in E^\infty_{-2,2}(S^1).
\]

Hence
\[
\pi_\ast\text{THH}(\mathbb{Z}/p)^{hS^1} \cong \mathbb{Z}_p[t, \mu_0] / (t\mu_0 = p)
\]
with \( t \in \pi_{-2} \) and \( \mu_0 \in \pi_2 \). Additively
\[
\pi_\ast\text{THH}(\mathbb{Z}/p)^{hS^1} \cong \begin{cases} 
\mathbb{Z}_p\{\mu_0^n\} & \text{for } i = 2n \geq 0, \\
\mathbb{Z}_p\{tm\} & \text{for } i = -2m \leq 0, \\
0 & \text{otherwise}.
\end{cases}
\]

**Proof.** The \( E^2 \)-term is concentrated in even total degrees, so all differentials are zero. We claim that the unit map \( S \to \text{THH}(\mathbb{Z}/p)^{hS^1} \) maps \( p \in \pi_0(S) \) to a class detected by \( t\mu_0 \) (times a unit in \( \mathbb{Z}/p \)) in \( E^\infty_{-2,2}(S^1) \).

The first terms of the \( S^1 \)-equivariant skeleton filtration of \( ES^1_+ = S(\mathbb{C}^\infty)_+ \) are
\[
S^1_+ \to S^3_+ \to \cdots \to ES^1_+,
\]
with \( S^3 = S(\mathbb{C}^2) \), and there is a homotopy cofiber sequence \( S^1_+ \to S^3_+ \to \Sigma^2(S^1_+) \to \Sigma(S^1_+) \).

There are induced maps
\[
\text{THH}(A)^{hS^1} \to \cdots \to F(S^3_+, \text{THH}(A))S^1 \xrightarrow{\pi} \text{THH}(A)
\]
and a homotopy (co-)fiber sequence
\[
\Sigma^{-2}\text{THH}(A) \to F(S^3_+, \text{THH}(A))S^1 \xrightarrow{\pi} \text{THH}(A) \xrightarrow{\sigma} \Sigma^{-1}\text{THH}(A).
\]
For brevity, let

\[ X = F(S^3_+, \text{THH}(A))^{S^1}. \]

It suffices to prove that \( p \) maps to a nonzero class in \( \pi_0 X \), since the image of \( p \) must be then be detected in \( E^{2,2}_2 = \mathbb{Z}/p\{t\mu_0\} \).

We do this by calculating \( H_* X \) and considering the map of Adams spectral sequences

\[ E_2^{*,*}(S) = \text{Ext}_{\mathcal{A}_*}^*(\mathbb{Z}/p, \mathbb{Z}/p) \longrightarrow \text{Ext}_{\mathcal{A}_*}^*(\mathbb{Z}/p, H_* X) = E_2^{*,*}(X). \]

There is a long exact sequence

\[ \cdots \longrightarrow H_{*+2} \text{THH}(A) \longrightarrow H_* X \longrightarrow H_* \text{THH}(A) \xrightarrow{\sigma} H_{*+1} \text{THH}(A) \longrightarrow \cdots \]

where \( \sigma \) is the homological \( \sigma \)-operator. For \( A = H\mathbb{Z}/p \), recall that \( H_* \text{THH}(\mathbb{Z}/p) = \mathcal{A}_* \otimes P(\mu_0) \), where \( \sigma(\tau_0) = \mu_0 \). It follows that \( H_0 X \cong \mathbb{Z}/p\{1\} \) and \( H_1 X \cong \mathbb{Z}/p\{t\tau_0\mu_0\} \). The class \( p \in \pi_0(S) \) is represented (up to sign) by the cohomology class in

\[ E_2^{1,1}(S) = \text{Ext}_{\mathcal{A}_*}^{1,1}(\mathbb{Z}/p, \mathbb{Z}/p) \]

of the 1-cocycle \( [\tau_0] \) in the normalized cobar complex

\[ 0 \to \mathbb{Z}/p \xrightarrow{d_1^1} \mathcal{A}_* \xrightarrow{d_2^1} \mathcal{A}_* \otimes \mathcal{A}_* \to \cdots. \]

Its image in \( E_2^{1,1}(X) = \text{Ext}_{\mathcal{A}_*}^{1,1}(\mathbb{Z}/p, H_* X) \) is the cohomology class of the 1-cocycle \( [\tau_0] \) in the normalized cobar complex

\[ 0 \to H_* X \xrightarrow{d_1^0} \mathcal{A}_* \otimes H_* X \xrightarrow{d_2^1} \mathcal{A}_* \otimes \mathcal{A}_* \otimes H_* X \to \cdots. \]

This cohomology class remains nonzero, because the 1-cocycle is not a 1-coboundary. Indeed, the only possible 0-cochain cobounding it (in internal degree 1) would be \( t\tau_0\mu_0 \in H_1 X \). However, a direct calculation shows that \( \nu(t\tau_0\mu_0) = 1 \otimes t\tau_0\mu_0 \), so \( d_1^0(t\tau_0\mu_0) = 0 \). Hence the class of \( [\tau_0] \) survives to \( E_\infty \), and represents a nonzero element in \( \pi_0(X) \).

\[ \square \]

**Proposition 8.2.** The map \( \hat{\Gamma}_1 : \text{THH}(\mathbb{Z}/p) \to \text{THH}(\mathbb{Z}/p)^{\text{T}} \) is \((-2)\)-coconnected. Hence each map \( \Gamma_n, \Gamma, \hat{\Gamma}_n \) and \( \hat{\Gamma} \) is \((-2)\)-coconnected.

**Corollary 8.3.**

\[ \pi_* \text{TF}(\mathbb{Z}/p; p) \cong \mathbb{Z}_p[\mu_0] \]

with

\[ \Gamma : \pi_* \text{TF}(\mathbb{Z}/p; p) \longrightarrow \pi_* \text{THH}(\mathbb{Z}/p)^{hS^1} \]

given by \( \Gamma(\mu_0) = \mu_0 \). Additively

\[ \pi_* \text{TF}(\mathbb{Z}/p; p) \cong \begin{cases} \mathbb{Z}_p[\mu_0^i] & \text{for } i = 2n \geq 0, \\ 0 & \text{otherwise}. \end{cases} \]
Proof of Proposition 8.2. Consider the $C_p$-Tate spectral sequence

$$
\tilde{E}^2_{*,*}(C_p) = H^{-*}(C_p, \pi_* THH(\mathbb{Z}/p)) \cong E(u_1) \otimes P(t, t^{-1}) \otimes P(\mu_0)
$$

$$
\Rightarrow \pi_* THH(\mathbb{Z}/p)^{tC_p}.
$$

By naturality with respect to

$$
\begin{array}{c}
\mathbb{Z}/p hS^1 \xrightarrow{E} \mathbb{Z}/p hC_p \xrightarrow{E^2} \mathbb{Z}/p hC_p
\end{array}
$$

the classes $t$ and $\mu_0$ are infinite cycles. We claim that $d^3(u_1 t^{-1}) = t \mu_0$, so that

$$
\tilde{E}^3_{*,*}(C_p) \cong P(t, t^{-1}).
$$

Hence this spectral sequence collapses at $E^3$, and

$$
\pi_* THH(\mathbb{Z}/p)^{tC_p} \cong \mathbb{Z}/p[t, t^{-1}].
$$

Finally, we claim that $\tilde{\Gamma}_1(\mu_0) \cong t^{-1}$. Then

$$
\tilde{\Gamma}_1: \mathbb{Z}/p[\mu_0] \longrightarrow \mathbb{Z}/p[t, t^{-1}]
$$

inverts $\mu_0$, and is $(-2)$-cocompact.

To show that $d^3(u_1 t^{-1}) = t \mu_0$, it is enough to show that $t \mu_0$ is a $d'$-boundary for some $r$, i.e., that the image of $p \in \pi_0 S$ in $\pi_0 THH(\mathbb{Z}/p)^{hS^1}$ maps to zero in $\pi_0 THH(\mathbb{Z}/p)^{tC_p}$. To see this, we use the factorization

$$
S \longrightarrow THH(\mathbb{Z}/p) \longrightarrow THH(\mathbb{Z}/p)^{tC_p}
$$

of the unit map. The image of $p \in \pi_0(S)$ is zero in $\pi_0 THH(\mathbb{Z}/p) = \mathbb{Z}/p$ hence must be mapped to zero by $\tilde{\Gamma}_1$, as claimed.

To show that $\tilde{\Gamma}_1(\mu_0) \cong t^{-1}$, Hesselholt-Madsen (1997) show that $\partial(\mu_0) \neq 0$ in $\pi_1 THH(\mathbb{Z}/p)^{hC_p} \cong \mathbb{Z}/p$. This uses the homotopy orbit spectral sequence

$$
H_s(C_p, \pi_0 THH(A)) \Rightarrow \pi_{s+t} THH(A)^{hC_p}
$$

to calculate $\pi_* THH(\mathbb{Z}/p)^{hC_p} = (\mathbb{Z}/p, \mathbb{Z}/p, \ldots)$, and the factorization

$$
S \xrightarrow{1\Gamma_1} THH(\mathbb{Z}/p)^{C_p} \xrightarrow{1\Gamma_1} THH(\mathbb{Z}/p)^{hC_p}
$$

to show that $\pi_0 THH(\mathbb{Z}/p)^{C_p} \cong \mathbb{Z}/p^2$. It follows that $\partial: \pi_1 THH(\mathbb{Z}/p); \mathbb{Z}/p) \rightarrow \pi_0 THH(\mathbb{Z}/p)^{hC_p}; \mathbb{Z}/p)$ maps the generator, with Hurewicz image $\bar{\sigma}_0$, to a non-zero class.

The mod $p$ Hurewicz image of $\mu_0$ is $\sigma_0$, so by a diagram chase it suffices to check that $s$ acts nontrivially on $\pi_0 THH(\mathbb{Z}/p)^{hC_p}$. This follows, e.g., by a comparison with the $S^1$-action on $S hC_p = BC_{p^+}$. 

\begin{theorem}
The $S^1$-Tate spectral sequence

$$
\tilde{E}^2_{*,*}(S^1) = P(t, t^{-1}) \otimes P(\mu_0) \Rightarrow \pi_* THH(\mathbb{Z}/p)^{tS^1}
$$

collapses at the $E^2$-term. The image of $p \in \pi_0(S) = \mathbb{Z}$ is represented by (a unit in $\mathbb{Z}/p$ times)

$$
t \mu_0 \in \tilde{E}^\infty_{*,*}(S^1).
$$

Hence

$$
\pi_* THH(\mathbb{Z}/p)^{tS^1} \cong \mathbb{Z}/p[t, t^{-1}].
$$
\end{theorem}
with \( t \in \pi_{-2} \). Additively

\[
\pi_i \THH(\mathbb{Z}/p)^{TS^1} \cong \begin{cases} \mathbb{Z}_p \{t^m\} & \text{for } i = -2m, \\ 0 & \text{otherwise.} \end{cases}
\]

Corollary 8.5. \( \hat{\Gamma}: \pi_* \TF(\mathbb{Z}/p; p) \to \pi_* \THH(\mathbb{Z}/p)^{TS^1} \) is given by

\[ \hat{\Gamma}(\mu_0) = t^{-1}. \]

Corollary 8.6. \( R^h: \pi_* \THH(\mathbb{Z}/p)^{hS^1} \to \pi_* \THH(\mathbb{Z}/p)^{TS^1} \) is given by

\[ R^h(\mu_0) = \mu_0 = pt^{-1}, \quad R^h(t) = t. \]

Hence \( R: \pi_* \TF(\mathbb{Z}/p; p) \to \pi_* \TF(\mathbb{Z}/p; p) \) is given by

\[ R(\mu_0) = p\mu_0. \]

Theorem 8.7 (Hesselholt-Madsen (1997)). \( \pi_*(\text{id} - R): \pi_* \TF(\mathbb{Z}/p; p) \to \TF(\mathbb{Z}/p; p) \) is zero for \( * = 0 \), and an isomorphism for \( * \neq 0 \). Hence

\[ \pi_i \TC(\mathbb{Z}/p; p) \cong \begin{cases} \mathbb{Z}_p & \text{for } i \in \{-1, 0\}, \\ 0 & \text{otherwise.} \end{cases} \]

The \( p \)-completed cyclotomic trace map \( \trc: K(\mathbb{Z}/p)_p \to TC(\mathbb{Z}/p; p) \) is \((-1)\)-coconnected.

9. The Nikolaus-Scholze reformulation

Note that \( (\THH(A)^{C_p})^{C_p^n} = \THH(A)^{C_p^n+1} \). By naturality of the comparison map \( \Gamma_n \) for the \( S^1 \)-map \( \hat{\Gamma}_1: \THH(A) \to \THH(A)^{C_p} \) we have a commutative square

\[
\begin{array}{ccc}
\THH(A)^{C_p^n} & \xrightarrow{\Gamma_{n+1}} & \THH(A)^{C_p^{n+1}} \\
\xrightarrow{\Gamma_n} & & \xrightarrow{G_n} \\
\THH(A)^{hC_p^n} & \xrightarrow{\hat{\Gamma}_{1C_p^n}} & (\THH(A)^{C_p})^{hC_p^n}
\end{array}
\]

for each \( n \geq 0 \). In the course of calculations, the map \( G_n \) had been seen to be an equivalence in many cases, including \( A = H\mathbb{Z}/p, H\mathbb{Z}, \ell, ku \) and \( S \). Nikolaus-Scholze prove that it is an equivalence for all connective (symmetric) ring spectra \( A \).
Passing to homotopy limits over $n$ using the $F$-maps, we obtain an implicitly $p$-completed commutative square

(5)

\[
\begin{array}{ccc}
TF(A,p) & \xrightarrow{\Gamma} & THH(A)^{tS^1} \\
| & & | \\
\Gamma & \downarrow & \Gamma \\
THH(A)^{hS^1} & \xrightarrow{\approx} & (THH(A)^{tC_p})^{hS^1} \\
\end{array}
\]

In the range where $\Gamma$ and $\hat{\Gamma}$ are equivalences, we can therefore replace id and $R: TF(A;p) \to TF(A;p)$ by $\hat{\Gamma}_1^{hS^1}$ and $GR^h$, respectively:

\[
\begin{array}{ccc}
TF(A;p) & \xrightarrow{id} & TF(A;p) \\
| & & | \\
\Gamma & \downarrow & \Gamma \\
THH(A)^{hS^1} & \xrightarrow{\approx} & (THH(A)^{tC_p})^{hS^1} \\
\end{array}
\]

(These diagrams were used in the TC-calculation by Ausoni, Bökstedt, Hesselholt, Madsen and Rognes.)

Switching notations to write $\phi_p$ for $\hat{\Gamma}_1$ and can for $R^h$, it follows that $TC(A;p)$ is equivalent to the homotopy equalizer of $\phi_p^{hS^1} = \hat{\Gamma}_1^{hS^1}$ and $G$ can $= GR^h$

\[
TC(A;p) \xrightarrow{\phi_p^{hS^1}} THH(A)^{hS^1} \xrightarrow{G \text{ can}} (THH(A)^{tC_p})^{hS^1},
\]

at least in the range where $\pi_*(\phi_p)$ is an isomorphism. A second insight by Nikolaus-Scholze is that this diagram is a homotopy equalizer in all degrees, i.e., without the restriction to degrees where $\phi_p$, $\Gamma$ and $\hat{\Gamma}$ are equivalences.

Hence this diagram can be taken as a revised definition of $TC(A;p)$ for connective (symmetric) ring spectra $A$. It has the advantage that genuinely $S^1$-equivariant homotopy type of $THH(A)$ only appears in the construction of $\phi_p$: $THH(A) \to THH(A)^{tC_p}$, and this can be achieved by $\infty$-categorical methods. The remaining steps, forming $C_p$-Tate constructions and $S^1$-homotopy fixed points only depend on the naively $S^1$-equivariant homotopy type of $THH(A)$, for which the preliminary construction as $|A \otimes S^*_p|$ suffices.

**Theorem 9.1.** (a) $\pi_* THH(\mathbb{Z}/p) = \mathbb{Z}/[\mu_0]$, and $\pi_* THH(\mathbb{Z}/p)^{tC_p} = \mathbb{Z}/[\mu_0, \mu_0^{-1}]$, with $\phi_p(\mu_0) = \mu_0$.

(b) $\pi_* THH(\mathbb{Z}[t,p]/(p \not\equiv t \mu_0))$ and $\pi_* (THH(\mathbb{Z}/p)^{tC_p})^{hS^1} = \mathbb{Z}[t, \mu_0, \mu_0^{-1}]/(p \not\equiv \mu_0) = \mathbb{Z}[\mu_0, \mu_0^{-1}]$, with $\phi_p^{hS^1}: t \mapsto t \equiv \mu_0^{-1}$ and $\mu_0 \mapsto \mu_0$.

(c) $\pi_* THH(\mathbb{Z}[t])^{hS^1} = \mathbb{Z}[t, t^{-1}, \mu_0]/(p \not\equiv \mu_0) = \mathbb{Z}[t, t^{-1}], \ \text{with \ can}(t) = t$, $\text{can}(\mu_0) = \mu_0$ and $G(t) \equiv \mu_0^{-1}$. Hence $G$ can: $t \mapsto \mu_0^{-1}$ and $\mu_0 \mapsto p \mu_0$.

**Theorem 9.2.** $\pi_* (\phi_p^{hS^1} - G \text{ can}) : \pi_* THH(\mathbb{Z}/p)^{hS^1} \to \pi_* (THH(\mathbb{Z}/p)^{tC_p})^{hS^1}$ is zero for $* = 0$ and an isomorphism for $* \neq 0$. Hence

$$
\pi_i TC(\mathbb{Z}/p, p) \cong \begin{cases} 
\mathbb{Z}_p & \text{for } i \in \{-1, 0\}, \\
0 & \text{otherwise}.
\end{cases}
$$
Lemma 10.1. The unit map \( (1998 \text{ and } 1999) \) for \( p \)

Proof. It suffices to prove that \( \mathbb{Z} \) closer to \( \mathbb{Z} \) calculations are the same for \( t \lambda \) similar to the argument for why \( \square \)

The proof uses a comparison of \( \alpha \) Ext \( \mathbb{THH} \) the class \( \mathbb{THH} \) Proposition 10.3 \( p \) from integral to mod \( \mathbb{THH} \) Proof. It suffices to prove that \( \mathbb{THH} \) coefficents. The case of \( \mathbb{THH} \) follows by naturality with respect to the Bockstein \( 1 \lambda \) \( \mathbb{THH} \) spectral sequence for \( 1 \theta \) \( \mathbb{THH} \) odd, and to Rognes \( 1 \mathbb{THH} \) for \( p = 2 \). The map \( \mathbb{THH}(\mathbb{Z}) \land S/p \rightarrow \mathbb{THH}(\mathbb{Z}_p) \land S/p \) is an equivalence, so the mod \( p \) calculations are the same for \( \mathbb{Z} \) and \( \mathbb{Z}_p \). We write \( \mathbb{THH}(\mathbb{Z}) \) for brevity, although the case \( \mathbb{Z}_p \) is conceptually closer to \( \mathbb{Z}/p \).

Let \( p \) be an odd prime, and write \( \pi_* X = \pi_*(X; \mathbb{Z}/p) = \pi_*(X \land S/p) \).

Lemma 10.2. The unit map \( S \rightarrow \mathbb{THH}(\mathbb{Z})^{hS^1} \) takes \( \alpha_1 \in \pi_{2p-3}(S) \) to class represented by (a unit in \( \mathbb{Z}/p \) times) \( t\lambda_1 \) in the \( S^1 \)-homotopy fixed point spectral sequence

\[
E^2_{*,*}(S^1) = \mathbb{Z}[t] \otimes \pi_\ast \mathbb{THH}(\mathbb{Z}) \Rightarrow \pi_\ast \mathbb{THH}(\mathbb{Z})^{hS^1}.
\]

Proof. It suffices to prove that \( \alpha_1 \) is detected in \( \pi_{2p-3}(X) \) where \( X = \mathbb{F}(S^3, \mathbb{THH}(\mathbb{Z}))^{S^1} \). The proof is similar to the argument for why \( p \) is detected in \( \pi_0 \mathbb{F}(S^3_+, \mathbb{THH}(\mathbb{Z}_p)) \), using the cobar representative \( \xi_1 \) for \( \alpha_1 \) in place of \( \hat{\pi}_0 \). The key point is that \( \sigma(\xi_1) \neq 0 \), so that \( \xi_1 \in H_* \mathbb{THH}(\mathbb{Z}) \) does not lift to \( H_* X \) in the long exact sequence (3). Hence \( \xi_1 \) does not become a coboundary in the cobar complex calculating Ext^\ast_{\mathbb{Z}/p}(\mathbb{Z}/p, H_* X) \). \( \square \)

Lemma 10.3. The unit map takes \( \alpha_1 \in \pi_{2p-3}(S) \) and \( v_1 \in \pi_{2p-3}(S) \) to classes represented by (units in \( \mathbb{Z}/p \) times) \( t\lambda_1 \) and \( t\mu_1 \) in the \( S^1 \)-homotopy fixed point spectral sequence

\[
E^2_{*,*}(S^1; \mathbb{Z}/p) = \mathbb{P}(t) \otimes \mathbb{E}(\lambda_1) \otimes \mathbb{P}(\mu_1) \Rightarrow \pi_\ast \mathbb{THH}(\mathbb{Z})^{hS^1}.
\]

Proof. It suffices to prove that \( \alpha_1 \) and \( v_1 \) are detected in \( \pi_*(X) \). The case of \( \alpha_1 \) follows by reduction from integral to mod \( p \) coefficients. The case of \( v_1 \) follows by naturality with respect to the Bockstein homomorphism, since \( \beta(v_1) = \alpha_1 \). \( \square \)

Proposition 10.3 (Bökstedt). There is a class \( \lambda^K_1 \in \pi_* K(\mathbb{Z}) \) with \( \text{tr}(\lambda^K_1) = \lambda_1 \) in \( \pi_* \mathbb{THH}(\mathbb{Z}) \).

Proof. The proof uses a comparison of \( A(\ast) = K(S) \) with \( K(\mathbb{Z}) \) along the linearization map \( S \rightarrow H\mathbb{Z} \). See Bökstedt-Madsen (1994) and Rognes (1998).

((Q: Is this a priori knowledge essential?))

Corollary 10.4. The class \( \lambda_1 \) is an infinite cycle in the \( S^1 \)-homotopy fixed point spectral sequence for \( \mathbb{THH}(\mathbb{Z}) \), hence also in the \( C_{p^n} \)-homotopy fixed point, \( S^1 \)-Tate and \( C_{p^n} \)-Tate spectral sequences.
Proof. The image of $\lambda^f$ under
\[ K(\mathbb{Z}) \xrightarrow{\text{tr}} TC(\mathbb{Z}; p) \to TF(\mathbb{Z}; p) \xrightarrow{\text{tr}} THH(\mathbb{Z})^{hS^1} \]
is mapped to $\lambda_1$ under the edge homomorphism
\[ F: THH(\mathbb{Z})^{hS^1} \to THH(\mathbb{Z}), \]
hence is represented by $\lambda_1 \in E_{0,2p-1}(S^1)$. \hfill $\square$

**Proposition 10.5.** The map $\tilde{\Gamma}_1 \land S/p: THH(\mathbb{Z}) \land S/p \to THH(\mathbb{Z})^{t_{C_p}} \land S/p$ is $(-1)$-coconnected.

**Corollary 10.6.** Each map $\Gamma_0 \land S/p, \Gamma \land S/p, \tilde{\Gamma}_0 \land S/p$ and $\tilde{\Gamma} \land S/p$ is $(-1)$-connected.

**Proof of Proposition 10.5.** Consider the $C_p$-Tate spectral sequence
\[ \tilde{E}_2^{2,1}(C_p; \mathbb{Z}/p) = H^{-2}(C_p; \pi_*THH(\mathbb{Z})) \cong E(u_1) \otimes P(t, t^{-1}) \otimes E(\lambda_1) \otimes P(\mu_1) \]
\[ \Longrightarrow \tilde{\pi}_*THH(\mathbb{Z})^{t_{C_p}}. \]
By naturality with respect to $THH(\mathbb{Z})^{hS^1} \to THH(\mathbb{Z})^{t_{C_p}}$, the images of the classes $\alpha_1$ and $v_1$ are represented by $t\lambda_1$ and $t\mu_1$, respectively. Since the unit map factors through
\[ \tilde{\Gamma}_1: THH(\mathbb{Z}) \to THH(\mathbb{Z})^{t_{C_p}} \]
and $\alpha_1$ and $v_1$ map to zero in $\tilde{\pi}_*THH(\mathbb{Z})$, the classes $t\lambda_1$ and $t\mu_1$ must be boundaries in the $C_p$-Tate spectral sequence. We must have $d^2(\lambda_1) = 0$ by naturality with respect to mod $p$ reduction. Hence the only possible differentials are
\[ d^{2p}(t^{1-p}) = t\lambda_1 \]
\[ d^{2p+1}(u_1t^{-p}) = t\mu_1. \]
Writing
\[ \tilde{E}_2^{2,1}(C_p; \mathbb{Z}/p) = E(u_1) \otimes P(t, t^{-1}) \otimes E(\lambda_1) \otimes P(t\mu_1) \]
the $d^{2p}$-differential cancels the $t^i$-terms for $p \nmid i$, i.e., for $v_p(i) = 0$. Thus
\[ \tilde{E}_2^{2p+1}(C_p; \mathbb{Z}/p) = E(u_1) \otimes P(t^p, t^{-p}) \otimes E(\lambda_1) \otimes P(t\mu_1) \]
and
\[ \tilde{E}_2^{2p+2}(C_p; \mathbb{Z}/p) = P(t^p, t^{-p}) \otimes E(\lambda_1). \]
There is no room for further differentials, so this is also $\tilde{E}_2^{\infty}(C_p; \mathbb{Z}/p)$. Hence
\[ \tilde{\pi}_*THH(\mathbb{Z})^{t_{C_p}} \cong E(\lambda_1) \otimes P(t^p, t^{-p}). \]
In the commutative square
\[ \tilde{\pi}_*THH(\mathbb{Z}) \xrightarrow{\tilde{\Gamma}_1} \tilde{\pi}_*THH(\mathbb{Z})^{t_{C_p}} \]
\[ \tilde{\pi}_*THH(\mathbb{Z}/p) \xrightarrow{\tilde{\Gamma}_1} \tilde{\pi}_*THH(\mathbb{Z}/p)^{t_{C_p}} \]
the class $\mu_1$ maps down to $\mu_0^p$ (since $\sigma\tilde{\Gamma}_1 = (\sigma\Gamma_1)^0$), which maps across to $t^{-p}$ (up to a unit in $\mathbb{Z}/p$). Hence $\tilde{\Gamma}_1(\mu_1) = t^{-p}$.

We claim that $\tilde{\Gamma}_1(\lambda_1) = \lambda_1$. Then the induced homomorphism in mod $p$ homotopy,
\[ \tilde{\Gamma}_1: E(\lambda_1) \otimes P(\mu_0) \to E(\lambda_1) \otimes P(t^p, t^{-p}), \]
inverts $\mu_1$ and is $(-1)$-connected.

To prove the claim, consider $\lambda_1 = \text{tr}_p(\lambda^f) \in \tilde{\pi}_*THH(\mathbb{Z})^{C_p}$. We have $R(\tilde{\lambda}_1) = \lambda_1$ and $F(\tilde{\lambda}_1) = \lambda_1$, so $\Gamma(\tilde{\lambda}_1)$ is represented by $\lambda_1$ in the $C_p$-homotopy fixed point spectral sequence. Hence $\tilde{\Gamma}_1(\lambda_1) = H^0\Gamma(\tilde{\lambda}_1)$ is represented by $\lambda_1$ in the $C_p$-Tate spectral sequence. \hfill $\square$

We write $p^b + \cdots + p^d$ for the truncated geometric series $\sum_{i=a}^{b} p^i$, and let $P_h(x) = P(x)/(x^h)$ be the truncated polynomial algebra of height $h$. 

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Theorem 10.7. The $S^1$-Tate spectral sequence
\[ \hat{E}^*_r(S^1; \mathbb{Z}/p) = \hat{H}^{-*}(S^1; \pi_* THH(\mathbb{Z})) \cong P(t, t^{-1}) \otimes E(\lambda_1) \otimes P(\mu_1) \]
\[ \implies \hat{\pi}_* THH(\mathbb{Z})^{tS^1} \]
has differentials
\[ d^r(p^k - p^{k+1}) \cong t^{p^k \cdot \lambda_1 \cdot (t \mu_1)}p^{k+\cdots+p} \]
with $r = 2(p^{k+1} + \cdots + p)$, for all $k \geq 0$. The $E^\infty$-term is
\[ \hat{E}^{\infty}_r(S^1; \mathbb{Z}/p) = E(\lambda_1) \otimes P(t, t^{-1}) \otimes E(\lambda_1) \otimes P(\mu_1) \]
\[ \implies \hat{\pi}_* THH(\mathbb{Z})^{C_{p^n}} \]
has $n$ even length differentials
\[ d^r(p^k - p^{k+1}) \cong t^{p^k \cdot \lambda_1 \cdot (t \mu_1)}p^{k+\cdots+p} \]
with $r = 2(p^{k+1} + \cdots + p)$, for all $0 \leq k < n$, followed by an odd length differential
\[ d^r(u_n t^{-p^n}) \cong (t \mu_1)p^{n-1+\cdots+1} \]
for $r = 2(p^n + \cdots + p) + 1$. This assumption holds for $n = 1$ by the proof of Proposition 10.5.

The intermediate terms in the $C_{p^n}$-Tate spectral sequence are then
\[ \hat{E}^*_r(C_{p^n}; \mathbb{Z}/p) = E(u_n) \otimes P(t^{p^k}, t^{-p^k}) \otimes E(\lambda_1) \otimes P(t \mu_1) \]
\[ \oplus \bigoplus_{v_p(i) = m < k} E(u_n) \otimes P_{p^{m} + \cdots + p}(t \mu_1)\{t^i \lambda_1 \mid v_p(i) = m\} \]
for $0 \leq k \leq n$ and $r = 2(p^k + \cdots + p) + 1$. For $k = 0$, this is the known formula for the $E^2$-term. Assuming the formula for some $0 \leq k < n$, the $d^r$-differential with $r = 2(p^{k+1} + \cdots + p)$ acts on the $t^i$-terms with $v_p(i) = k$, replacing
\[ E(u_n) \otimes P(t^{p^k}, t^{-p^k}) \otimes E(\lambda_1) \otimes P(t \mu_1) \]
with the direct sum of
\[ E(u_n) \otimes P(t^{p^{k+1}}, t^{-p^{k+1}}) \otimes E(\lambda_1) \otimes P(t \mu_1) \]
and
\[ E(u_n) \otimes P_{p^{m} + \cdots + p}(t \mu_1)\{t^i \lambda_1 \mid v_p(i) = k\} \].
This verifies the formula for $k + 1$. By finite induction it also holds for $k = n$.

The odd length $d^r$-differential replaces
\[ E(u_n) \otimes P(t^{p^n}, t^{-p^n}) \otimes E(\lambda_1) \otimes P(t \mu_1) \]
with
\[ P(t^{p^n}, t^{-p^n}) \otimes E(\lambda_1) \otimes P_{p^{n-1} + \cdots + 1}(t \mu_1) \).
Hence
\[ \hat{E}^*_r(C_{p^n}; \mathbb{Z}/p) = P(t^{p^n}, t^{-p^n}) \otimes E(\lambda_1) \otimes P_{p^{n-1} + \cdots + 1}(t \mu_1) \]
\[ \oplus \bigoplus_{v_p(i) = m < n} E(u_n) \otimes P_{p^{m} + \cdots + p}(t \mu_1)\{t^i \lambda_1 \mid v_p(i) = m\} \]
for $r = 2(p^n + \cdots + 1)$. This is also the $E^\infty$-term, because there is no room for further differentials.
The $C_p$-homotopy spectral sequence

$$E^2_{*,*}(C_p; \mathbb{Z}/p) = H^{-*}(C_p; \pi_* \text{THH}(\mathbb{Z})) \cong E(u_n) \otimes P(t) \otimes E(\lambda) \otimes P(\mu_1)$$

$$\implies \pi_* \text{THH}(\mathbb{Z})^{hC_p}$$

is obtained by truncating the (upper half-plane) $C_p$-Tate spectral sequence to calculate $\bar{\pi}_* \text{THH}(\mathbb{Z})^{hC_p}$. Since

$$\pi_* \text{THH}(\mathbb{Z})^{hC_p}$$

is algebraically simpler to describe the localized (left half-plane) spectral sequence

$$\mu_1^{-1} E^2_{*,*}(C_p; \mathbb{Z}/p) = H^{-*}(C_p; \mu_1^{-1} \pi_* \text{THH}(\mathbb{Z})) \cong E(u_n) \otimes P(t) \otimes E(\lambda) \otimes P(\mu_1, \mu_1^{-1})$$

$$\implies \pi_* (\text{THH}(\mathbb{Z})^{1}_{C_p})^{hC_p}$$

Since $\mu_1^{-1} \pi_* \text{THH}(\mathbb{Z}) \cong \pi_* \text{THH}(\mathbb{Z})^{1}_{C_p}$, this is the same as the $C_p$-homotopy fixed point spectral sequence for $\text{THH}(\mathbb{Z})^{1}_{C_p}$. Since the localization map $\pi_* \text{THH}(\mathbb{Z}) \to \mu_1^{-1} \pi_* \text{THH}(\mathbb{Z})$ is $(-1)$-coconnected, the spectral sequences $E_{*,*}(C_p; \mathbb{Z}/p)$ and $\mu_1^{-1} E_{*,*}(C_p; \mathbb{Z}/p)$ agree in total degrees $\ast \geq 0$. Hence we can use the localized spectral sequence to calculate $\pi_* \text{THH}(\mathbb{Z})^{hC_p}$ in degrees $\ast \geq 0$, which then also calculates $\pi_* \text{TF}(\mathbb{Z}; p)$ is this range of degrees. (The negative homotopy groups of $\text{TF}(\mathbb{Z}; p)$ are zero.)

The $C_p$-homotopy fixed point spectral sequence $E_{*,*}(C_p; \mathbb{Z}/p)$ has $n$ even length differentials

$$d^r(t^p) = \lambda_1 \cdot (t_1^p) + \cdots + p$$

with $r = 2(p^{k+1} + \cdots + p)$, for all $0 \leq k < n$, followed by an odd length differential

$$d^r(u_n) = p^\ast \cdot (t_1^p)^{n-1} + \cdots + 1$$

for $r = 2(p^n + \cdots + p) + 1$. Hence the localized $C_p$-homotopy fixed point spectral sequence

$$\mu_1^{-1} E_{*,*}(C_p; \mathbb{Z}/p) \implies \pi_* (\text{THH}(\mathbb{Z})^{1}_{C_p})^{hC_p}$$

has $n$ even length differentials

$$d^r(\mu_1^p) = \lambda_1 \cdot (t_1^p) + \cdots + p \cdot \mu_1^p \cdot p^{k+1}$$

with $r = 2(p^{k+1} + \cdots + p)$, for all $0 \leq k < n$, followed by an odd length differential

$$d^r(u_n \mu_1^p) = (t_1^p)^{n-1} + \cdots + 1$$

for $r = 2(p^n + \cdots + p) + 1$.

The intermediate terms are then

$$\mu_1^{-1} E^r_{*,*}(C_p; \mathbb{Z}/p) = E(u_n) \otimes E(\lambda) \otimes P(t_1^p) \otimes P(\mu_1^p, \mu_1^{-1} \mu_1^p)$$

$$\oplus \bigoplus_{v_p(j) = m \leq k} E(u_n) \otimes P_{p^m + \cdots + p}(t_1^p) \{ \lambda_1 \mu_1^j \mid v_p(j) = m \}.$$
\[
\mu_1^{-1}E^r(C_p^n; \mathbb{Z}/p) = E(\lambda_1) \otimes P_{p^n + \ldots + n} (t \mu_1) \otimes P(\mu_1^n, \mu_1^{-p^n}) \\
\oplus \bigoplus_{v_p(j)=m<n} E(u_n) \otimes P_{p^n + \ldots + p}(t \mu_1) \{\lambda_1 \mu_1^j \mid v_p(j) = m\}
\]
for \( r = 2(p^n + \ldots + 1) \). This is also the \( E^\infty \)-term, because there is no room for further differentials. As previously discussed it is also the \( E^\infty \)-term of the \( C_p^n \)-homotopy fixed point spectral sequence, in total degress \( * \geq 0 \).

Counting classes, we obtain

\[
\dim_{\mathbb{Z}/p} \pi_*THH(\mathbb{Z})^{hC_p^n} = \begin{cases} 
  n & \text{for } i \neq 0, -1 \mod 2p^{n+1}, \\
  n + 1 & \text{for } i \equiv 0, -1 \mod 2p^{n+1},
\end{cases}
\]
assuming that \( i \geq 0 \). By Corollary 10.6, the same formula gives \( \dim_{\mathbb{Z}/p} \pi_*THH(\mathbb{Z})^{tC_p^n} \), for \( i \geq 0 \). Hence we know the rank of the abutment of the \( C_p^{n+1} \)-Tate spectral sequence

\[
E^2_{*,*}(C_p^{n+1}; \mathbb{Z}/p) = \tilde{H}^{-*}(C_p^{n+1}; \pi_*THH(\mathbb{Z})) \cong E(u_{n+1}) \otimes P(t, t^{-1}) \otimes E(\lambda_1) \otimes P(\mu_1)
\]

\[\Rightarrow \pi_*THH(\mathbb{Z})^{tC_p^{n+1}}.\]

We use this to establish the \( n + 1 \) case of the inductive hypothesis.

By naturality with respect to the Frobenius and Verschiebung maps

\[
F: THH(\mathbb{Z})^{tC_p^n} \longrightarrow THH(\mathbb{Z})^{tC_p^n}
\]
\[
V: THH(\mathbb{Z})^{tC_p^n} \longrightarrow THH(\mathbb{Z})^{tC_p^n+1}
\]
it follows from the differential structure of the \( C_p^n \)-Tate spectral sequence that the \( C_p^{n+1} \)-Tate spectral sequence has \( n \) even length differentials

\[
d^r(t^{p^k-p^{k+1}}) = t^{p^k} \cdot \lambda_1 \cdot (t \mu_1)^{p^k+\ldots+p}
\]
with \( r = 2(p^{k+1} + \ldots + p) \), for \( 0 \leq k < n \). Thereafter, the odd length differential

\[
d^r(u_{n+1} t^{-p^n}) = 0
\]
vanishes, for \( r = 2(p^n + \ldots + p) + 1 \). Hence

\[
E^r_{*,*}(C_p^{n+1}; \mathbb{Z}/p) = E(u_{n+1}) \otimes P(t^{p^n}, t^{-p^n}) \otimes E(\lambda_1) \otimes P(\mu_1)
\]

\[\oplus \bigoplus_{v_p(i)=m<n} E(u_{n+1}) \otimes P_{p^n + \ldots + p}(t \mu_1) \{t^i \lambda_1 \mid v_p(i) = m\}
\]
for \( r = 2(p^n + \ldots + 1) \).

It remains to show that there one more even length differential

\[d^r(t^{p^n-p^{n+1}}) = t^{p^n} \cdot \lambda_1 \cdot (t \mu_1)^{p^n+\ldots+p}\]
with \( r = 2(p^{n+1} + \ldots + p) \), followed by a new odd length differential

\[d^r(u_{n+1} t^{-p^{n+1}}) = (t \mu_1)^{p^{n+1}+\ldots+1}\]
for \( r = 2(p^{n+1} + \ldots + p) + 1 \).

The first target class,

\[t^{p^n} \cdot \lambda_1 \cdot (t \mu_1)^{p^n+\ldots+p},\]
is in total degree \( 2p^{n+1} - 2p^n - 1 \equiv 0, -1 \mod 2p^{n+1} \), where there are a total of \( n \) permanent cycles (generators of the \( E^\infty \)-term). These are of the form \( t^{i} \cdot \lambda_1 \cdot \mu_1^j \) with

\[i = p^n - p^{n+1} + p^{k+1} + \ldots + p \]
\[j = p^k + \ldots + p\]
for \( 0 \leq k < n \). Thus the next infinite cycle \( t^{p^n} \cdot \lambda_1 \cdot (t \mu_1)^{p^n+\ldots+p} \), which is the case \( k = n \) of the formulas above, must be a boundary. Checking bidegrees, the only possible source of a differential hitting this class is \( t^{p^n-p^{n+1}} \). This establishes the asserted even length differential (6).
Next, we claim that there is a differential

\[
d''((u_{n+1}t^{-p^{n+1}} \cdot \lambda_1) \simeq \lambda_1 \cdot (t^{\mu_1})^{p^n + \cdots + 1}
\]

with \( r = 2(p^{n+1} + \cdots + p) + 1 \), which implies the asserted odd length differential (7). The total degree of \( \lambda_1 \cdot (t^{\mu_1})^{p^n + \cdots + 1} \) is \( 2p^{n+1} + 2p - 3 \not\equiv 0 \mod 2p^{n+1} \), so there are precisely \( n \) permanent cycles in this total degree. These are the classes \( t^i \cdot \lambda_1 \cdot \mu_1^j \) with

\[
i = -p^{n+1} + p^{k+1} + \cdots + 1
\]

\[
j = p^k + \cdots + 1
\]

for \( 0 \leq k < n \). Thus the next infinite cycle \( t^p \cdot \lambda_1 \cdot \mu_1^p \cdot t^{\mu_1} = 2(t^{\mu_1})^{p^n + \cdots + 1} \), corresponding to \( k = n \), must be a boundary. For bidegree reasons the only possible source of a differential hitting this class is \( u_{n+1}t^{-p^{n+1}} \cdot \lambda_1 \), and this establishes the asserted odd length differential (8). \( \square \)

**Theorem 10.8.** The \( S^1 \)-homotopy fixed point spectral sequence

\[
E^2_{\infty}(S^1; \mathbb{Z}/p) = H^*(S^1; \pi_*THH(\mathbb{Z})) \cong P(t) \otimes E(\lambda_1) \otimes P(\mu_1)
\]

agrees in non-negative total degrees with the localized \( S^1 \)-homotopy fixed point spectral sequence

\[
\mu_1^{-1}E^2_{\infty}(S^1; \mathbb{Z}/p) = H^*(S^1; \mu^{-1}_1 \pi_*THH(\mathbb{Z})) \cong P(t) \otimes E(\lambda_1) \otimes P(\mu_1, \mu_1^{-1})
\]

\[\implies \pi_*THH(\mathbb{Z})^{hS^1} \]

which has differentials

\[
d''(\mu_1^p) = \lambda_1 \cdot (t^{\mu_1})^{p^k + \cdots + p} \cdot \mu_1^{-p^k + 1}
\]

with \( r = 2(p^{k+1} + \cdots + p) \), for all \( k \geq 0 \). The localized \( E^\infty \)-term is

\[
\mu_1^{-1}E^\infty_{\infty}(S^1; \mathbb{Z}/p) = E(\lambda_1) \otimes P(t^{\mu_1}) \otimes \bigoplus_{v_p(j)=m} P_{p^{m+1}+\cdots+p}(t^{\mu_1})\{\lambda_1 \mu_1^j\}
\]

**Theorem 10.9.** \( \pi_*TF(\mathbb{Z}; p) \) and \( \pi_*TP(\mathbb{Z}) = \pi_*THH(\mathbb{Z})^{hS^1} \) agree in degrees \( s \geq 0 \) with

\[\pi_*THH(\mathbb{Z})^{hS^1} = E(\lambda_1) \otimes P(v_1) \oplus \bigoplus_{v_p(j)=m} P_{p^{m+1}+\cdots+p}(v_1)\{\lambda_1 \mu_1^j\} \]

and

\[\pi_*THH(\mathbb{Z})^{TS^1} = E(\lambda_1) \otimes P(v_1) \oplus \bigoplus_{v_p(j)=m} P_{p^{m+1}+\cdots+p}(v_1)\{t^i \lambda_1\} \]

**Proof.** \( v_1 \in \pi_*S(S) \) is detected by \( t^{\mu_1} \). \( \square \)

In non-negative degrees the restriction map \( R: TF(\mathbb{Z}; p) \rightarrow TF(\mathbb{Z}; p) \) agrees with \( R^h: THH(\mathbb{Z})^{hS^1} \rightarrow THH(\mathbb{Z})^{TS^1} \). At the level of \( E^\infty \)-terms, this is the homomorphism

\[
E(\lambda_1) \otimes P(t^{\mu_1}) \oplus \bigoplus_{v_p(j)=m} P_{p^{m+1}+\cdots+p}(t^{\mu_1})\{\lambda_1 \mu_1^j\} \xrightarrow{\text{R}} E(\lambda_1) \otimes P(t^{\mu_1}) \oplus \bigoplus_{v_p(j)=m} P_{p^{m+1}+\cdots+p}(t^{\mu_1})\{t^i \lambda_1\}
\]

where the target corresponds under \( G \) to

\[
E(\lambda_1) \otimes P(t^{\mu_1}) \oplus \bigoplus_{v_p(j)=m} P_{p^{m+1}+\cdots+p}(t^{\mu_1})\{\lambda_1 \mu_1^j\}.
\]

On the summand \( E(\lambda_1) \otimes P(t^{\mu_1}) \) the restriction map agrees with the identity, also in \( \pi_*TF(\mathbb{Z}; p) \), since \( \lambda_1 \) and \( t^{\mu_1} \) are in the image from \( \pi_*K(\mathbb{Z}) \).

The summand \( P_{p^{m+1}+\cdots+p}(t^{\mu_1})\{\lambda_1 \mu_1^j\} \), with \( v_p(j) = m \geq 0 \), is concentrated in total degrees

\[
2p - 1 + 2pj \leq s \leq 2p(j + p^{m+1} - 1) + 1.
\]
These are all negative if $j \leq -p^{m+1}$. For $j \geq 0$ the summand maps by multiplication by $(\mu_1)^j$ to $P_{p^{m+1}+\cdots+p}(\mu_1)\{\lambda_1t^{-j}\}$. This is zero for $j \geq 2p^m$. ((The case $j = p^m$ is exceptional. Is $R$ nilpotent on this summand?)) In the remaining cases $j = -p^md$ with $1 \leq d \leq p-1$. The $P^d$-map

$$P_{p^{m+1}+\cdots+p}(\mu_1)\{\lambda_1\mu_1^{-p^md}\}_{* \geq 0} \to P_{p^{m+1}+\cdots+p}(\mu_1)\{\lambda_1\mu_1^{-p^md}\}_{* \geq 0}$$

is surjective, because $(p^{m+1}+\cdots+p) - p^md \geq (p^m + \cdots + p)$. For $m \geq 1$ the target corresponds under $G$ to $P_{p^{m+1}+\cdots+p}(\mu_1)\{\lambda_1\mu_1^{-p^m-1d}\}$, i.e., the summand corresponding to $j/p$ in place of $j$.

These calculations can be lifted from the $E^\infty$-level to the abutment. (The de Rham–Witt formalism helps to structure these lifts.) Granting this, we can recognize the components

$$1 - R: \prod_{m \geq 0} P_{p^{m+1}+\cdots+p}(v_1)\{\lambda_1\mu_1^{-p^md}\}_{* \geq 0} \to \prod_{m \geq 0} P_{p^{m+1}+\cdots+p}(v_1)\{\lambda_1\mu_1^{-p^md}\}_{* \geq 0}$$

for $1 \leq d \leq p-1$ as the homomorphisms with kernel

$$\lim_m P_{p^{m+1}+\cdots+p}(v_1)\{\lambda_1\mu_1^{-p^md}\}_{* \geq 0} \cong P(v_1)\{\lambda_1\mu_1^{-p^md}\}_{* \geq 0} \cong P(v_1)\{t^d\lambda_1\}$$

and cokernel $\text{Rlim}_m(\cdot) = 0$.

**Theorem 10.10** (Bökstedt–Madsen (1994, 1995)). Let $p$ be any odd prime.

$$\pi_* TC(\mathbb{Z};p) \cong P(v_1) \otimes E(\partial, \lambda_1) \oplus P(v_1)\{t^d\lambda_1 \mid 1 \leq d \leq p-1\}$$

$$\cong P(v_1)\{\partial, 1, t^0\lambda_1, \ldots, t\lambda_1, \partial\lambda_1, \lambda_1\}$$

is a free $P(v_1) = \mathbb{Z}/p[v_1]$-module of rank $p+3$.

**Corollary 11.** Let $p$ be any odd prime, let $ku$ be the connective complex $K$-theory spectrum, and let $j$ be the connective image-of-$J$ spectrum. After $p$-adic completion there is an equivalence

$$K(\mathbb{Z}_p) \cong_p j \vee \Sigma j \vee \Sigma^3 ku.$$

**Sketch proof.** One first constructs maps $S \vee \Sigma S \to K(\mathbb{Z}_p)$, and factors them through $j \vee \Sigma j$. The homotopy cofiber has the mod $p$ homotopy of $\Sigma ku$, including the $v_1$-action, and this characterizes this spectrum, by Rognes (1993). □

The analogous results for $p = 2$ were obtained in Rognes (1998/1999).

**References**


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