TOPOLOGICAL CYCLIC HOMOLOGY OF THE PRIME FIELDS AND THE INTEGERS

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1. Topological cyclotomy

For a commutative (symmetric or orthogonal) ring spectrum A we initially defined its topological Hochschild homology

$$THH(A) = |A \odot S^1_\bullet|$$

as the topological realization of the cyclic bar construction $[q] \mapsto A \odot S^1_q = A \wedge A \wedge \cdots \wedge A$. This is a commutative ring spectrum with S^1-action. Bökstedt, Hsiang and Madsen (1993) gave a modified definition that makes $THH(A)$ a "genuine" S^1-equivariant orthogonal ring spectrum. In particular, for each finite subgroup $C_r \subset S^1$ one can form the C_r-fixed point spectrum $THH(A)^{C_r}$. For $q|r$ we have $C_q \subset C_r$, and there is a natural (Frobenius) map

$$F\colon THH(A)^{C_r} \longrightarrow THH(A)^{C_q}$$

that forgets part of the invariance. In the stable homotopy category, there is also a transfer (Verschiebung) map

$$V\colon THH(A)^{C_q} \longrightarrow THH(A)^{C_r}\,.$$

Let $ES^1 = S(\mathbb{C}^\infty)$ be a free, contractible S^1-CW complex. It is also a free, contractible C_r-CW space for each $C_r \subset S^1$. Let $\tilde{E}S^1 = S^{\mathbb{C}^\infty}$ be the mapping cone of the collapse map $ES^1_+ \to S^0$. Let p be a prime. The geometric C_p-fixed point spectrum of $THH(A)$ can (provisionally) be defined as

$$THH(A)^{\varphi C_p} = \Phi^{C_p} THH(A) = [\tilde{E}S^1 \wedge THH(A)]^{C_r}\,.$$

This has a residual S^1/C_p-action. There is a non-obvious, "cyclotomic", equivalence

$$THH(A)^{\varphi C_p} \simeq THH(A)\,,$$

which is equivariant with respect to the p-th root isomorphism $S^1/C_p \cong S^1$. The inclusion $S^0 \to \tilde{E}S^1$ induces an S^1/C_p-equivariant map

$$THH(A)^{C_p} \to THH(A)^{\varphi C_p}\,.$$

Passing to C_q-fixed points, with $pq = r$, we obtain a natural restriction map

$$R\colon THH(A)^{C_r} \longrightarrow THH(A)^{C_q}\,.$$

The F- and R-maps commute. Restricting attention to prime powers $r = p^n$ for $n \geq 0$, we obtain a (non-commutative) diagram

$$\dots \underset{R}{\overset{F}{\rightrightarrows}} THH(A)^{C_{p^{n+1}}} \underset{R}{\overset{F}{\rightrightarrows}} THH(A)^{C_{p^n}} \underset{R}{\overset{F}{\rightrightarrows}} THH(A)^{C_{p^{n-1}}} \underset{R}{\overset{F}{\rightrightarrows}} \dots$$

There is a trace map $\mathrm{tr}\colon K(A) \to THH(A)$ from the algebraic K-theory of A, which admits lifts

$$\mathrm{tr}_{p^n} : K(A) \longrightarrow THH(A)^{C_{p^n}}$$

for all $n \geq 0$. These are compatible (up to homotopy) with the R- and F-maps, hence induce a map to the homotopy limit of the diagram above. This homotopy limit

$$TC(A; p) = \operatorname*{holim}_{n,R,F} THH(A)^{C_{p^n}}$$

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is the (p-typical) topological cyclic homology of A. The lifted map

$$\mathrm{trc}\colon K(A) \longrightarrow TC(A;p)$$

is the (p-typical) cyclotomic trace map. It is a sensitive invariant of algebraic K-theory, especially for connective A with $\pi_0(A)$ close to $\mathbb{Z}/p$, $\mathbb{Z}_p$ or similar rings. See work by Bökstedt (ca. 1985, unpublished), Goodwillie (1986), Bökstedt-Madsen (1993 and 1994), McCarthy (1997), Dundas (1997), Hesselholt-Madsen (1997 and 2003), Rognes (1998 and 1999), Ausoni-Rognes (2002), etc.

2. Topological Frobenius and restriction homology

Concentrating on the sequence of F-maps, let

$$TF(A;p) = \operatorname*{holim}_{n,F} THH(A)^{C_{p^n}} .$$

The R-maps then induce a self-map

$$R\colon TF(A;p) \longrightarrow TF(A;p) .$$

Concentrating on the sequence of R-maps, let

$$TR(A;p) = \operatorname*{holim}_{n,R} THH(A)^{C_{p^n}} .$$

The F-maps then induce a self-map

$$F\colon TR(A;p) \longrightarrow TF(R;p) .$$

There are homotopy equivalences of ring spectra

$$\mathrm{hoeq}(\mathrm{id}, R) \simeq TC(A;p) \simeq \mathrm{hoeq}(\mathrm{id}, F) .$$

Additively,

$$\mathrm{hoeq}(\mathrm{id}, R\colon TF(A;p) \to TF(A;p)) \simeq \mathrm{hofib}(\mathrm{id} - R\colon TF(A;p) \to TF(A;p))$$

and

$$\mathrm{hoeq}(\mathrm{id}, F\colon TR(A;p) \to TR(A;p)) \simeq \mathrm{hofib}(\mathrm{id} - F\colon TR(A;p) \to TR(A;p)) .$$

In particular, there is a short exact sequence

$$0 \to \operatorname*{Rlim}_{n,F} \pi_{*+1} THH(A)^{C_{p^n}} \longrightarrow \pi_* TF(A;p) \longrightarrow \lim_{n,F} \pi_* THH(A)^{C_{p^n}} \to 0$$

and a long exact sequence

$$\dots \to \pi_{*+1} TF(A;p) \xrightarrow{\partial} \pi_* TC(A;p) \longrightarrow \pi_* TF(A;p) \xrightarrow{\mathrm{id}-R} \pi_* TF(A;p) \to \dots$$

which we can use to calculate $\pi_* TF(A;p)$ and $\pi_* TC(A;p)$ in some cases.

3. Homotopy fixed points

Let

$$THH(A)^{hC_r} = F(ES^1_+, THH(A))^{C_r}$$

be the C_r-homotopy fixed points of $THH(A)$. The collapse map $ES^1_+ \to S^0$ induces a comparison map

$$THH(A)^{C_r} \longrightarrow THH(A)^{hC_r} .$$

When $r = p^n$ we denote this map by Γ_n. Let

$$THH(A)^{hS^1} = F(ES^1_+, THH(A))^{S^1}$$

be the S^1-homotopy fixed points of $THH(A)$. The natural map

$$THH(A)^{hS^1} \longrightarrow \operatorname*{holim}_{n,F} THH(A)^{hC_{p^n}}$$

becomes an equivalence after p-completion. Hence there is a natural comparison map

$$\Gamma\colon TF(A;p) \longrightarrow THH(A)^{hS^1}_p$$

making the following diagram commute.

$$\begin{array}{ccc} TF(A;p) & = & \operatorname{holim}_{n,F} THH(A)^{C_{p^n}} \\ \Gamma \downarrow & & \downarrow \operatorname{holim}_{n,F} \Gamma_n \\ THH(A)^{hS^1}_p & \xrightarrow{\simeq} & \operatorname{holim}_{n,F} THH(A)^{hC_{p^n}}_p \end{array}$$

The skeleton filtration of ES^1 leads to an algebra spectral sequence

$$E^2_{s,t}(C_r) = H^{-s}(C_r; \pi_t THH(A)) \Longrightarrow \pi_{s+t} THH(A)^{hC_r}$$

(group cohomology) called the homotopy fixed point spectral sequence. The C_r-action on $\pi_* THH(A)$ extends over the connected group S^1, hence is trivial. There is a similar spectral sequence

$$E^2_{s,t}(C_r; \mathbb{Z}/p) = H^{-s}(C_r; \pi_t(THH(A); \mathbb{Z}/p)) \Longrightarrow \pi_{s+t}(THH(A)^{hC_r}; \mathbb{Z}/p)$$

for homotopy with $\mathbb{Z}/p$-coefficients, which is an algebra spectral sequence for $p \geq 5$, and sometimes for $p = 3$ or $p = 2$. Here $\pi_*(X) = \pi_*(X \wedge S/p)$, where S/p is the mod p Moore spectrum. This is a ring spectrum in the homotopy category for $p \geq 5$. In many cases $\pi_*(\Gamma_n)$ or $\pi_*(\Gamma_n; \mathbb{Z}/p)$ is an isomorphism for $*$ sufficiently large, in which case the C_{p^n}-homotopy fixed point spectral sequence gives a way to access the homotopy groups or mod p homotopy groups of the fixed point spectrum $THH(A)^{C_{p^n}}$.

The Frobenius maps $F\colon THH(A)^{hC_{p^n}} \to THH(A)^{hC_{p^{n-1}}}$ induce a map of algebra spectral sequences

$$F\colon E^2_{*,*}(C_{p^n}) \longrightarrow E^2_{*,*}(C_{p^{n-1}})$$

converging to

$$F\colon \pi_*(THH(A)^{hC_{p^n}}) \longrightarrow \pi_*(THH(A)^{hC_{p^{n-1}}}) .$$

Passing to the limit over n, there is also an algebra spectral sequence

$$E^2_{s,t}(S^1) = H^{-s}(S^1; \pi_t THH(A)) \Longrightarrow \pi_{s+t} THH(A)^{hS^1}$$

(group cohomology, suitably defined), with p-complete and mod p variants. When $\pi_*(\Gamma)$ is an isomorphism for $*$ sufficiently large, this gives a way to access the homotopy groups of $TF(A;p)$, usually after p-completion or with mod p coefficients.

The Verschiebung maps $V\colon THH(A)^{hC_{p^{n-1}}} \to THH(A)^{hC_{p^n}}$ also induce a map of spectral sequences

$$V\colon E^2_{*,*}(C_{p^{n-1}}) \longrightarrow E^2_{*,*}(C_{p^n})$$

converging to

$$V\colon \pi_*(THH(A)^{hC_{p^{n-1}}}) \longrightarrow \pi_*(THH(A)^{hC_{p^n}}) .$$

4. Tate constructions

Let

$$THH(A)^{tC_r} = \hat{\mathbb{H}}(C_r, THH(A)) = [\tilde{E}S^1 \wedge F(ES^1_+, THH(A))]^{C_r}$$

be the C_r-Tate construction for $THH(A)$. The collapse map $ES^1_+ \to S^0$ induces a comparison map

$$THH(A)^{C_q} \simeq [THH(A)^{\varphi C_p}]^{C_q} = [\tilde{E}S^1 \wedge THH(A)]^{C_r} \longrightarrow THH(A)^{tC_r}$$

for $pq = r$. When $r = p^n$ we denote this map by $\hat{\Gamma}_n$. Nikolaus-Scholze call $\hat{\Gamma}_1\colon THH(A) \to THH(A)^{tC_p}$ the Frobenius map φ_p. The same map was denoted $\hat{\gamma}\colon T(A) \to \hat{T}(A)$, e.g. by Lunøe-Nielsen and Rognes. Let

$$THH(A)^{tS^1} = \hat{\mathbb{H}}(S^1, THH(A)) = [\tilde{E}S^1 \wedge F(ES^1_+, THH(A))]^{S^1}$$

be the S^1-Tate construction on $THH(A)$, which Hesselholt (2016, arXiv) denotes $TP(A)$. The natural map

$$THH(A)^{tS^1} \longrightarrow \operatorname*{holim}_{n,F} THH(A)^{tC_{p^n}}$$

becomes an equivalence after p-completion. Hence there is a natural comparison map

$$\hat{\Gamma}\colon TF(A;p) \longrightarrow THH(A)^{tS^1}_p$$

making the following diagram commute.

$$\begin{array}{ccc} TF(A;p) & = & \operatorname{holim}_{n,F} THH(A)^{C_{p^{n-1}}} \\ \downarrow{\scriptstyle \hat\Gamma} & & \downarrow{\scriptstyle \operatorname{holim}_{n,F}\hat\Gamma_n} \\ THH(A)_p^{tS^1} & \xrightarrow{\simeq} & \operatorname{holim}_{n,F} THH(A)_p^{tC_{p^n}} \end{array}$$

The skeleton filtration of ES^1 and the induced filtration of $\tilde{E}S^1$ lead to an algebra spectral sequence

$$\hat{E}^2_{s,t}(C_r) = \hat{H}^{-s}(C_r; \pi_t THH(A)) \Longrightarrow \pi_{s+t} THH(A)^{tC_r}$$

(Tate cohomology) called the Tate spectral sequence. There is a similar spectral sequence

$$\hat{E}^2_{s,t}(C_r, \mathbb{Z}/p) = \hat{H}^{-s}(C_r; \pi_t(THH(A); \mathbb{Z}/p)) \Longrightarrow \pi_{s+t}(THH(A)^{tC_r}; \mathbb{Z}/p)$$

for homotopy with $\mathbb{Z}/p$-coefficients. In many cases $\pi_*(\hat\Gamma_n)$ or $\pi_*(\hat\Gamma_n; \mathbb{Z}/p)$ is an isomorphism for $*$ sufficiently large, in which case the C_{p^n}-Tate spectral sequence gives a way to access the homotopy groups or mod p homotopy groups of the fixed point spectrum $THH(A)^{C_{p^{n-1}}}$.

The Frobenius maps $F\colon THH(A)^{tC_{p^n}} \to THH(A)^{tC_{p^{n-1}}}$ induce a map of algebra spectral sequences

$$F\colon \hat{E}^2_{*,*}(C_{p^n}) \longrightarrow \hat{E}^2_{*,*}(C_{p^{n-1}})$$

converging to

$$F\colon \pi_*(THH(A)^{tC_{p^n}}) \longrightarrow \pi_*(THH(A)^{tC_{p^{n-1}}}) .$$

Passing to the limit over n, there is also an algebra spectral sequence

$$\hat{E}^2_{s,t}(S^1) = \hat{H}^{-s}(S^1; \pi_t THH(A)) \Longrightarrow \pi_{s+t} THH(A)^{tS^1}$$

(Tate cohomology, suitably defined), with p-complete and mod p variants. When $\pi_*(\hat\Gamma)$ is an isomorphism for $*$ sufficiently large, this gives a second way to access the homotopy groups of $TF(A;p)$, usually after p-completion or with mod p coefficients.

The Verschiebung maps $V\colon THH(A)^{tC_{p^{n-1}}} \to THH(A)^{tC_{p^n}}$ also induce a map of spectral sequences

$$V\colon \hat{E}^2_{*,*}(C_{p^{n-1}}) \longrightarrow \hat{E}^2_{*,*}(C_{p^n})$$

converging to

$$V\colon \pi_*(THH(A)^{tC_{p^{n-1}}}) \longrightarrow \pi_*(THH(A)^{tC_{p^n}}) .$$

5. The restriction and homotopy restriction maps

By taking the smash product of the S^1-equivariant homotopy cofiber sequence

$$ES^1_+ \longrightarrow S^0 \longrightarrow \tilde{E}S^1 \longrightarrow \Sigma(ES^1_+)$$

and the natural S^1-map

$$THH(A) \longrightarrow F(ES^1_+, THH(A))$$

and passing to C_{p^n}-fixed points, we obtain a vertical map of horizontal homotopy cofiber sequences:

(1)
$$\begin{array}{ccccccc} THH(A)_{hC_{p^n}} & \xrightarrow{N} & THH(A)^{C_{p^n}} & \xrightarrow{R} & THH(A)^{C_{p^{n-1}}} & \xrightarrow{\partial} & \Sigma THH(A)_{hC_{p^n}} \\ \| & & \downarrow{\scriptstyle \Gamma_n} & & \downarrow{\scriptstyle \hat\Gamma_n} & & \| \\ THH(A)_{hC_{p^n}} & \xrightarrow{N^h} & THH(A)^{hC_{p^n}} & \xrightarrow{R^h} & THH(A)^{tC_{p^n}} & \xrightarrow{\partial^h} & \Sigma THH(A)_{hC_{p^n}} \end{array}$$

Note that the middle square is homotopy (co-)cartesian. It follows by induction that each spectrum $THH(A)^{C_{p^n}}$ is connective (assuming that A is connective). There is a map of spectral sequences

$$R^h\colon E^2_{s,t}(C_{p^n}) = H^{-s}(C_{p^n}; \pi_t THH(A)) \longrightarrow \hat{E}^2_{s,t}(C_{p^n}) = \hat{H}^{-s}(C_{p^n}; \pi_t THH(A))$$

converging to

$$R^h\colon \pi_* THH(A)^{hC_{p^n}} \longrightarrow \pi_* THH(A)^{tC_{p^n}} .$$

Nikolaus-Scholze call R^h the canonical map can. In the range where $\pi_*(\Gamma_n)$ and $\pi_*(\hat{\Gamma}_n)$ are isomorphisms, this is identified with

$$R\colon \pi_* THH(A)^{C_{p^n}} \longrightarrow \pi_* THH(A)^{C_{p^{n-1}}} .$$

Passing to homotopy limits over n using the F-maps, we obtain an implicitly p-completed vertical map of horizontal homotopy cofiber sequences:

$$\begin{array}{ccccccc} \Sigma THH(A)_{hS^1} & \xrightarrow{N} & TF(A;p) & \xrightarrow{R} & TF(A;p) & \xrightarrow{\partial} & \Sigma^2 THH(A)_{hS^1} \\ \| & & \downarrow{\scriptstyle \Gamma} & & \downarrow{\scriptstyle \hat{\Gamma}} & & \| \\ \Sigma THH(A)_{hS^1} & \xrightarrow{N^h} & THH(A)^{hS^1} & \xrightarrow{R^h} & THH(A)^{tS^1} & \xrightarrow{\partial^h} & \Sigma^2 THH(A)_{hS^1} \end{array} \tag{2}$$

Again there is a map of spectral sequences

$$R^h\colon E^2_{s,t}(S^1) = H^{-s}(S^1; \pi_t THH(A)) \longrightarrow \hat{E}^2_{s,t}(S^1) = \hat{H}^{-s}(S^1; \pi_t THH(A))$$

converging to

$$R^h\colon \pi_* THH(A)^{hS^1} \longrightarrow \pi_* THH(A)^{tS^1} .$$

In the range where $\pi_*(\hat{\Gamma}_1)$ is an isomorphism, this is identified with

$$R\colon \pi_* TF(A;p) \longrightarrow TF(A;p) .$$

6. The homotopy limit property for finite cyclic groups

We say that a map $f\colon X \to Y$ is k-coconnected if $\pi_*(f)\colon \pi_*(X) \to \pi_*(Y)$ is injective for $* = k$ and an isomorphism for $* > k$. Note that $\Gamma_n\colon THH(A)_p^{C_{p^n}} \to THH(A)_p^{hC_{p^n}}$ is k-coconnected if and only if $\hat{\Gamma}_n\colon THH(A)_p^{C_{p^{n-1}}} \to THH(A)_p^{tC_{p^n}}$ is k-coconnected.

Theorem 6.1 (Tsalidis)**.** *Let A be bounded below with $H_*(A;\mathbb{Z}/p)$ of finite type. If $\Gamma_1\colon THH(A)_p^{C_p} \to THH(A)_p^{hC_p}$ is k-coconnected, then $\Gamma_n\colon THH(A)_p^{C_{p^n}} \to THH(A)_p^{hC_{p^n}}$ is k-coconnected for each $n \geq 1$.*

This was proved by Tsalidis (1998?), generalizing earlier work by Ravenel. Later joint work by Bökstedt, Bruner, Lunøe-Nielsen and Rognes (2014) generalized the result further.

Theorem 6.2. *Let A be bounded below with $H_*(A;\mathbb{Z}/p)$ of finite type. If $\Gamma_1\colon THH(A)^{C_p} \wedge S/p \to THH(A)^{hC_p} \wedge S/p$ is k-coconnected, then $\Gamma_n\colon THH(A)^{C_{p^n}} \wedge S/p \to THH(A)^{hC_{p^n}} \wedge S/p$ is k-coconnected for each $n \geq 1$.*

7. Group and Tate cohomology

Let p be an odd prime. The group- and Tate cohomology of C_{p^n} with coefficients in $\mathbb{Z}/p$ are given by

$$H^*(C_{p^n};\mathbb{Z}/p) = E(u_n) \otimes P(t)$$
$$\hat{H}^*(C_{p^n};\mathbb{Z}/p) = E(u_n) \otimes P(t, t^{-1})$$

with $u_n \in H^1 \cong \hat{H}^1$ and $t \in H^2 \cong \hat{H}^2$. Here $E(u_n) = \mathbb{Z}/p[u_n]/(u_n^2)$ is the exterior algebra, $P(t) = \mathbb{Z}/p[t]$ is the polynomial algebra, and $P(t,t^{-1}) = \mathbb{Z}/p[t,t^{-1}]$ is the Laurent polynomial algebra. The canonical homomorphism

$$R^h\colon H^*(C_{p^n};\mathbb{Z}/p) \longrightarrow \hat{H}^*(C_{p^n};\mathbb{Z}/p)$$

inverts t.

The inclusion $C_{p^{n-1}} \to C_{p^n}$ induces homomorphisms

$$F\colon H^*(C_{p^n};\mathbb{Z}/p) \longrightarrow H^*(C_{p^{n-1}};\mathbb{Z}/p)$$
$$F\colon \hat{H}^*(C_{p^n};\mathbb{Z}/p) \longrightarrow \hat{H}^*(C_{p^{n-1}};\mathbb{Z}/p)$$

given by $F(u_n) = 0$ and $F(t) = t$. In the limiting case

$$H^*(S^1;\mathbb{Z}/p) = P(t)$$
$$\hat{H}^*(S^1;\mathbb{Z}/p) = P(t,t^{-1})$$

with $t \in H^2 \cong \hat{H}^2$.

There are also Verschiebung (transfer) homomorphisms

$$V\colon H^*(C_{p^{n-1}};\mathbb{Z}/p) \longrightarrow H^*(C_{p^n};\mathbb{Z}/p)$$
$$V\colon \hat{H}^*(C_{p^{n-1}};\mathbb{Z}/p) \longrightarrow \hat{H}^*(C_{p^n};\mathbb{Z}/p)$$

given by $V(u_{n-1}) = u_n$ and $V(t) = 0$.

8. Calculations for $\mathbb{Z}/p$

Consider the case $A = H\mathbb{Z}/p$. Recall that $\pi_* THH(\mathbb{Z}/p) = P(\mu_0) = \mathbb{Z}/p[\mu_0]$, where $\mu_0 \in \pi_2 THH(\mathbb{Z}/p)$ has mod p Hurewicz image $\sigma\bar{\tau}_0 \in H_2(THH(\mathbb{Z}/p))$.

Theorem 8.1. *The S^1-homotopy fixed point spectral sequence*

$$E^2_{*,*}(S^1) = P(t) \otimes P(\mu_0) \Longrightarrow \pi_* THH(\mathbb{Z}/p)^{hS^1}$$

collapses at the E^2-term. There are additive extensions, so that the image of $p \in \pi_0(S) = \mathbb{Z}$ is represented by (a unit in $\mathbb{Z}/p$ times)

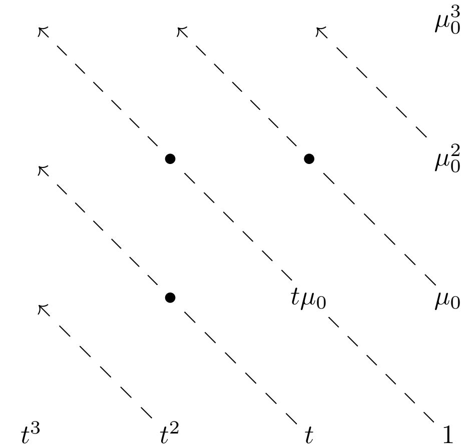
$$t\mu_0 \in E^\infty_{-2,2}(S^1)\,.$$

Hence

$$\pi_* THH(\mathbb{Z}/p)^{hS^1} \cong \mathbb{Z}_p[t,\mu_0]/(t\mu_0 \doteq p)$$

with $t \in \pi_{-2}$ and $\mu_0 \in \pi_2$. Additively

$$\pi_i THH(\mathbb{Z}/p)^{hS^1} \cong \begin{cases} \mathbb{Z}_p\{\mu_0^n\} & \text{for } i = 2n \geq 0, \\ \mathbb{Z}_p\{t^m\} & \text{for } i = -2m \leq 0, \\ 0 & \text{otherwise.} \end{cases}$$



Proof. The E^2-term is concentrated in even total degrees, so all differentials are zero. We claim that the unit map $S \to THH(\mathbb{Z}/p)^{hS^1}$ maps $p \in \pi_0(S)$ to a class detected by $t\mu_0$ (times a unit in $\mathbb{Z}/p$) in $E^\infty_{-2,2}(S^1)$.

The first terms of the S^1-equivariant skeleton filtration of $ES^1_+ = S(\mathbb{C}^\infty)_+$ are

$$S^1_+ \to S^3_+ \to \cdots \to ES^1_+\,,$$

with $S^3 = S(\mathbb{C}^2)$, and there is a homotopy cofiber sequence

$$S^1_+ \longrightarrow S^3_+ \longrightarrow \Sigma^2(S^1_+) \longrightarrow \Sigma(S^1_+)\,.$$

There are induced maps

$$THH(A)^{hS^1} \to \cdots \to F(S^3_+, THH(A))^{S^1} \xrightarrow{\pi} THH(A)$$

and a homotopy (co-)fiber sequence

$$\Sigma^{-2} THH(A) \longrightarrow F(S^3_+, THH(A))^{S^1} \xrightarrow{\pi} THH(A) \xrightarrow{\sigma} \Sigma^{-1} THH(A)\,.$$

For brevity, let

$$X = F(S^3_+, THH(A))^{S^1}\ .$$

It suffices to prove that p maps to a nonzero class in $\pi_0 X$, since the image of p must be then be detected in $E^\infty_{-2,2} = \mathbb{Z}/p\{t\mu_0\}$.

We do this by calculating H_*X and considering the map of Adams spectral sequences

$$E_2^{*,*}(S) = \operatorname{Ext}^{*,*}_{\mathscr{A}_*}(\mathbb{Z}/p, \mathbb{Z}/p) \longrightarrow \operatorname{Ext}^{*,*}_{\mathscr{A}_*}(\mathbb{Z}/p, H_*X) = E_2^{*,*}(X)\,.$$

There is a long exact sequence

$$(3) \qquad \cdots \to H_{*+2}THH(A) \to H_*X \to H_*THH(A) \xrightarrow{\sigma} H_{*+1}THH(A) \to \dots$$

where σ is the homological σ-operator. For $A = H\mathbb{Z}/p$, recall that $H_*THH(\mathbb{Z}/p) = \mathscr{A}_* \otimes P(\mu_0)$, where $\sigma(\bar\tau_0) = \mu_0$. It follows that $H_0X \cong \mathbb{Z}/p\{1\}$ and $H_1X \cong \mathbb{Z}/p\{t\bar\tau_0\mu_0\}$. The class $p \in \pi_0(S)$ is represented (up to sign) by the cohomology class in

$$E_2^{1,1}(S) = \operatorname{Ext}^{1,1}_{\mathscr{A}_*}(\mathbb{Z}/p, \mathbb{Z}/p)$$

of the 1-cocycle $[\bar\tau_0]$ in the normalized cobar complex

$$0 \to \mathbb{Z}/p \xrightarrow{d_1^0} \bar{\mathscr{A}}_* \xrightarrow{d_1^2} \bar{\mathscr{A}}_* \otimes \bar{\mathscr{A}}_* \to \dots\,.$$

Its image in $E_2^{1,1}(X) = \operatorname{Ext}^{1,1}_{\mathscr{A}_*}(\mathbb{Z}/p, H_*X)$ is the cohomology class of the 1-cocycle $[\bar\tau_0]1$ in the normalized cobar complex

$$0 \to H_*X \xrightarrow{d_1^0} \bar{\mathscr{A}}_* \otimes H_*X \xrightarrow{d_1^2} \bar{\mathscr{A}}_* \otimes \bar{\mathscr{A}}_* \otimes H_*X \to \dots\,.$$

This cohomology class remains nonzero, because the 1-cocycle is not a 1-coboundary. Indeed, the only possible 0-cochain cobounding it (in internal degree 1) would be $t\bar\tau_0\mu_0 \in H_1X$. However, a direct calculation shows that $\nu(t\bar\tau_0\mu_0) = 1 \otimes t\bar\tau_0\mu_0$, so $d_1^0(t\bar\tau_0\mu_0) = 0$. Hence the class of $[\bar\tau_0]1$ survives to E_∞, and represents a nonzero element in $\pi_0(X)$. □

Proposition 8.2. *The map $\hat\Gamma_1 \colon THH(\mathbb{Z}/p) \to THH(\mathbb{Z}/p)^{tC_p}$ is (-2)-coconnected. Hence each map Γ_n, Γ, $\hat\Gamma_n$ and $\hat\Gamma$ is (-2)-coconnected.*

Corollary 8.3.

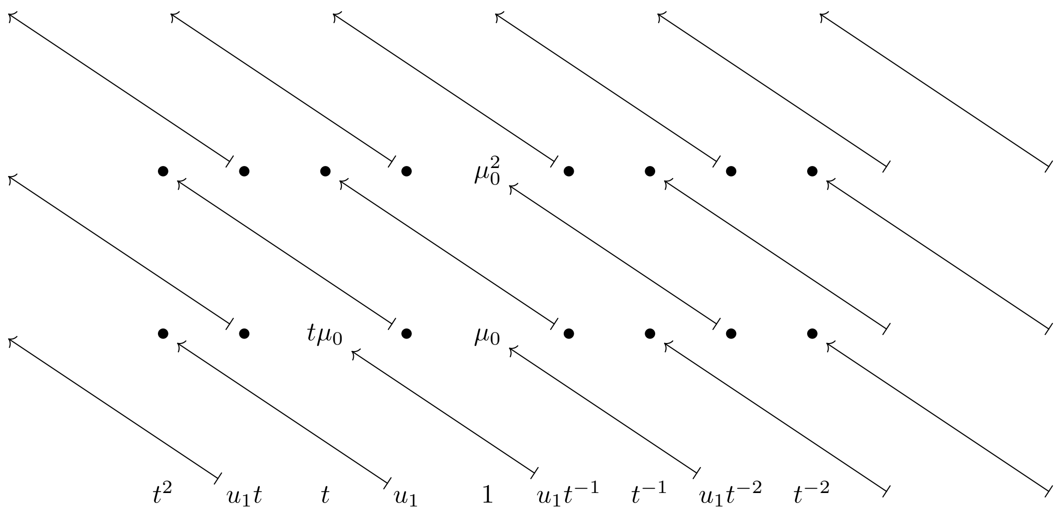
$$\pi_*TF(\mathbb{Z}/p;p) \cong \mathbb{Z}_p[\mu_0]$$

with

$$\Gamma \colon \pi_*TF(\mathbb{Z}/p;p) \longrightarrow \pi_*THH(\mathbb{Z}/p)^{hS^1}$$

given by $\Gamma(\mu_0) = \mu_0$. Additively

$$\pi_iTF(\mathbb{Z}/p;p) \cong \begin{cases} \mathbb{Z}_p\{\mu_0^n\} & \text{for } i = 2n \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$



Proof of Proposition 8.2. Consider the C_p-Tate spectral sequence

$$\hat{E}^2_{*,*}(C_p) = \hat{H}^{-*}(C_p, \pi_* THH(\mathbb{Z}/p)) \cong E(u_1) \otimes P(t, t^{-1}) \otimes P(\mu_0)$$
$$\Longrightarrow \pi_* THH(\mathbb{Z}/p)^{tC_p} .$$

By naturality with respect to

$$THH(\mathbb{Z}/p)^{hS^1} \xrightarrow{F} THH(\mathbb{Z}/p)^{hC_p} \xrightarrow{R^h} THH(\mathbb{Z}/p)^{tC_p}$$

the classes t and μ_0 are infinite cycles. We claim that $d^3(u_1 t^{-1}) \doteq t\mu_0$, so that

$$\hat{E}^3_{*,*}(C_p) \cong P(t, t^{-1}) .$$

Hence this spectral sequence collapses at E^3, and

$$\pi_* THH(\mathbb{Z}/p)^{tC_p} \cong \mathbb{Z}/p[t, t^{-1}] .$$

Finally, we claim that $\hat{\Gamma}_1(\mu_0) \doteq t^{-1}$. Then

$$\hat{\Gamma} \colon \mathbb{Z}/p[\mu_0] \longrightarrow \mathbb{Z}/p[t, t^{-1}]$$

inverts μ_0, and is (-2)-coconnected.

To show that $d^3(u_1 t^{-1}) \doteq t\mu_0$, it is enough to show that $t\mu_0$ is a d^r-boundary for some r, i.e., that the image of $p \in \pi_0 S$ in $\pi_0 THH(\mathbb{Z}/p)^{hS^1}$ maps to zero in $\pi_0 THH(\mathbb{Z}/p)^{tC_p}$. To see this, we use the factorization

$$S \longrightarrow THH(\mathbb{Z}/p) \xrightarrow{\hat{\Gamma}_1} THH(\mathbb{Z}/p)^{tC_p}$$

of the unit map. The image of $p \in \pi_0(S)$ is zero in $\pi_0 THH(\mathbb{Z}/p) = \mathbb{Z}/p$ hence must be mapped to zero by $\hat{\Gamma}_1$, as claimed.

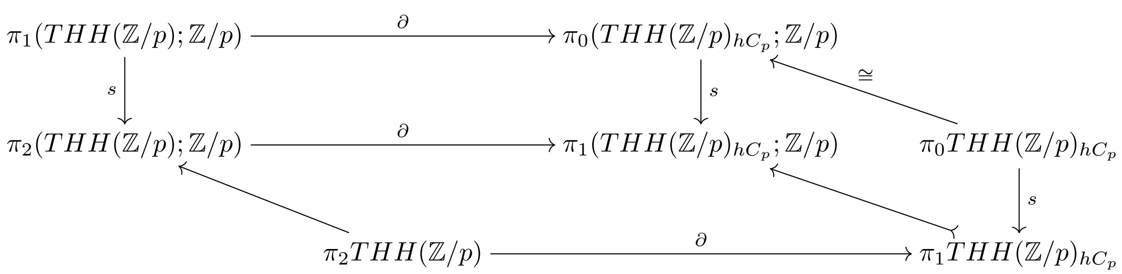
To show that $\hat{\Gamma}_1(\mu_0) \doteq t^{-1}$, Hesselholt-Madsen (1997) show that $\partial(\mu_0) \neq 0$ in $\pi_1 THH(\mathbb{Z}/p)_{hC_p} \cong \mathbb{Z}/p$. This uses the homotopy orbit spectral sequence

$$H_s(C_{p^n}, \pi_t THH(A)) \Longrightarrow \pi_{s+t} THH(A)_{hC_{p^n}}$$

to calculate $\pi_* THH(\mathbb{Z}/p)_{hC_p} = (\mathbb{Z}/p, \mathbb{Z}/p, \dots)$, and the factorizaton

$$S \to THH(\mathbb{Z}/p)^{C_p} \xrightarrow{\Gamma_1} THH(\mathbb{Z}/p)^{hC_p}$$

to show that $\pi_0 THH(\mathbb{Z}/p)^{C_p} \cong \mathbb{Z}/p^2$. It follows that $\partial \colon \pi_1(THH(\mathbb{Z}/p); \mathbb{Z}/p) \to \pi_0(THH(\mathbb{Z}/p)_{hC_p}; \mathbb{Z}/p)$ maps the generator, with Hurewicz image $\bar{\tau}_0$, to a non-zero class.



The mod p Hurewicz image of μ_0 is $\sigma\bar{\tau}_0$, so by a diagram chase it suffices to check that s acts nontrivially on $\pi_0 THH(\mathbb{Z}/p)_{hC_p}$. This follows, e.g., by a comparison with the S^1-action on $S_{hC_p} = BC_{p+}$. □

Theorem 8.4. *The S^1-Tate spectral sequence*

$$\hat{E}^2_{*,*}(S^1) = P(t, t^{-1}) \otimes P(\mu_0) \Longrightarrow \pi_* THH(\mathbb{Z}/p)^{tS^1}$$

collapses at the E^2-term. The image of $p \in \pi_0(S) = \mathbb{Z}$ is represented by (a unit in $\mathbb{Z}/p$ times)

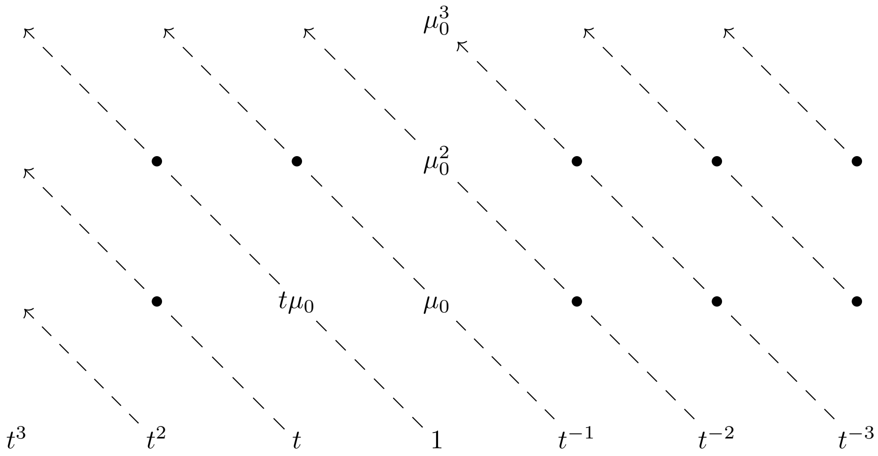
$$t\mu_0 \in \hat{E}^{\infty}_{-2,2}(S^1) .$$

Hence

$$\pi_* THH(\mathbb{Z}/p)^{tS^1} \cong \mathbb{Z}_p[t, t^{-1}]$$

with $t \in \pi_{-2}$. *Additively*

$$\pi_i THH(\mathbb{Z}/p)^{tS^1} \cong \begin{cases} \mathbb{Z}_p\{t^m\} & \text{for } i = -2m, \\ 0 & \text{otherwise.} \end{cases}$$



Corollary 8.5. $\hat{\Gamma} \colon \pi_* TF(\mathbb{Z}/p; p) \to \pi_* THH(\mathbb{Z}/p)^{tS^1}$ *is given by*

$$\hat{\Gamma}(\mu_0) \doteq t^{-1}\,.$$

Corollary 8.6. $R^h \colon \pi_* THH(\mathbb{Z}/p)^{hS^1} \to \pi_* THH(\mathbb{Z}/p)^{tS^1}$ *is given by*

$$R^h(\mu_0) = \mu_0 \doteq pt^{-1}$$
$$R^h(t) = t\,.$$

Hence $R \colon \pi_* TF(\mathbb{Z}/p; p) \to \pi_* TF(\mathbb{Z}/p; p)$ *is given by*

$$R(\mu_0) \doteq p\mu_0\,.$$

Theorem 8.7 (Hesselholt-Madsen (1997))**.** $\pi_*(\mathrm{id} - R) \colon \pi_* TF(\mathbb{Z}/p; p) \to TF(\mathbb{Z}/p; p)$ *is zero for* $* = 0$, *and an isomorphism for* $* \neq 0$. *Hence*

$$\pi_i TC(\mathbb{Z}/p; p) \cong \begin{cases} \mathbb{Z}_p & \text{for } i \in \{-1, 0\}, \\ 0 & \text{otherwise.} \end{cases}$$

The p*-completed cyclotomic trace map* $\mathrm{trc} \colon K(\mathbb{Z}/p)_p \to TC(\mathbb{Z}/p; p)$ *is* (-1)*-coconneted.*

9. The Nikolaus-Scholze reformulation

Note that $(THH(A)^{tC_p})^{C_{p^n}} = THH(A)^{tC_{p^{n+1}}}$. By naturality of the comparison map Γ_n for the S^1-map $\hat{\Gamma}_1 \colon THH(A) \to THH(A)^{tC_p}$ we have a commutative square

(4)
$$\begin{array}{ccc} THH(A)^{C_{p^n}} & \xrightarrow{\hat{\Gamma}_{n+1}} & THH(A)^{tC_{p^{n+1}}} \\ \Gamma_n \downarrow & & \simeq \downarrow G_n \\ THH(A)^{hC_{p^n}} & \xrightarrow{\hat{\Gamma}_1^{hC_{p^n}}} & (THH(A)^{tC_p})^{hC_{p^n}} \end{array}$$

for each $n \geq 0$. In the course of calculations, the map G_n had been seen to be an equivalence in many cases, including $A = H\mathbb{Z}/p$, $H\mathbb{Z}$, ℓ, ku and S. Nikolaus-Scholze prove that it is an equivalence for all connective (symmetric) ring spectra A.

Passing to homotopy limits over n using the F-maps, we obtain an implicitly p-completed commutative square

(5)
$$\begin{tikzcd} TF(A,p) \arrow[r, "\hat\Gamma"] \arrow[d, "\Gamma"'] & THH(A)^{tS^1} \arrow[d, "\simeq"', "G"] \\ THH(A)^{hS^1} \arrow[r, "\hat\Gamma_1^{hS^1}"] & (THH(A)^{tC_p})^{hS^1} \end{tikzcd}$$

In the range where Γ and $\hat\Gamma$ are equivalences, we can therefore replace id and $R\colon TF(A;p) \to TF(A;p)$ by $\hat\Gamma_1^{hS^1}$ and GR^h, respectively:

$$\begin{tikzcd} TF(A;p) \arrow[r, "\mathrm{id}"] \arrow[d, "\Gamma"'] & TF(A;p) \arrow[d, "\hat\Gamma"] \\ THH(A)^{hS^1} \arrow[rd, "\hat\Gamma_1^{hS^1}"'] & THH(A)^{tS^1} \arrow[d, "\simeq"', "G"] \\ & (THH(A)^{tC_p})^{hS^1} \end{tikzcd} \qquad \begin{tikzcd} TF(A;p) \arrow[r, "R"] \arrow[d, "\Gamma"'] & TF(A;p) \arrow[d, "\hat\Gamma"] \\ THH(A)^{hS^1} \arrow[r, "R^h"] \arrow[rd, "GR^h"'] & THH(A)^{tS^1} \arrow[d, "\simeq"', "G"] \\ & (THH(A)^{tC_p})^{hS^1} \end{tikzcd}$$

(These diagrams were used in the TC-calculations by Ausoni, Bökstedt, Hesselholt, Madsen and Rognes.)

Switching notations to write φ_p for $\hat\Gamma_1$ and can for R^h, it follows that $TC(A;p)$ is equivalent to the homotopy equalizer of $\varphi_p^{hS^1} = \hat\Gamma_1^{hS^1}$ and $G\,\mathrm{can} = GR^h$

$$TC(A;p) \longrightarrow THH(A)^{hS^1} \overset{\varphi_p^{hS^1}}{\underset{G\,\mathrm{can}}{\rightrightarrows}} (THH(A)^{tC_p})^{hS^1},$$

at least in the range where $\pi_*(\varphi_p)$ is an isomorphism. A second insight by Nikolaus-Scholze is that this diagram is a homotopy equalizer in all degrees, i.e., without the restriction to degrees where φ_p, Γ and $\hat\Gamma$ are equivalences.

Hence this diagram can be taken as a revised definition of $TC(A;p)$ for connective (symmetric) ring spectra A. It has the advantage that genuinely S^1-equivariant homotopy type of $THH(A)$ only appears in the construction of $\varphi_p\colon THH(A) \to THH(A)^{tC_p}$, and this can be achieved by ∞-categorical methods. The remaining steps, forming C_p-Tate constructions and S^1-homotopy fixed points only depend on the naively S^1-equivariant homotopy type of $THH(A)$, for which the preliminary construction as $|A \odot S^1_\bullet|$ suffices.

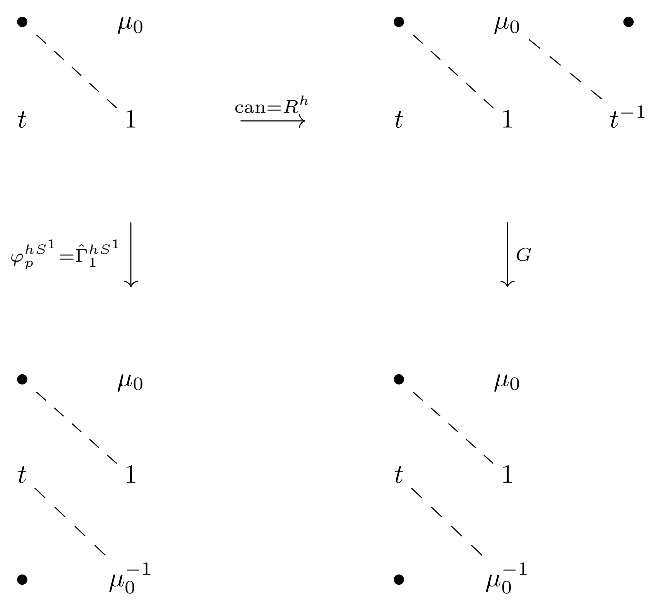
Theorem 9.1. *(a)* $\pi_*THH(\mathbb{Z}/p) = \mathbb{Z}/p[\mu_0]$ *and* $\pi_*THH(\mathbb{Z}/p)^{tC_p} = \mathbb{Z}/p[\mu_0, \mu_0^{-1}]$, *with* $\varphi_p(\mu_0) = \mu_0$.

(b) $\pi_*THH(\mathbb{Z}/p)^{hS^1} = \mathbb{Z}_p[t, \mu_0]/(p \doteq t\mu_0)$ *and* $\pi_*(THH(\mathbb{Z}/p)^{tC_p})^{hS^1} = \mathbb{Z}_p[t, \mu_0, \mu_0^{-1}]/(p \doteq t\mu_0) = \mathbb{Z}_p[\mu_0, \mu_0^{-1}]$, *with* $\varphi_p^{hS^1}: t \mapsto t \doteq p\mu_0^{-1}$ *and* $\mu_0 \mapsto \mu_0$.

(c) $\pi_*THH(\mathbb{Z}/p)^{tS^1} = \mathbb{Z}_p[t, t^{-1}, \mu_0]/(p \doteq t\mu_0) = \mathbb{Z}_p[t, t^{-1}]$, *with* $\mathrm{can}(t) = t$, $\mathrm{can}(\mu_0) = \mu_0$ *and* $G(t) \doteq \mu_0^{-1}$. *Hence* $G\,\mathrm{can}\colon t \,\dot\mapsto\, \mu_0^{-1}$ *and* $\mu_0 \,\dot\mapsto\, p\mu_0$.

Theorem 9.2. $\pi_*(\varphi_p^{hS^1} - G\,\mathrm{can})\colon \pi_*THH(\mathbb{Z}/p)^{hS^1} \to \pi_*(THH(\mathbb{Z}/p)^{tC_p})^{hS^1}$ *is zero for* $* = 0$ *and an isomorphism for* $* \neq 0$. *Hence*

$$\pi_i TC(\mathbb{Z}/p;p) \cong \begin{cases} \mathbb{Z}_p & \text{for } i \in \{-1, 0\}, \\ 0 & \text{otherwise.} \end{cases}$$



10. Calculations for $\mathbb{Z}$

The calculation of $\pi_*(TC(\mathbb{Z};p);\mathbb{Z}/p)$ is due to Bökstedt-Madsen (1994 and 1995) for p odd, and to Rognes (1998 and 1999) for $p=2$. The map $THH(\mathbb{Z}) \wedge S/p \to THH(\mathbb{Z}_p) \wedge S/p$ is an equivalence, so the mod p calculations are the same for $\mathbb{Z}$ and $\mathbb{Z}_p$. We write $THH(\mathbb{Z})$ for brevity, although the case $\mathbb{Z}_p$ is conceptually closer to $\mathbb{Z}/p$.

Let p be an odd prime, and write $\bar{\pi}_* X = \pi_*(X;\mathbb{Z}/p) = \pi_*(X \wedge S/p)$.

Lemma 10.1. *The unit map $S \to THH(\mathbb{Z})^{hS^1}$ takes $\alpha_1 \in \pi_{2p-3}(S)$ to class represented by (a unit in $\mathbb{Z}/p$ times) $t\lambda_1$ in the S^1-homotopy fixed point spectral sequence*

$$E^2_{*,*}(S^1) = \mathbb{Z}[t] \otimes \pi_* THH(\mathbb{Z}) \Longrightarrow \pi_* THH(\mathbb{Z})^{hS^1}\,.$$

Proof. It suffices to prove that α_1 is detected in $\pi_{2p-3}(X)$ where $X = F(S^3, THH(\mathbb{Z}))^{S^1}$. The proof is similar to the argument for why p is detected in $\pi_0 F(S^3_+, THH(\mathbb{Z}/p))^{S^1}$, using the cobar representative $[\bar{\xi}_1]$ for α_1 in place of $\bar{\tau}_0$. The key point is that $\sigma(\bar{\xi}_1) \neq 0$, so that $\bar{\xi}_1 \in H_* THH(\mathbb{Z})$ does not lift to $H_* X$ in the long exact sequence (3). Hence $[\bar{\xi}_1]$ does not become a coboundary in the cobar complex calculating $\mathrm{Ext}^{*,*}_{\mathscr{A}_*}(\mathbb{Z}/p, H_* X)$. □

Lemma 10.2. *The unit map takes $\alpha_1 \in \bar{\pi}_{2p-3}(S)$ and $v_1 \in \bar{\pi}_{2p-2}(S)$ to classes represented by (units in $\mathbb{Z}/p$ times) $t\lambda_1$ and $t\mu_1$ in the S^1-homotopy fixed point spectral sequence*

$$E^2_{*,*}(S^1;\mathbb{Z}/p) = P(t) \otimes E(\lambda_1) \otimes P(\mu_1) \Longrightarrow \bar{\pi}_* THH(\mathbb{Z})^{hS^1}\,.$$

Proof. It suffices to prove that α_1 and v_1 are detected in $\bar{\pi}_*(X)$. The case of α_1 follows by reduction from integral to mod p coefficients. The case of v_1 follows by naturality with respect to the Bockstein homomorphism, since $\beta(v_1) = \alpha_1$. □

Proposition 10.3 (Bökstedt)**.** *There is a class $\lambda_1^K \in \bar{\pi}_* K(\mathbb{Z})$ with $\mathrm{tr}(\lambda_1^K) = \lambda_1$ in $\bar{\pi}_* THH(\mathbb{Z})$.*

Proof. The proof uses a comparison of $A(*) = K(S)$ with $K(\mathbb{Z})$ along the linearization map $S \to H\mathbb{Z}$. See Bökstedt-Madsen (1994) and Rognes (1998). □

((Q: Is this a priori knowledge essential?))

Corollary 10.4. *The class λ_1 is an infinite cycle in the S^1-homotopy fixed point spectral sequence for $THH(\mathbb{Z})$, hence also in the C_{p^n}-homotopy fixed point, S^1-Tate and C_{p^n}-Tate spectral sequences.*

Proof. The image of λ_1^K under

$$K(\mathbb{Z}) \xrightarrow{\text{trc}} TC(\mathbb{Z};p) \longrightarrow TF(\mathbb{Z};p) \xrightarrow{\Gamma} THH(\mathbb{Z})^{hS^1}$$

is mapped to λ_1 under the edge homomorphism

$$F\colon THH(\mathbb{Z})^{hS^1} \longrightarrow THH(\mathbb{Z}),$$

hence is represented by $\lambda_1 \in E^\infty_{0,2p-1}(S^1)$. □

Proposition 10.5. *The map $\hat\Gamma_1 \wedge S/p\colon THH(\mathbb{Z}) \wedge S/p \to THH(\mathbb{Z})^{tC_p} \wedge S/p$ is (-1)-coconnected.*

Corollary 10.6. *Each map $\Gamma_n \wedge S/p$, $\Gamma \wedge S/p$, $\hat\Gamma_n \wedge S/p$ and $\hat\Gamma \wedge S/p$ is (-1)-connected.*

Proof of Proposition 10.5. Consider the C_p-Tate spectral sequence

$$\begin{aligned}\hat E^2_{*,*}(C_p;\mathbb{Z}/p) = \hat H^{-*}(C_p;\bar\pi_* THH(\mathbb{Z})) &\cong E(u_1)\otimes P(t,t^{-1})\otimes E(\lambda_1)\otimes P(\mu_1)\\ &\Longrightarrow \bar\pi_* THH(\mathbb{Z})^{tC_p}.\end{aligned}$$

By naturality with respect to $THH(\mathbb{Z})^{hS^1} \to THH(\mathbb{Z})^{tC_p}$ the images of the classes α_1 and v_1 are represented by $t\lambda_1$ and $t\mu_1$, respectively. Since the unit map factors through

$$\hat\Gamma_1\colon THH(\mathbb{Z}) \longrightarrow THH(\mathbb{Z})^{tC_p}$$

and α_1 and v_1 map to zero in $\bar\pi_* THH(\mathbb{Z})$, the classes $t\lambda_1$ and $t\mu_1$ must be boundaries in the C_p-Tate spectral sequence. We must have $d^2(\lambda_1)=0$ by naturality with respect to mod p reduction. Hence the only possible differentials are

$$\begin{aligned} d^{2p}(t^{1-p}) &\doteq t\lambda_1\\ d^{2p+1}(u_1 t^{-p}) &\doteq t\mu_1 .\end{aligned}$$

Writing

$$\hat E^2_{*,*}(C_p;\mathbb{Z}/p) = E(u_1)\otimes P(t,t^{-1})\otimes E(\lambda_1)\otimes P(t\mu_1)$$

the d^{2p}-differential cancels the t^i-terms for $p\nmid i$, i.e., for $v_p(i)=0$. Thus

$$\hat E^{2p+1}_{*,*}(C_p;\mathbb{Z}/p) = E(u_1)\otimes P(t^p,t^{-p})\otimes E(\lambda_1)\otimes P(t\mu_1)$$

and

$$\hat E^{2p+2}_{*,*}(C_p;\mathbb{Z}/p) = P(t^p,t^{-p})\otimes E(\lambda_1).$$

There is no room for further differentials, so this is also $\hat E^\infty_{*,*}(C_p;\mathbb{Z}/p)$. Hence

$$\bar\pi_* THH(\mathbb{Z})^{tC_p} \cong E(\lambda_1)\otimes P(t^p,t^{-p}).$$

In the commutative square

$$\begin{array}{ccc} \bar\pi_* THH(\mathbb{Z}) & \xrightarrow{\hat\Gamma_1} & \bar\pi_* THH(\mathbb{Z})^{tC_p} \\ \downarrow & & \downarrow \\ \bar\pi_* THH(\mathbb{Z}/p) & \xrightarrow{\hat\Gamma_1} & \bar\pi_* THH(\mathbb{Z}/p)^{tC_p} \end{array}$$

the class μ_1 maps down to μ_0^p (since $\sigma\bar\tau_1 = (\sigma\bar\tau_0)^p$), which maps across to t^{-p} (up to a unit in $\mathbb{Z}/p$). Hence $\hat\Gamma_1(\mu_1)\doteq t^{-p}$.

We claim that $\hat\Gamma_1(\lambda_1)=\lambda_1$. Then the induced homomorphism in mod p homotopy,

$$\hat\Gamma_1\colon E(\lambda_1)\otimes P(\mu_0) \longrightarrow E(\lambda_1)\otimes P(t^p,t^{-p}),$$

inverts μ_1 and is (-1)-connected.

To prove the claim, consider $\tilde\lambda_1 = \mathrm{tr}_p(\lambda_1^K) \in \bar\pi_* THH(\mathbb{Z})^{C_p}$. We have $R(\tilde\lambda_1)=\lambda_1$ and $F(\tilde\lambda_1)=\lambda_1$, so $\Gamma_1(\tilde\lambda_1)$ is represented by λ_1 in the C_p-homotopy fixed point spectral sequence. Hence $\hat\Gamma_1(\lambda_1) = R^h\Gamma_1(\tilde\gamma_1)$ is represented by λ_1 in the C_p-Tate spectral sequence. □

We write $p^b+\dots+p^a$ for the truncated geometric series $\sum_{i=a}^b p^i$, and let $P_h(x)=P(x)/(x^h)$ be the truncated polynomial algebra of height h.

Theorem 10.7. *The S^1-Tate spectral sequence*

$$\hat{E}^2_{*,*}(S^1;\mathbb{Z}/p) = \hat{H}^{-*}(S^1;\bar{\pi}_* THH(\mathbb{Z})) \cong P(t,t^{-1}) \otimes E(\lambda_1) \otimes P(\mu_1) \Longrightarrow \bar{\pi}_* THH(\mathbb{Z})^{tS^1}$$

has differentials

$$d^r(t^{p^k - p^{k+1}}) \doteq t^{p^k} \cdot \lambda_1 \cdot (t\mu_1)^{p^k+\dots+p}$$

with $r = 2(p^{k+1}+\dots+p)$, for all $k \geq 0$. The E^∞-term is

$$\hat{E}^\infty_{*,*}(S^1;\mathbb{Z}/p) = E(\lambda_1) \otimes P(t\mu_1) \oplus \bigoplus_{v_p(i)=m} P_{p^m+\dots+p}(t\mu_1)\{t^i\lambda_1\}\,.$$

Proof of Theorem 10.7. We determine the structure of the C_{p^n}-Tate and C_{p^n}-homotopy fixed point spectral sequences for $THH(\mathbb{Z})$, by induction on $n \geq 1$. Assume that the C_{p^n}-Tate spectral sequence

$$\hat{E}^2_{*,*}(C_{p^n};\mathbb{Z}/p) = \hat{H}^{-*}(C_{p^n};\bar{\pi}_* THH(\mathbb{Z})) \cong E(u_n) \otimes P(t,t^{-1}) \otimes E(\lambda_1) \otimes P(\mu_1) \Longrightarrow \bar{\pi}_* THH(\mathbb{Z})^{tC_{p^n}}$$

has n even length differentials

$$d^r(t^{p^k - p^{k+1}}) \doteq t^{p^k} \cdot \lambda_1 \cdot (t\mu_1)^{p^k+\dots+p}$$

with $r = 2(p^{k+1}+\dots+p)$, for all $0 \leq k < n$, followed by an odd length differential

$$d^r(u_n t^{-p^n}) \doteq (t\mu_1)^{p^{n-1}+\dots+1}$$

for $r = 2(p^n+\dots+p)+1$. This assumption holds for $n=1$ by the proof of Proposition 10.5.

The intermediate terms in the C_{p^n}-Tate spectral sequence are then

$$\hat{E}^r_{*,*}(C_{p^n};\mathbb{Z}/p) = E(u_n) \otimes P(t^{p^k}, t^{-p^k}) \otimes E(\lambda_1) \otimes P(t\mu_1) \oplus \bigoplus_{v_p(i)=m<k} E(u_n) \otimes P_{p^m+\dots+p}(t\mu_1)\{t^i\lambda_1 \mid v_p(i)=m\}$$

for $0 \leq k \leq n$ and $r = 2(p^k+\dots+p)+1$. For $k=0$, this is the known formula for the E^2-term. Assuming the formula for some $0 \leq k < n$, the d^r-differential with $r = 2(p^{k+1}+\dots+p)$ acts on the t^i-terms with $v_p(i)=k$, replacing

$$E(u_n) \otimes P(t^{p^k}, t^{-p^k}) \otimes E(\lambda_1) \otimes P(t\mu_1)$$

with the direct sum of

$$E(u_n) \otimes P(t^{p^{k+1}}, t^{-p^{k+1}}) \otimes E(\lambda_1) \otimes P(t\mu_1)$$

and

$$E(u_n) \otimes P_{p^k+\dots+p}(t\mu_1)\{t^i\lambda_1 \mid v_p(i)=k\}\,.$$

This verifies the formula for $k+1$. By finite induction it also holds for $k=n$.

The odd length d^r-differential replaces

$$E(u_n) \otimes P(t^{p^n}, t^{-p^n}) \otimes E(\lambda_1) \otimes P(t\mu_1)$$

with

$$P(t^{p^n}, t^{-p^n}) \otimes E(\lambda_1) \otimes P_{p^{n-1}+\dots+1}(t\mu_1)\,.$$

Hence

$$\hat{E}^r_{*,*}(C_{p^n};\mathbb{Z}/p) = P(t^{p^n}, t^{-p^n}) \otimes E(\lambda_1) \otimes P_{p^{n-1}+\dots+1}(t\mu_1) \oplus \bigoplus_{v_p(i)=m<n} E(u_n) \otimes P_{p^m+\dots+p}(t\mu_1)\{t^i\lambda_1 \mid v_p(i)=m\}$$

for $r = 2(p^n+\dots+1)$. This is also the E^∞-term, because there is no room for further differentials.

The C_{p^n}-homotopy spectral sequence

$$E^2_{*,*}(C_{p^n};\mathbb{Z}/p) = H^{-*}(C_{p^n};\bar\pi_* THH(\mathbb{Z})) \cong E(u_n)\otimes P(t)\otimes E(\lambda_1)\otimes P(\mu_1) \Longrightarrow \bar\pi_* THH(\mathbb{Z})^{hC_{p^n}}$$

is obtained by truncating the (upper half-plane) C_{p^n}-Tate spectral sequence to the second quadrant. It is algebraically simpler to describe the localized (left half-plane) spectral sequence

$$\mu_1^{-1}E^2_{*,*}(C_{p^n};\mathbb{Z}/p) = H^{-*}(C_{p^n};\mu_1^{-1}\bar\pi_* THH(\mathbb{Z})) \cong E(u_n)\otimes P(t)\otimes E(\lambda_1)\otimes P(\mu_1,\mu_1^{-1}) \Longrightarrow \bar\pi_*(THH(\mathbb{Z})^{tC_p})^{hC_{p^n}}\,.$$

Since $\mu_1^{-1}\bar\pi_* THH(\mathbb{Z}) \cong \bar\pi_* THH(\mathbb{Z})^{tC_p}$, this is the same as the C_{p^n}-homotopy fixed point spectral sequence for $THH(\mathbb{Z})^{tC_p}$. Since the localization map $\bar\pi_* THH(\mathbb{Z}) \to \mu_1^{-1}\bar\pi_* THH(\mathbb{Z})$ is (-1)-coconnected, the spectral sequences $E^r_{*,*}(C_{p^n};\mathbb{Z}/p)$ and $\mu_1^{-1}E^r_{*,*}(C_{p^n};\mathbb{Z}/p)$ agree in total degrees $* \ge 0$. Hence we can use the localized spectral sequence to calculate $\bar\pi_* THH(\mathbb{Z})^{hC_{p^n}}$ in degrees $* \ge 0$, which then also calculates $\bar\pi_* TF(\mathbb{Z},p)$ is this range of degrees. (The negative homotopy groups of $TF(\mathbb{Z},p)$ are zero.)

The C_{p^n}-homotopy fixed point spectral sequence $E^r_{*,*}(C_{p^n};\mathbb{Z}/p)$ has n even length differentials

$$d^r(t^{p^k}) \doteq t^{p^k+p^{k+1}}\cdot\lambda_1\cdot(t\mu_1)^{p^k+\dots+p}$$

with $r = 2(p^{k+1}+\dots+p)$, for all $0\le k<n$, followed by an odd length differential

$$d^r(u_n) \doteq t^{p^n}\cdot(t\mu_1)^{p^{n-1}+\dots+1}$$

for $r = 2(p^n+\dots+p)+1$. Hence the localized C_{p^n}-homotopy fixed point spectral sequence

$$\mu_1^{-1}E^r_{*,*}(C_{p^n};\mathbb{Z}/p) \Longrightarrow \bar\pi_*(THH(\mathbb{Z})^{tC_p})^{hC_{p^n}}$$

has n even length differentials

$$d^r(\mu_1^{p^k}) \doteq \lambda_1\cdot(t\mu_1)^{p^{k+1}+\dots+p}\cdot\mu_1^{p^k-p^{k+1}}$$

with $r = 2(p^{k+1}+\dots+p)$, for all $0\le k<n$, followed by an odd length differential

$$d^r(u_n\mu_1^{p^n}) \doteq (t\mu_1)^{p^n+\dots+1}$$

for $r = 2(p^n+\dots+p)+1$.

The intermediate terms are then

$$\mu_1^{-1}E^r_{*,*}(C_{p^n};\mathbb{Z}/p) = E(u_n)\otimes E(\lambda_1)\otimes P(t\mu_1)\otimes P(\mu_1^{p^k},\mu_1^{-p^k}) \oplus \bigoplus_{v_p(j)=m<k} E(u_n)\otimes P_{p^{m+1}+\dots+p}(t\mu_1)\{\lambda_1\mu_1^j \mid v_p(j)=m\}\,.$$

for $k=0$ and $r=2$, as well as for $0<k\le n$ and $r=2(p^k+\dots+p)+1$. For $k=0$, this is the known formula for the E^2-term. Assuming the formula for some $0\le k<n$, the d^r-differential with $r=2(p^{k+1}+\dots+p)$ acts on the μ_1^j-terms with $v_p(j)=k$, replacing

$$E(u_n)\otimes E(\lambda_1)\otimes P(t\mu_1)\otimes P(\mu_1^{p^k},\mu_1^{-p^k})$$

with the direct sum of

$$E(u_n)\otimes E(\lambda_1)\otimes P(t\mu_1)\otimes P(\mu_1^{p^{k+1}},\mu_1^{-p^{k+1}})$$

and

$$E(u_n)\otimes P_{p^{k+1}+\dots+p}(t\mu_1)\{\lambda_1\mu_1^j \mid v_p(j)=k\}\,.$$

This verifies the formula for $k+1$. By finite induction it also holds for $k=n$.

The odd length d^r-differential replaces

$$E(u_n)\otimes E(\lambda_1)\otimes P(t\mu_1)\otimes P(\mu_1^{p^n},\mu_1^{-p^n})$$

with

$$E(\lambda_1)\otimes P_{p^n+\dots+1}(t\mu_1)\otimes P(\mu_1^{p^n},\mu_1^{-p^n})\,.$$

Hence

$$\mu_1^{-1}E^r(C_{p^n};\mathbb{Z}/p) = E(\lambda_1)\otimes P_{p^n+\dots+1}(t\mu_1)\otimes P(\mu_1^{p^n},\mu_1^{-p^n})$$
$$\oplus \bigoplus_{v_p(j)=m<n} E(u_n)\otimes P_{p^{m+1}+\dots+p}(t\mu_1)\{\lambda_1\mu_1^j \mid v_p(j)=m\}$$

for $r = 2(p^n+\dots+1)$. This is also the E^∞-term, because there is no room for further differentials. As previously discussed it is also the E^∞-term of the C_{p^n}-homotopy fixed point spectral sequence, in total degress $* \ge 0$.

Counting classes, we obtain

$$\dim_{\mathbb{Z}/p} \bar\pi_i THH(\mathbb{Z})^{hC_{p^n}} = \begin{cases} n & \text{for } i \not\equiv 0,-1 \mod 2p^{n+1}, \\ n+1 & \text{for } i \equiv 0,-1 \mod 2p^{n+1}, \end{cases}$$

assuming that $i \ge 0$. By Corollary 10.6, the same formula gives $\dim_{\mathbb{Z}/p} \bar\pi_i TF(\mathbb{Z};p)$ and $\dim_{\mathbb{Z}/p} \bar\pi_i THH(\mathbb{Z})^{tC_{p^{n+1}}}$, for $i \ge 0$. Hence we know the rank of the abutment of the $C_{p^{n+1}}$-Tate spectral sequence

$$\hat E^2_{*,*}(C_{p^{n+1}};\mathbb{Z}/p) = \hat H^{-*}(C_{p^{n+1}};\bar\pi_* THH(\mathbb{Z})) \cong E(u_{n+1})\otimes P(t,t^{-1})\otimes E(\lambda_1)\otimes P(\mu_1)$$
$$\Longrightarrow \bar\pi_* THH(\mathbb{Z})^{tC_{p^{n+1}}}\,.$$

We use this to establish the $n+1$ case of the inductive hypothesis.

By naturality with respect to the Frobenius and Verschiebung maps

$$F\colon THH(\mathbb{Z})^{tC_{p^{n+1}}} \longrightarrow THH(\mathbb{Z})^{tC_{p^n}}$$
$$V\colon THH(\mathbb{Z})^{tC_{p^n}} \longrightarrow THH(\mathbb{Z})^{tC_{p^{n+1}}}$$

it follows from the differential structure of the C_{p^n}-Tate spectral sequence that the $C_{p^{n+1}}$-Tate spectral sequence has n even length differentials

$$d^r(t^{p^k-p^{k+1}}) \doteq t^{p^k}\cdot\lambda_1\cdot(t\mu_1)^{p^k+\dots+p}$$

with $r = 2(p^{k+1}+\dots+p)$, for $0 \le k < n$. Thereafter, the odd length differential

$$d^r(u_{n+1}t^{-p^n}) = 0$$

vanishes, for $r = 2(p^n+\dots+p)+1$. Hence

$$\hat E^r_{*,*}(C_{p^{n+1}};\mathbb{Z}/p) = E(u_{n+1})\otimes P(t^{p^n},t^{-p^n})\otimes E(\lambda_1)\otimes P(t\mu_1)$$
$$\oplus \bigoplus_{v_p(i)=m<n} E(u_{n+1})\otimes P_{p^m+\dots+p}(t\mu_1)\{t^i\lambda_1 \mid v_p(i)=m\}$$

for $r = 2(p^n+\dots+1)$.

It remains to show that there one more even length differential

$$d^r(t^{p^n-p^{n+1}}) \doteq t^{p^n}\cdot\lambda_1\cdot(t\mu_1)^{p^n+\dots+p} \tag{6}$$

with $r = 2(p^{n+1}+\dots+p)$, followed by a new odd length differential

$$d^r(u_{n+1}t^{-p^{n+1}}) \doteq (t\mu_1)^{p^n+\dots+1} \tag{7}$$

for $r = 2(p^{n+1}+\dots+p)+1$.

The first target class,

$$t^{p^n}\cdot\lambda_1\cdot(t\mu_1)^{p^n+\dots+p}\,,$$

is in total degree $2p^{n+1}-2p^n-1 \not\equiv 0,-1 \mod 2p^{n+1}$, where there are a total of n permanent cycles (generators of the E^∞-term). These are of the form $t^i\cdot\lambda_1\cdot\mu_1^j$ with

$$i = p^n - p^{n+1} + p^{k+1} + \dots + p$$
$$j = p^k + \dots + p$$

for $0 \le k < n$. Thus the next infinite cycle $t^{p^n}\cdot\lambda_1\cdot(t\mu_1)^{p^n+\dots+p}$, which is the case $k=n$ of the formulas above, must be a boundary. Checking bidegrees, the only possible source of a differential hitting this class is $t^{p^n-p^{n+1}}$. This establishes the asserted even length differential (6).

Next, we claim that there is a differential

$$d^r(u_{n+1}t^{-p^{n+1}} \cdot \lambda_1) \doteq \lambda_1 \cdot (t\mu_1)^{p^n+\cdots+1} \tag{8}$$

with $r = 2(p^{n+1} + \cdots + p) + 1$, which implies the asserted odd length differential (7). The total degree of $\lambda_1 \cdot (t\mu_1)^{p^n+\cdots+1}$ is $2p^{n+1} + 2p - 3 \not\equiv 0, -1 \mod 2p^{n+1}$, so there are precisely n permanent cycles in this total degree. These are the classes $t^i \cdot \lambda_1 \cdot \mu_1^j$ with

$$i = -p^{n+1} + p^{k+1} + \cdots + 1$$
$$j = p^k + \cdots + 1$$

for $0 \le k < n$. Thus the next infinite cycle $t^{p^n+\cdots+1} \cdot \lambda_1 \cdot \mu_1^{p^n+\cdots+1} = \lambda_1 \cdot (t\mu_1)^{p^n+\cdots+1}$, corresponding to $k = n$, must be a boundary. For bidegree reasons the only possible source of a differential hitting this class is $u_{n+1}t^{-p^{n+1}} \cdot \lambda_1$, and this establishes the asserted odd length differential (8). □

Theorem 10.8. *The S^1-homotopy fixed point spectral sequence*

$$E^2_{*,*}(S^1; \mathbb{Z}/p) = H^{-*}(S^1; \bar\pi_* THH(\mathbb{Z})) \cong P(t) \otimes E(\lambda_1) \otimes P(\mu_1)$$
$$\Longrightarrow \bar\pi_* THH(\mathbb{Z})^{hS^1} .$$

agrees in non-negative total degrees with the localized S^1-homotopy fixed point spectral sequence

$$\mu_1^{-1}E^2_{*,*}(S^1; \mathbb{Z}/p) = H^{-*}(S^1; \mu_1^{-1}\bar\pi_* THH(\mathbb{Z})) \cong P(t) \otimes E(\lambda_1) \otimes P(\mu_1, \mu_1^{-1})$$
$$\Longrightarrow \bar\pi_*(THH(\mathbb{Z})^{tC_p})^{hS^1} ,$$

which has differentials

$$d^r(\mu_1^{p^k}) \doteq \lambda_1 \cdot (t\mu_1)^{p^k+\cdots+p} \cdot \mu_1^{p^k-p^{k+1}}$$

with $r = 2(p^{k+1} + \cdots + p)$, for all $k \ge 0$. The localized E^∞-term is

$$\mu_1^{-1}E^\infty_{*,*}(S^1; \mathbb{Z}/p) = E(\lambda_1) \otimes P(t\mu_1)$$
$$\oplus \bigoplus_{v_p(j)=m} P_{p^{m+1}+\cdots+p}(t\mu_1)\{\lambda_1\mu_1^j\} .$$

Theorem 10.9. *$\bar\pi_* TF(\mathbb{Z}; p)$ and $\bar\pi_* TP(\mathbb{Z}) = \bar\pi_* THH(\mathbb{Z})^{hS^1}$ agree in degrees $* \ge 0$ with*

$$\bar\pi_*(THH(\mathbb{Z})^{tC_p})^{hS^1} \cong E(\lambda_1) \otimes P(v_1) \oplus \bigoplus_{v_p(j)=m} P_{p^{m+1}+\cdots+p}(v_1)\{\lambda_1\mu_1^j\}$$

and

$$\bar\pi_* THH(\mathbb{Z})^{tS^1} \cong E(\lambda_1) \otimes P(v_1) \oplus \bigoplus_{v_p(i)=m} P_{p^m+\cdots+p}(v_1)\{t^i\lambda_1\} .$$

Proof. $v_1 \in \bar\pi_*(S)$ is detected by $t\mu_1$. □

In non-negative degrees the restriction map $R\colon TF(\mathbb{Z}; p) \to TF(\mathbb{Z}; p)$ agrees with $R^h\colon THH(\mathbb{Z})^{hS^1} \to THH(\mathbb{Z})^{tS^1}$. At the level of E^∞-terms, this is the homomorphism

$$E(\lambda_1) \otimes P(t\mu_1) \oplus \bigoplus_{v_p(j)=m} P_{p^{m+1}+\cdots+p}(t\mu_1)\{\lambda_1\mu_1^j\} \xrightarrow{R^h} E(\lambda_1) \otimes P(t\mu_1) \oplus \bigoplus_{v_p(i)=m} P_{p^m+\cdots+p}(t\mu_1)\{t^i\lambda_1\}$$

where the target corresponds under G to

$$E(\lambda_1) \otimes P(t\mu_1) \oplus \bigoplus_{v_p(j)=m} P_{p^{m+1}+\cdots+p}(t\mu_1)\{\lambda_1\mu_1^j\} .$$

On the summand $E(\lambda_1) \otimes P(t\mu_1)$ the restriction map agrees with the identity, also in $\bar\pi_* TF(\mathbb{Z}; p)$, since λ_1 and $t\mu_1$ are in the image from $\bar\pi_* K(\mathbb{Z})$.

The summand $P_{p^{m+1}+\cdots+p}(t\mu_1)\{\lambda_1\mu_1^j\}$, with $v_p(j) = m \ge 0$, is concentrated in total degrees

$$2p - 1 + 2pj \le * \le 2p(j + p^{m+1} - 1) + 1 .$$

These are all negative if $j \leq -p^{m+1}$. For $j \geq 0$ the summand maps by multiplication by $(t\mu_1)^j$ to $P_{p^m+\cdots+p}(t\mu_1)\{\lambda_1 t^{-j}\}$. This is zero for $j \geq 2p^m$. ((The case $j = p^m$ is exceptional. Is R nilpotent on this summand?)) In the remaining cases $j = -p^m d$ with $1 \leq d \leq p-1$. The R^h-map

$$P_{p^{m+1}+\cdots+p}(t\mu_1)\{\lambda_1\mu_1^{-p^m d}\}_{*\geq 0} \longrightarrow P_{p^m+\cdots+p}(t\mu_1)\{t^{p^m d}\lambda_1\}_{*\geq 0}$$

is surjective, because $(p^{m+1} + \cdots + p) - p^m d \geq (p^m + \cdots + p)$. For $m \geq 1$ the target corresponds under G to $P_{p^m+\cdots+p}(t\mu_1)\{\lambda_1\mu_1^{-p^{m-1}d}\}$, i.e., the summand corresponding to j/p in place of j.

These calculations can be lifted from the E^∞-level to the abutment. (The de Rham–Witt formalism helps to structure these lifts.) Granting this, we can recognize the components

$$1 - R\colon \prod_{m\geq 0} P_{p^{m+1}+\cdots+p}(v_1)\{\lambda_1\mu_1^{-p^m d}\}_{*\geq 0} \longrightarrow \prod_{m\geq 0} P_{p^{m+1}+\cdots+p}(v_1)\{\lambda_1\mu_1^{-p^m d}\}_{*\geq 0}$$

for $1 \leq d \leq p-1$ as the homomorphisms with kernel

$$\lim_m P_{p^{m+1}+\cdots+p}(v_1)\{\lambda_1\mu_1^{-p^m d}\}_{*\geq 0} \cong P(v_1)\{\lambda_1\mu_1^{-p^m d}\}_{*\geq 0} \cong P(v_1)\{t^d\lambda_1\}$$

and cokernel $\mathrm{Rlim}_m(-) = 0$.

Theorem 10.10 (Bökstedt-Madsen (1994, 1995))**.** *Let p be any odd prime.*

$$\begin{aligned}\bar{\pi}_* TC(\mathbb{Z};p) &\cong P(v_1) \otimes E(\partial, \lambda_1) \oplus P(v_1)\{t^d\lambda_1 \mid 1 \leq d \leq p-1\} \\ &\cong P(v_1)\{\partial, 1, t^{p-1}\lambda_1, \dots, t\lambda_1, \partial\lambda_1, \lambda_1\}\end{aligned}$$

is a free $P(v_1) = \mathbb{Z}/p[v_1]$-module of rank $p+3$.

Corollary 10.11. *Let p be any odd prime, let ku be the connective complex K-theory spectrum, and let j be the connective image-of-J spectrum. After p-adic completion there is an equivalence*

$$K(\mathbb{Z}_p) \simeq_p j \vee \Sigma j \vee \Sigma^3 ku\,.$$

Sketch proof. One first constructs maps $S \vee \Sigma S \to K(\mathbb{Z}_p)$, and factors them through $j \vee \Sigma j$. The homotopy cofiber has the mod p homotopy of Σku, including the v_1-action, and this characterizes this spectrum, by Rognes (1993). □

The analogous results for $p = 2$ were obtained in Rognes (1998/1999).

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