TOPOLOGICAL CYCLIC HOMOLOGY OF $\mathbb{S}$-ALGEBRAS

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November 16th–23rd 1998

1. Some categories.

Let Γ^{op} be the category of finite pointed sets $k_+ = \{0, 1, \dots, k\}$ for $k \geq 0$, and base-point preserving functions. We often write $* = 0_+$ and $S^0 = 1_+$.

Let $I \subset \Gamma^{op}$ be the subcategory of injective functions $k_+ \to l_+$. Its admits a wedge sum $\vee \colon I \times I \to I$ taking (k_+, l_+) to their concatenation $(k+l)_+$.

Let $\mathcal{C}$ be a small $\Gamma\mathcal{S}_*$-category. In the case when $\mathcal{C}$ has only one object $*$ we let $A = \underline{\mathcal{C}}(*, *) \in \Gamma\mathcal{S}_*$. The composition makes A a monoid in $(\Gamma\mathcal{S}_*, \wedge, \mathbb{S})$, which is what we also call an $\mathbb{S}$-algebra.

Let Λ be Connes' cyclic category, with $ob\Lambda = ob\Delta = \{[q] \mid q \geq 0\}$, and morphism sets

$$\Lambda([p], [q]) = \Delta([p], [q]) \times C_{p+1} \,.$$

As a category, Λ is generated by the morphisms $\delta_i \colon [q-1] \to [q]$ and $\sigma_j \colon [q+1] \to [q]$ in Δ, together with the cyclic structure maps $\tau_q \colon [q] \to [q]$ of order $q+1$, subject to a list of relations, which we omit.

A cyclic set X is a functor $X \colon \Lambda^{op} \to \mathcal{E}ns$, and similarly for cyclic objects in other categories. We denote the images of δ_i, σ_j and τ_q by d_i, s_j and t_q, respectively. The omitted relations in Λ translate to the following *cyclic identities*, in addition to the simplicial identities:

$$\begin{aligned}
d_i t_q &= t_{q-1} d_{i-1} \qquad && 0 < i \leq q \\
d_0 t_q &= d_q && \\
s_j t_q &= t_{q+1} s_{j-1} \qquad && 0 < j \leq q \\
s_0 t_q &= t_{q+1}^2 s_q && \\
t_q^{q+1} &= 1 &&
\end{aligned}$$

By restriction to $\Delta^{op} \subset \Lambda^{op}$, a cyclic object determines an underlying simplicial object.

2. Topological Hochschild homology.

For $x = k_+ \in obI$ let

$$S^x = \bigwedge_{x \setminus *} S^1 \cong S^k \,,$$

and let $\Omega^x Y = \underline{\mathcal{S}_*}(S^x, Y) \cong \Omega^k X$ be the simplicial mapping space. For any $(q+1)$-tuple $x = (x_0, x_1, \dots, x_q)$ let $\vee x = x_0 \vee x_1 \vee \dots \vee x_q$.

For each $q \geq 0$ and $x = (x_0, x_1, \dots, x_q) \in I^{q+1}$ let

$$V(\mathcal{C}, x) = \bigvee_{c_0,\dots,c_q} \underline{\mathcal{C}}(c_0, c_q)(S^{x_0}) \wedge \underline{\mathcal{C}}(c_1, c_0)(S^{x_1}) \wedge \cdots \wedge \underline{\mathcal{C}}(c_q, c_{q-1})(S^{x_q}) .$$

In the case of an $\mathbb{S}$-algebra A this simplifies to

$$V(A, x) = A(S^{x_0}) \wedge A(S^{x_1}) \wedge \cdots \wedge A(S^{x_q}) .$$

The association

$$x \mapsto \Omega^{\vee x} V(\mathcal{C}, x)$$

is a functor $I^{q+1} \to \mathcal{S}_*$. Its homotopy colimit defines the simplicial space

$$THH(\mathcal{C})_q = \operatorname*{hocolim}_{x \in I^{q+1}} \Omega^{\vee x} V(\mathcal{C}, x) .$$

This can be extended to a simplicial Γ-space, by defining

$$k_+ \mapsto THH(\mathcal{C})_q(k_+) = \operatorname*{hocolim}_{x \in I^{q+1}} \Omega^{\vee x}(V(\mathcal{C}, x) \wedge k_+) .$$

In the case of an $\mathbb{S}$-algebra, its associated spectrum is stably equivalent to that of $A \wedge \cdots \wedge A = A^{\wedge(q+1)}$.

There is a cyclic category $[q] \mapsto I^{q+1}$, with simplicial structure maps $d_i \colon I^{q+1} \to I^q$ given by

$$d_i(x_0, \dots, x_q) = \begin{cases} (x_0, \dots, x_i \vee x_{i+1}, \dots, x_q) & \text{for } 0 \leq i < q, \\ (x_q \vee x_0, x_1, \dots, x_{q-1}) & \text{for } i = q, \end{cases}$$

together with

$$s_j(x_0, \dots, x_q) = (x_0, \dots, x_j, *, x_{j+1}, \dots, x_q) .$$

The extension to a cyclic structure is given by the cyclic permutation

$$t_q(x_0, \dots, x_q) = (x_q, x_0, \dots, x_{q-1}) .$$

Likewise there are maps

$$\begin{aligned} d_i &\colon \Omega^{\vee x} V(\mathcal{C}, x) \to \Omega^{\vee d_i(x)} V(\mathcal{C}, d_i(x)) \\ s_j &\colon \Omega^{\vee x} V(\mathcal{C}, x) \to \Omega^{\vee s_j(x)} V(\mathcal{C}, s_j(x)) \\ t_q &\colon \Omega^{\vee x} V(\mathcal{C}, x) \to \Omega^{\vee t_q(x)} V(\mathcal{C}, t_q(x)) \end{aligned}$$

for $0 \leq i, j \leq q$. Here d_i uses the composition

$$\underline{\mathcal{C}}(c_i, c_{i-1})(S^{x_i}) \wedge \underline{\mathcal{C}}(c_{i+1}, c_i)(S^{x_{i+1}}) \xrightarrow{\circ} \underline{\mathcal{C}}(c_{i+1}, c_{i-1})(S^{x_i \vee x_{i+1}})$$

for $0 < i < q$. When $i = 0$ the same formula applies, interpreting c_{-1} as c_q. The last face map d_q also involves twist maps cyclically permuting the smash factors in $V(\mathbb{S}, x) = S^{x_0} \wedge \cdots \wedge S^{x_q} \cong S^{\vee x}$ and $V(\mathcal{C}, x)$. Next, s_j inserts the unit map below into $V(\mathcal{C}, x)$:

$$S^0 \xrightarrow{1_{c_j}} \underline{\mathcal{C}}(c_j, c_j)(S^0)$$

taking the non-basepoint to the identity map on c_j. The cyclic map t_q cyclically permutes the smash factors in $V(\mathbb{S}, x) \cong S^{\vee x}$ and in $V(\mathcal{C}, x)$.

These structures induce maps

$$\begin{aligned} d_i &\colon THH(\mathcal{C})_q \to THH(\mathcal{C})_{q-1} \\ s_j &\colon THH(\mathcal{C})_q \to THH(\mathcal{C})_{q+1} \\ t_q &\colon THH(\mathcal{C})_q \to THH(\mathcal{C})_q \end{aligned}$$

making $[q] \mapsto THH(\mathcal{C})_q$ a cyclic space, denoted $THH(\mathcal{C})$. This also extends to a cyclic Γ-space $([q], k_+) \mapsto THH(\mathcal{C})_q(k_+)$. In more detail, an r-simplex in the nerve of I^{q+1} is a chain $x^0 \leftarrow \cdots \leftarrow x^r = x$ of composable maps, where each x^s is a $(q+1)$-tuple $(x_0^s, \dots, x_q^s)$ for $0 \le s \le r$. Each morphism $\phi\colon [p] \to [q]$ in Λ induces a map $\phi^*\colon THH(\mathcal{C})_q \to THH(\mathcal{C})_p$ taking

$$(x^0 \leftarrow \cdots \leftarrow x^r = x, f\colon S^{\vee x} \to V(\mathcal{C}, x))$$

to

$$(\phi^*(x^0) \leftarrow \cdots \leftarrow \phi^*(x^r) = \phi^*(x), \phi^*(f)\colon S^{\vee \phi^*(x)} \to V(\mathcal{C}, \phi^*(x)))$$

(Here $\delta_i^* = d_i$, $\sigma_j^* = s_j$ and $\tau_q^* = t_q$.)

3. The circle action.

The geometric realization of a cyclic set $X\colon \Lambda^{op} \to \mathcal{E}ns$ has a natural S^1-action. Similarly for cyclic spaces, cyclic Γ-spaces, etc.

The orbit of a 0-simplex $a \in X_0$ is the 1-simplex $t_1 s_0(a)$, which has both ends at a, since $d_0 t_1 s_0(a) = d_1 s_0(a) = a$ and $d_1 t_1 s_0(a) = d_0 s_0(a) = a$. The S^1-action moves the 0-simplex a in $|X|$ evenly along the 1-simplex $t_1 s_0(a)$ in $|X|$.

The orbit of a 1-simplex $b \in X_1$, is the union of the two 2-simplices $t_2 s_1(b)$ and $t_2^2 s_0(b)$ along their common edge $d_2 t_2 s_1(b) = t_1(b) = d_0 t_2^2 s_0(b)$. The resulting square has $d_0 t_2 s_1(b) = b = d_2 t_2^2 s_0(b)$ at two sides, hence is really a cylinder. Its remaining two sides are $d_1 t_2 s_1(b) = t_1 s_0(d_0(b))$ and $d_1 t_2^2 s_0(b) = t_1 s_0(d_1(b))$, i.e., the orbits of the two end-points of b. The S^1-action moves the 1-simplex b in $|X|$ evenly across this square combining $t_2 s_1(b)$ and $t_2^2 s_0(b)$ in $|X|$.

More generally, a q-simplex $c \in X_q$ is moved across the union of $(q+1)$ simplices, each of dimension $(q+1)$, given by $t_{q+1}^{q+1-j} s_j(c)$ for $0 \le j \le q$. The power of t_{q+1} is chosen to map the degenerate simplex $s_j(c)$ to one that is generally not degenerate. These simplices fit together to a copy of $S^1 \times \Delta^q$, mapped into $|X|$, describing the orbit of c under the S^1-action.

What are the S^1-fixed points of this action on $|X|$? A 0-simplex a is fixed only if the 1-simplex $t_1 s_0(a)$ is degenerate, i.e., if and only if

$$t_1 s_0(a) = s_0(a)\,.$$

For a nondegenerate q-simplex $c \in X_q$, an interior point ξ of $\{c\} \times \Delta^q$ is moved across an interior part of the simplex $d = t_{q+1}^{q+1} s_0(c)$. If d is nondegenerate, this part of the orbit is not constant, so ξ is not in the fixed point set. We cannot have $d = s_j(e)$ for $j \neq 0$ because then $c = d_0 s_j(e)$ would be degenerate. If $d = s_0(e)$ we have $e = d_0(d) = c$ and for $q \ge 1$ no interior orbit in $S^1 \times \Delta^q$ is contained in a line through the 0th vertex of d. Thus ξ is not in the fixed point set in this case either.

Lemma 3.1. *Let $X\colon \Lambda^{op} \to \mathcal{E}ns$ be a cyclic set. Its S^1-fixed point set is the discrete subset*
$$|X|^{S^1} = \{a \in X_0 \mid t_1 s_0(a) = s_0(a)\}\,.$$

4. Circle fixed points.

The cyclic structure on $THH(\mathcal{C})$ thus determines an S^1-action on its geometric realization $|THH(\mathcal{C})|$. Its S^1-fixed point set is given by the lemma above. Explicitly, a point in $THH(\mathcal{C})_0$ is a tuple

$$a = (x^0 \leftarrow \dots \leftarrow x^r = x, f\colon S^x \to \bigvee_{c \in ob\mathcal{C}} \underline{\mathcal{C}}(c,c)(S^x))$$

with $x^s \in I$ for $0 \le s \le r$. It degenerates to

$$\begin{aligned} s_0(a) = ((x^0,*) \leftarrow \dots \leftarrow (x^r,*) = (x,*),\\ f \wedge 1_c\colon S^{x\vee *} \to \bigvee_{c \in ob\mathcal{C}} \underline{\mathcal{C}}(c,c)(S^x) \wedge \underline{\mathcal{C}}(c,c)(S^0) \subseteq V(\mathcal{C},(x,*))) \end{aligned}$$

whose cyclic twist is

$$\begin{aligned} t_1 s_0(a) = ((*,x^0) \leftarrow \dots \leftarrow (*,x^r) = (*,x),\\ 1_c \wedge f\colon S^{*\vee x} \to \bigvee_{c \in ob\mathcal{C}} \underline{\mathcal{C}}(c,c)(S^0) \wedge \underline{\mathcal{C}}(c,c)(S^x) \subseteq V(\mathcal{C},(*,x)))\,. \end{aligned}$$

These are equal if and only if all $x^r = *$, so $x = *$ with $S^x = S^0$, and $f\colon S^0 \to \bigvee_{c \in ob\mathcal{C}} \underline{\mathcal{C}}(c,c)(S^0)$ takes the non-basepoint to $1_c \in \underline{\mathcal{C}}(c,c)(S^0)$ for some $c \in ob\mathcal{C}$. (The base point identifies with the identity map when there is a zero object $c = *$ in $\mathcal{C}$. We now assume this.) This identifies to a copy of $ob\mathcal{C}$. Taking the Γ-space structure into account leads to $THH(\mathcal{C})(k_+)^{S^1} \cong ob\mathcal{C} \wedge k_+$.

Lemma 4.1. $THH(\mathcal{C})^{S^1} = ob\mathcal{C}$ *as spaces, and* $THH(\mathcal{C})^{S^1} = \Sigma^\infty(ob\mathcal{C})$ *as* Γ*-spaces.*

5. Frobenius maps and TF.

Let $C_r \subset S^1$ be the cyclic subgroup of order r. It turns out that the C_r-cyclic fixed point sets of $THH(\mathcal{C})$ are more accessible than the S^1-fixed point sets. Let p be a prime. We think of $C_{p^{n-1}}$ as a subgroup of C_{p^n} of index p. The C_{p^n}-fixed points for the circle action on $THH(\mathcal{C})$ are contained in the $C_{p^{n-1}}$-fixed points, by neglect of structure.

The *Frobenius map*

$$F = F_p\colon THH(\mathcal{C})^{C_{p^n}} \to THH(\mathcal{C})^{C_{p^{n-1}}}$$

is defined as the inclusion between these fixed point sets, interpreting a point in $THH(\mathcal{C})$ that is fixed by C_{p^n} as in particular being fixed by $C_{p^{n-1}}$.

These assemble to a sequential limit diagram

$$\dots \xrightarrow{F} THH(\mathcal{C})^{C_{p^n}} \xrightarrow{F} THH(\mathcal{C})^{C_{p^{n-1}}} \xrightarrow{F} \dots \xrightarrow{F} THH(\mathcal{C})^{C_p} \xrightarrow{F} THH(\mathcal{C})\,.$$

Replacing each map by a fibration and taking the limit, or more precisely taking the homotopy limit of this diagram, defines the functor TF:

$$TF(\mathcal{C},p) = \operatorname*{holim}_n THH(\mathcal{C})^{C_{p^n}}$$

where the maps in the limit are the Frobenius maps $F = F_p$.

There are natural maps

$$ob\mathcal{C} \xrightarrow{trF_p} TF(\mathcal{C}, p) \to THH(\mathcal{C})$$

including the S^1-fixed points of $THH(\mathcal{C})$ into the limit, and thus the homotopy limit, of the C_{p^n}-fixed points. (The notation trF_p is not standard, and may be confused with a transfer map.)

Alternatively one may consider the diagram $r \mapsto THH(\mathcal{C})^{C_r}$ indexed by the multiplicative monoid of natural numbers, i.e., the category with objects natural numbers $r \geq 1$, and a morphism from r to s precisely with r is a multiple of s. Let

$$TF(\mathcal{C}) = \operatorname*{holim}_r THH(\mathcal{C})^{C_r}$$

where the map $THH(\mathcal{C})^{C_r} \to THH(\mathcal{C})^{C_s}$ where r is a multiple of s is the inclusion as subspaces of $THH(\mathcal{C})$. Then $TF(\mathcal{C})$ is the fiber product over $THH(\mathcal{C})$ of the $TF(\mathcal{C}, p)$ for all primes p, and there is a natural map $TF(\mathcal{C}) \to TF(\mathcal{C}, p)$ inducing a homotopy equivalence after p-adic completion. Thus $TF(\mathcal{C})$ does not really carry any more information than the collection of all the $TF(\mathcal{C}, p)$'s.

There are canonical map

$$\gamma \colon THH(\mathcal{C})^{C_r} \to THH(\mathcal{C})^{hC_r} = \underline{\mathcal{S}_*}(EC_{r+}, THH(\mathcal{C}))^{C_r}$$

from the fixed points to the homotopy fixed points for the C_r-action on $THH(\mathcal{C})$. We obtain maps

$$TF(\mathcal{C}, p) = \operatorname*{holim}_n THH(\mathcal{C})^{C_{p^n}} \xrightarrow{\gamma} \operatorname*{holim}_n THH(\mathcal{C})^{hC_{p^n}} .$$

After p-adic completion, the natural map

$$THH(\mathcal{C})^{hS^1} \to \operatorname*{holim}_n THH(\mathcal{C})^{hC_{p^n}}$$

is a homotopy equivalence. Hence there is a map

$$TF(\mathcal{C})^\wedge_p \simeq TF(\mathcal{C}, p)^\wedge_p \to THH(\mathcal{C})^{hS^1}{}^\wedge_p$$

which in some cases is a homotopy equivalence, or a homotopy equivalence on connective covers. Hence TF may be thought of as close to the S^1-homotopy fixed points of THH. In this sense, TF is close to a topological negative cyclic homology.

Even if $THH(\mathcal{C})$ is of finite type, i.e., each homotopy group is finitely generated, it is usually not the case that $TF(\mathcal{C}, p)$ and $THH(\mathcal{C})^{hS^1}$ are of finite type. Hence we shall seek to reduce the size of $TF(\mathcal{C})$ further, by taking into account more structure available in $THH(\mathcal{C})$.

6. Edgewise subdivision.

The S^1-action on $THH(\mathcal{C})$ is not simplicial, which seems to make it hard to analyze the structure of the C_r-fixed points for $C_r \subset S^1$. Edgewise subdivision is a method to replace $THH(\mathcal{C})$ by another simplicial space $sd_r THH(\mathcal{C})$, which admits a simplicial C_r-action, such that there is a natural homeomorphism of geometric realizations

$$D\colon |sd_r THH(\mathcal{C})| \xrightarrow{\cong} |THH(\mathcal{C})|$$

identifying the simplicial C_r-action on the left with the C_r-action on the right that comes from restricting the S^1-action to the subgroup $C_r \subset S^1$. Hence there is a homeomorphism

$$D^{C_r}\colon |(sd_r THH(\mathcal{C}))^{C_r}| \xrightarrow{\cong} |THH(\mathcal{C})|^{C_r}$$

and $(sd_r THH(\mathcal{C}))^{C_r}$ provides a simplicial model for the C_r-fixed points.

Let the functor $sd_r\colon \Delta \to \Delta$ be given by

$$sd_r([q]) = [(q+1)r-1] \cong [q] \coprod \cdots \coprod [q]$$

(r summands on the right), and

$$sd_r(\phi) = \phi \coprod \cdots \coprod \phi$$

for morphisms $\phi\colon [p] \to [q]$ in Δ.

Then a simplicial set $X\colon \Delta^{op} \to \mathcal{E}ns$ defines a new simplicial set $sd_r X = X \circ sd_r\colon \Delta^{op} \to \mathcal{E}ns$, with

$$(sd_r X)_q = X_{(q+1)r-1} \,.$$

The ith face map is

$$sd_r(d_i) = d_i \circ d_{i+q+1} \circ \cdots \circ d_{i+(q+1)(r-1)} \,,$$

and the jth degeneracy map is

$$sd_r(s_j) = s_j \circ s_{j+q+1} \circ \cdots \circ s_{j+(q+1)(r-1)}$$

for $0 \le i, j \le q$. We call $sd_r X$ the *r-fold edgewise subdivision* of X. Similarly for simplicial objects in other categories.

If X is cyclic, i.e., extends to $X\colon \Lambda^{op} \to \mathcal{E}ns$, then $sd_r X$ gets a simplicial C_r-action given in degree q by the cyclic structure map $t^{q+1}_{(q+1)r-1}$.

$$T_q = t^{q+1}_{(q+1)r-1}\colon (sd_r X)_q = X_{(q+1)r-1} \to X_{(q+1)r-1} = (sd_r X)_q \,.$$

A check of the cyclic identities shows that $T_q\colon (sd_r X)_q \to (sd_r X)_q$ is simplicial, and that $T_q^r = 1$.

The homeomorphism $D\colon |sd_r X| \to |X|$ is induced by maps

$$(sd_r X)_q \times \Delta^q = X_{(q+1)r-1} \times \Delta^q \to X_{(q+1)r-1} \times \Delta^{(q+1)r-1}$$

embedding Δ^q diagonally into the r-fold join

$$\Delta^q * \cdots * \Delta^q \cong \Delta^{(q+1)r-1} \,.$$

In [BHM] it is shown that these maps induce a homeomorphism, and that the simplicial C_r-action on the left agrees with the C_r-action on the right restricted from the S^1-action. The name 'edgewise subdivision' stems from the fact that each edge of X is divided into r edges in $sd_r X$.

7. Restriction maps and TR.

Consider C_p as a subgroup of C_{p^n}, with quotient group $C_{p^{n-1}}$. The *restriction map*

$$R = R_p \colon THH(\mathcal{C})^{C_{p^n}} \to THH(\mathcal{C})^{C_{p^{n-1}}}$$

is defined by applying $C_{p^{n-1}}$-fixed points to the geometric realization of a simplicial S^1-equivariant map

$$R_p \colon sd_p THH(\mathcal{C})^{C_p} \to THH(\mathcal{C}) .$$

On q-simplices, this is a map

$$(R_p)_q \colon \left(\operatorname*{hocolim}_{x \in I^{p(q+1)}} \Omega^{\vee x} V(\mathcal{C}, x) \right)^{C_p} \to \operatorname*{hocolim}_{y \in I^{q+1}} \Omega^{\vee y} V(\mathcal{C}, y) .$$

An r-simplex in the homotopy colimit on the left is a chain of maps $x^0 \leftarrow \cdots \leftarrow x^r = x = (x_0, \dots, x_{p(q+1)-1})$ in $I^{p(q+1)}$, together with a map

$$f \colon S^{\vee x} \cong S^{x_0} \wedge \cdots \wedge S^{p(q+1)-1} \to \bigvee_{c_0, \dots, c_{p(q+1)-1}} \underline{\mathcal{C}}(c_0, c_{p(q+1)-1})(S^{x_0}) \wedge \cdots \wedge \underline{\mathcal{C}}(c_{p(q+1)-1}, c_{p(q+1)-2})(S^{x_{p(q+1)-1}}) .$$

The generator of the C_p-action permutes the factors in $I^{p(q+1)}$ by cyclically shifting them $(q+1)$ positions to the right, and similarly for the $p(q+1)$ smash product factors in $V(\mathcal{C}, x)$. The source of $(R_p)_q$ consists of the C_p-invariant chains $x^0 \leftarrow \cdots \leftarrow x^r = x$, together with the C_p-equivariant maps f as above.

A $p(q+1)$-tuple $x \in I^{p(q+1)}$ is C_p-invariant precisely when it has the form $\Delta_p(y) = (y, \dots, y)$ for $y \in I^{q+1}$. Here $\Delta_p \colon I^{q+1} \to I^{p(q+1)}$ is the p-fold diagonal embedding. Thus we may assume the C_p-invariant chain $x^0 \leftarrow \cdots \leftarrow x^r = x$ arises by applying Δ_p to a chain $y^0 \leftarrow \cdots \leftarrow y^r = y = (y_0, \dots, y_q)$ in I^{q+1}. So

$$x = \Delta(y) = (y_0, \dots, y_q, \dots, y_0, \dots, y_q)$$

is y repeated p times.

A C_p-equivariant map $f \colon X \to Y$ induces a map $f^{C_p} \colon X^{C_p} \to Y^{C_p}$ by restriction to the C_p-fixed point spaces. This is the core of the construction of the restriction maps. We apply this with

$$X = S^{\vee x} \cong S^{y_0} \wedge \cdots \wedge S^{y_q} \wedge \cdots \wedge S^{y_0} \wedge \cdots \wedge S^{y_q} \cong (S^{y_0} \wedge \cdots \wedge S^{y_q})^{\wedge p}$$

Here the generator of C_p cyclically permutes the p wedge factors in the last expression, so the C_p-fixed points are a diagonal copy of $S^{\vee y}$:

$$X^{C_p} \cong S^{\vee y} \cong S^{y_0} \wedge \cdots \wedge S^{y_q} .$$

We also let

$$Y = \bigvee_{c_0, \dots, c_{p(q+1)-1}} \underline{\mathcal{C}}(c_0, c_{p(q+1)-1})(S^{y_0}) \wedge \cdots \wedge \underline{\mathcal{C}}(c_{p(q+1)-1}, c_{p(q+1)-2})(S^{y_q}) .$$

The C_p-fixed points are the summands with $c_i = c_{q+1+i} = \cdots = c_{(p-1)(q+1)+i}$ for $0 \le i \le q$, and wedge factors repeating periodically every $(q+1)$ factors, i.e.,

$$Y^{C_p} \cong \bigvee_{c_0,\dots,c_q} \underline{\mathcal{C}}(c_0, c_q)(S^{y_0}) \wedge \cdots \wedge \underline{\mathcal{C}}(c_q, c_{q-1})(S^{y_q}) \cong V(\mathcal{C}, y)\,.$$

Thus R_p takes the C_p-invariant (r,q)-simplex determined by $x^0 \leftarrow \cdots \leftarrow x^r = x$ and $f\colon S^{\vee x} \to V(\mathcal{C}, x)$ to the (r,q)-simplex in $THH(\mathcal{C})$ determined by $y^0 \leftarrow \cdots \leftarrow y^r = y$ and $f^{C_p}\colon (S^{\vee x})^{C_p} \to V(\mathcal{C}, x)^{C_p}$, identified with a map $f^{C_p}\colon S^{\vee y} \to V(\mathcal{C}, y)$.

The resulting map $R_p\colon THH(\mathcal{C})^{C_p} \to THH(\mathcal{C})$ is a cyclic map, hence S^1-equivariant. Taking $C_{p^{n-1}}$-fixed points for $n \ge 1$ defines the various restriction maps, as displayed above. They assemble to a sequential limit diagram

$$\dots \xrightarrow{R} THH(\mathcal{C})^{C_{p^n}} \xrightarrow{R} THH(\mathcal{C})^{C_{p^{n-1}}} \xrightarrow{R} \dots \xrightarrow{R} THH(\mathcal{C})^{C_p} \xrightarrow{R} THH(\mathcal{C})\,.$$

Replacing each map by a fibration and taking the limit, or more precisely taking the homotopy limit of this diagram, defines the functor TR:

$$TR(\mathcal{C}, p) = \operatorname*{holim}_n THH(\mathcal{C})^{C_{p^n}}$$

where the maps in the limit are the restriction maps $R = R_p$.

Since the restriction maps arise by taking fixed points of an S^1-equivariant map, they commute with the forgetful Frobenius maps. Also the inclusion of $ob\mathcal{C}$ into $THH(\mathcal{C})$ as the S^1-fixed points, taking $c \in ob\mathcal{C}$ to the identity map on c, commutes with the restriction maps. Hence there are natural maps

$$ob\mathcal{C} \xrightarrow{trR_p} TR(\mathcal{C}, p) \to THH(\mathcal{C})$$

including the S^1-fixed points into the limit, and thus the homotopy limit, of the C_{p^n}-fixed points. (Again this notation is nonstandard, but less likely to confuse.)

8. The fundamental cofibration sequence.

For each $n \ge 1$ there is a cofiber sequence of spectra

$$THH(\mathcal{C})_{hC_{p^n}} \xrightarrow{N} THH(\mathcal{C})^{C_{p^n}} \xrightarrow{R} THH(\mathcal{C})^{C_{p^{n-1}}}\,.$$

Hence a $\Gamma\mathcal{S}_*$-functor $F\colon \mathcal{C} \to \mathcal{D}$ inducing a homotopy equivalence on THH will also induce a homotopy equivalence on the C_{p^n}-fixed point subspectra of THH, for each $n \ge 0$. Let us return to this when we discuss equivariant homotopy theory in more detail in connection with the calculation of TC for spaces.

9. Topological cyclic homology TC.

Let p be a prime, and let $\mathbb{I}_p$ be the category with objects $1, p, \dots, p^n, \dots$ for $n \ge 0$, and commuting morphisms $r, f\colon p^n \to p^{n-1}$ for all $n \ge 1$. Thus there are $(k+1)$ distinct morphisms $p^{n+k} \to p^n$, given as the various composites $r^i f^j$ for $i + j = k$. Then

$$p^n \mapsto THH(\mathcal{C})^{C_{p^n}}$$

defines a functor, taking r to the restriction map $R = R_p$ and f to the Frobenius map $F = F_p$. We define the p-primary *topological cyclic homology* of the $\Gamma\mathcal{S}_*$-category $\mathcal{C}$ to be

$$TC(\mathcal{C}, p) = \operatorname*{holim}_{p^n \in \mathbb{I}_p} THH(\mathcal{C})^{C_{p^n}} .$$

This comes enriched as a Γ-space by taking into account the Γ-space enrichment of $THH(\mathcal{C})$ and its fixed points.

Alternatively, TC may be described as a homotopy equalizer for maps between sequential homotopy limits, thus avoiding the details of how this more complicated homotopy limit is defined.

Since the R- and F-maps commute, the R-maps induce a self-map of $TF(\mathcal{C}, p)$, and the F-maps induce a self map of $TR(\mathcal{C}, p)$. There are homotopy equalizer diagrams

$$TC(\mathcal{C}, p) \xrightarrow{\pi} TF(\mathcal{C}, p) \underset{1}{\overset{R}{\rightrightarrows}} TF(\mathcal{C}, p)$$

and

$$TC(\mathcal{C}, p) \xrightarrow{\pi} TR(\mathcal{C}, p) \underset{1}{\overset{F}{\rightrightarrows}} TR(\mathcal{C}, p) .$$

Thus TC is homotopy equivalent to the homotopy fiber of $R - 1$ acting on TF, or of $F - 1$ acting on TR.

10. The cyclotomic trace map for symmetric monoidal $\Gamma\mathcal{S}_*$-categories.

The inclusion of $ob\mathcal{C}$ into the various C_{p^n}-fixed points of $THH(\mathcal{C})$ determines a map into the limit of the functor above, and thus into the homotopy limit:

$$trc_p \colon ob\mathcal{C} \to TC(\mathcal{C}, p)$$

refining the maps trF_p, trR_p and tr to $TF(\mathcal{C}, p)$, $TR(\mathcal{C}, p)$ and $THH(\mathcal{C})$, respectively. When $\mathcal{C}$ is a symmetric monoidal $\Gamma\mathcal{S}_*$-category with weak equivalences $wU\mathcal{C} \subseteq U\mathcal{C}$, its algebraic K-theory category $K(\mathcal{C}, w) = hoN^w\bar{H}\mathcal{C}$ is a $\Gamma\mathcal{S}_*$-category with objects $obK(\mathcal{C}, w)$ the ordinary Γ-space defining the algebraic K-theory of $\mathcal{C}$ with respect to the weak equivalences. There results a cyclotomic trace map

$$trc_p \colon obK(\mathcal{C}, w) \to TC(K(\mathcal{C}, w), p) .$$

When $wU\mathcal{C}$ is a subcategory of strong equivalences, the inclusion $\bar{H}\mathcal{C} \to K(\mathcal{C}, w)$ induces a homotopy equivalence on THH, and thus on p-primary TC by the fundamental cofiber sequence and induction. Also $TC(\bar{H}\mathcal{C}, p)$ is a very special Γ-space in the $\bar{H}$-direction, so there are stable equivalences of $bi\Gamma$-spaces

$$\Sigma^\infty TC(\mathcal{C}, p) \xrightarrow{\simeq} TC(\bar{H}\mathcal{C}, p) \xrightarrow{\simeq} TC(K(\mathcal{C}, w), p) .$$

Combined, these yield a stable map

$$trc_p \colon obK(\mathcal{C}, w) \to TC(\mathcal{C}, p)$$

called the *cyclotomic trace map* for $\mathcal{C}$.

11. The cyclotomic trace map for $\mathbb{S}$-algebras.

When A is an $\mathbb{S}$-algebra, i.e., a monoid in $(\Gamma\mathcal{S}_*, \wedge, \mathbb{S})$, we can associate to it a symmetric monoidal $\Gamma\mathcal{S}_*$-category Φ_A of finitely generated free A-modules, or rather, their fibrant replacements, and the Γ-spaces of A-module maps between them. These admit a notion of strong equivalence w, for which the morphisms in $wU\Phi_A$ are maps inducing stable equivalences between the objects of Φ_A. ((Check ?))

Then $K(A)$ is defined as $obK(\Phi_A, w)$, and the inclusion $TC(A,p) \to TC(\Phi_A, p)$ is a homotopy equivalence by Morita equivalence for THH and the fundamental cofiber sequences. Thus there is a stable map

$$trc_p \colon K(A) \to TC(A,p)$$

called the *cyclotomic trace map* for A. Its homotopy class is well defined, and natural in the $\mathbb{S}$-algebra A.

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