NOTES ON TOPOLOGICAL CYCLIC HOMOLOGY AND THE CYCLOTOMIC TRACE MAP

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We report on the construction of topological Hochschild homology, topological cyclic homology and the cyclotomic trace map from algebraic K-theory, following Bökstedt, Hsiang and Madsen's paper (*The cyclotomic trace and algebraic K-theory of spaces*, Invent. math. **111** (1993), 465–540), and Madsen's book (*Algebraic K-theory and traces*, Aarhus University preprint, vol. 26, 1995). Some of these ideas go back to a letter from Goodwillie to Waldhausen.

1. The Dennis trace map

Recall the *bar construction*. For a topological monoid M there is a simplicial monoid $N_\bullet M$ called the nerve of M, with $N_q M = M^q$ and face and degeneracy maps given by

$$\begin{aligned} d_0([m_1|\dots|m_q]) &= [m_2|\dots|m_q] \\ d_i([m_1|\dots|m_q]) &= [m_1|\dots|m_i m_{i+1}|\dots|m_q] \qquad \text{for } 0<i<q \\ d_q([m_1|\dots|m_q]) &= [m_1|\dots|m_{q-1}] \\ s_j([m_1|\dots|m_q]) &= [m_1|\dots|m_j|\,1\,|m_{j+1}|\dots|m_q]\,. \end{aligned}$$

Its geometric realization is

$$BM = \coprod_{q\ge 0} M^q \times \Delta^q/\sim$$

with $\sim$ generated by the simplicial structure. The inclusion of the simplicial 1-skeleton $\Sigma M \cong BM^{(1)} \to BM$ has an adjoint map $M \to \Omega BM$ which is a weak equivalence if M is *grouplike* (the monoid $\pi_0(M)$ is a group), and is a group completion in general.

Let A be ring, always associative with unit. The disjoint union

$$\coprod_{k\ge 0} BGL_k(A)$$

is a topological monoid, with product maps induced by the block sum maps

$$GL_k(A) \times GL_l(A) \to GL_{k+l}(A)\,.$$

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Its group completion

$$K(A) = \Omega B\Big(\coprod_{k\geq 0} BGL_k(A)\Big) \simeq \mathbb{Z} \times BGL(A)^+$$

is the algebraic K-theory space of A, and its homotopy groups $\pi_i K(A) = K_i(A)$ are the higher algebraic K-groups of A.

Next recall the *cyclic construction.* It is a simplicial abelian group $Z_\bullet A$ with $Z_q A = A \otimes \cdots \otimes A = A^{\otimes q+1}$ ($q+1$ factors) and face and degeneracy maps

$$\begin{aligned} d_i(a_0 \otimes \cdots \otimes a_q) &= a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_q \qquad \text{for } 0 \leq i < q \\ d_q(a_0 \otimes \cdots \otimes a_q) &= a_q a_0 \otimes \cdots \otimes a_{q-1} \\ s_j(a_0 \otimes \cdots \otimes a_q) &= a_0 \otimes \cdots \otimes a_j \otimes 1 \otimes a_{j+1} \otimes \cdots \otimes a_q \,. \end{aligned}$$

Its associated chain complex is the Hochschild complex for A, whose homology groups are the *Hochschild homology groups* of A:

$$HH_i(A) = H_i(Z_* A)\,.$$

The geometric realization of a simplicial abelian group is a product of Eilenberg–Mac Lane spaces, whose homotopy groups are the homology groups of the associated chain complex (by a theorem of Moore). We denote the geometric realization of the cyclic construction by $HH(A)$.

Also recall the *cyclic nerve construction.* Let M be a topological monoid and define a simplicial space $N^{cy}_\bullet M$ by $N^{cy}_q M = M^{q+1}$ with face and degeneracy maps similar to those in the cyclic construction. (We omit the formulas.) Let $N^{cy}M$ be its geometric realization. There is a natural projection map $\pi\colon N^{cy}_\bullet M \to N_\bullet M$ that forgets the zeroth coordinate.

The Dennis trace map is induced by maps

$$BGL_k(A) \to N^{cy} GL_k(A) \to HH(M_k(A))$$

induced by simplicial maps

$$S_\bullet \circ I_\bullet \colon N_\bullet GL_k(A) \to N^{cy}_\bullet GL_k(A) \to Z_\bullet M_k(A)$$

given by

$$\begin{aligned} I_q &\colon [g_1|\dots|g_q] \mapsto (g_0, g_1, \dots, g_q) \\ S_q &\colon (g_0, g_1, \dots, g_q) \mapsto g_0 \otimes g_1 \otimes \cdots \otimes g_q \end{aligned}$$

where we let $g_0 = (g_1 \dots g_q)^{-1} \in GL_k(A)$. Note that S_q includes matrices in $GL_k(A)$ into $M_k(A)$. The map $I_\bullet$ is a section to the natural projection π.

Next Hochschild homology satisfies Morita invariance, meaning that the *multi-trace map* $\mathrm{Tr}^{(k)}_\bullet \colon Z_\bullet M_k(A) \to Z_\bullet A$ given by

$$g_0 \otimes \cdots \otimes g_q \mapsto \sum_{i_0,\dots,i_q=1}^{k} (g_0)_{i_0 i_1} \otimes \cdots \otimes (g_q)_{i_q i_0}$$

induces a homotopy equivalence $HH(M_k(A)) \to HH(A)$.

For example, when $k = 1$ the composite map takes the 1-simplex $[g_1]$ in $BGL_1(A)$ to the 1-simplex $g_1^{-1} \otimes g_1$ in $HH(A)$.

The composites $\mathrm{Tr}_\bullet^{(k)} \circ S_\bullet \circ I_\bullet$ determine a map

$$\coprod_{k \geq 0} BGL_k(A) \to HH(A)$$

which (modulo a slight π_0-adjustment) extends over the group completion to define the *Dennis trace map*

$$\mathrm{tr}\colon K(A) \to HH(A)\,.$$

Unfortunately the induced homomorphisms $K_i(A) \to HH_i(A)$ are often zero. For example when $A = \mathbb{Z}$, the algebraic K-groups $K_i(\mathbb{Z})$ are highly nontrivial, while $HH_i(\mathbb{Z}) = \mathbb{Z}$ for $i = 0$ and zero otherwise. In effect, the linearization from $GL_k(A)$ to $M_k(A)$ is too drastic to capture rational information. However torsion information, and thus completed information, can be detected, and more so when working "topologically" rather than homologically.

2. Functors with smash product

Let Top_* be the category of based spaces and continuous maps. All such spaces are based at $*$.

A *functor with assembly map* is a continuous functor $L\colon \mathrm{Top}_* \to \mathrm{Top}_*$ such that $L(*) = *$. There is then a (left) assembly map

$$\sigma_{X,Y}\colon X \wedge L(Y) \to L(X \wedge Y)$$

natural in $X, Y \in \mathrm{Top}_*$, satisfying $\sigma_{X,Y\wedge Z} \circ (1_X \wedge \sigma_{Y,Z}) = \sigma_{X\wedge Y,Z}$. There is also a (right) assembly map

$$\sigma'_{X,Y}\colon L(X) \wedge Y \to L(X \wedge Y)$$

equal to $L(T) \circ \sigma_{Y,X} \circ T$, where the T-maps switch factors.

A *functor with smash product* (FSP for short) is a functor L as above, equipped with natural transformations

$$\begin{aligned} \eta_X\colon X &\to L(X) && \text{(unit)} \\ \mu_{X,Y}\colon L(X) \wedge L(Y) &\to L(X \wedge Y) && \text{(product)} \end{aligned}$$

compatible with the assembly maps. This means that $\eta_{X\wedge Y} = \sigma_{X,Y} \circ (\eta_X \wedge 1_Y)$, $\eta_{X\wedge Y} = \sigma'_{X,Y} \circ (1_X \wedge \eta_Y)$, $\sigma_{X,Y} = \mu_{X,Y} \circ (\eta_X \wedge 1_{L(Y)})$ and $\sigma'_{X,Y} = \mu_{X,Y} \circ (1_{L(X)} \wedge \eta_Y)$. The product is assumed to be associative, meaning that $\mu_{X\wedge Y,Z} \circ (\mu_{X,Y} \wedge 1_{L(Z)}) = \mu_{X,Y\wedge Z} \circ (1_{L(X)} \wedge \mu_{Y,Z})$.

Let I be the full subcategory of the category of finite sets and injective functions, with objects $\mathbf{n} = \{1, 2, \ldots, n\}$ for all integers $n \geq 0$. There is an imbedding $I \to \mathrm{Top}_*$ mapping $\mathbf{n}$ to $S^n = S^1 \wedge \cdots \wedge S^1$ (n factors). We will only use the values of FSPs restricted to the image of this imbedding.

The spaces $L_n = L(S^n)$ and maps $\sigma'_{S^n,S^1}\colon \Sigma L_n = L(S^n)\wedge S^1 \to L(S^{n+1}) = L_{n+1}$ define a prespectrum L^S underlying the FSP L. The maps $\eta_{S^n}\colon S^n \to L_n$ and $\mu_{S^m,S^n}\colon L_m \wedge L_n \to L_{m+n}$ make L^S into a unital ring (pre-)spectrum. Its homotopy groups are $\pi_i L^S = \mathrm{colim}_n\, \pi_{i+n} L_n$, which for $i \geq 0$ are the homotopy groups of the underlying infinite loop space

$$Q(L) = \operatorname*{hocolim}_n \Omega^n L_n \,.$$

The unit and product maps make $\pi_* L^S$ into a graded (unital, associative) ring.

Here are two important examples of FSPs.

Let A be ring. Define the FSP $\tilde{A}$ by the Dold–Thom construction

$$\tilde{A}(X) = \{\textstyle\sum_i a_i \cdot x_i \mid a_i \in A, x_i \in X\}/\sim$$

with $a \cdot * \sim 0$ and $0 \cdot x \sim 0$ for all $a \in A$ and $x \in X$. This is the configuration space of points in X with labels in A. This functor is continuous, has an obvious unit map $X \to \tilde{A}(X)$ taking x to $1 \cdot x$, and a product map $\tilde{A}(X) \wedge \tilde{A}(Y) \to \tilde{A}(X \wedge Y)$ taking $(\sum_i a_i \cdot x_i) \wedge (\sum_j b_j \cdot y_j)$ to $\sum_{i,j} a_i b_j \cdot x_i \wedge y_j$. The product $a_i b_j$ is formed in the ring A. Evaluated on spheres $\tilde{A}(S^n)$ is an Eilenberg–Mac Lane space of type (A, n), with $\pi_{i+n}\tilde{A}(S^n) = A$ for $i = 0$ and 0 otherwise. Hence $\tilde{A}^S$ is the Eilenberg–Mac Lane spectrum of A, with homotopy groups $\pi_* \tilde{A}^S = A$ concentrated in degree 0, as graded rings. We think of $\tilde{A}$ as a model for the ring A lifted to the level of FSPs, with the Eilenberg–Mac Lane spectrum HA for A as the intermediate ring spectrum.

Let M be a topological monoid. Define the FSP $\tilde{M}$ by

$$\tilde{M}(X) = M_+ \wedge X \,.$$

This functor is also continuous, has a unit map $X \to \tilde{M}(X)$ taking x to $1 \wedge x$, and a product map $\tilde{M}(X) \wedge \tilde{M}(Y) \to \tilde{M}(X \wedge Y)$ taking $m \wedge x \wedge n \wedge y$ to $mn \wedge x \wedge y$, where the product mn is formed in the topological monoid M. The corresponding prespectrum $\tilde{G}^S$ has nth space $M_+ \wedge S^n = \Sigma^n_+ M$ and underlying infinite loop space $Q(M_+)$, with homotopy groups $\pi_* Q(M_+) = \pi^S_* M_+$ equal to the unreduced stable homotopy of M. Hence $\tilde{M}$ is an FSP with the suspension spectrum of M as its associated ring spectrum.

We will assume that our FSPs are *connective*, meaning that each $L(S^n)$ is $(n-1)$-connected, and that the assembly maps $\Sigma L(S^n) \to L(S^{n+1})$ are $(2n-c)$-connected for some fixed c.

3. Algebraic K-theory

We can now define the algebraic K-theory of an FSP.

Let $M_k(L)$ be the $k \times k$ matrix FSP over L defined by

$$M_k(L)(X) = F(\mathbf{k}_+, \mathbf{k}_+ \wedge L(X)) \simeq \prod_{j=1}^{k} \bigvee_{i=1}^{k} L(X) \,.$$

These are really spaces of matrices with at most one non-basepoint entry from $L(X)$ in each column. The product map

$$M_k(L)(X) \wedge M_k(L)(Y) \xrightarrow{\mu_{X,Y}} M_k(X \wedge Y)$$

takes $f \wedge g$ with $f \colon \mathbf{k}_+ \to \mathbf{k}_+ \wedge L(X)$ and $g \colon \mathbf{k}_+ \to \mathbf{k}_+ \wedge L(Y)$ to the composite

$$\mathbf{k}_+ \xrightarrow{g} \mathbf{k}_+ \wedge L(Y) \xrightarrow{f \wedge 1_{L(Y)}} \mathbf{k}_+ \wedge L(X) \wedge L(Y) \xrightarrow{1_{\mathbf{k}_+} \wedge \mu_{X,Y}} \mathbf{k}_+ \wedge L(X \wedge Y)\,.$$

The unit is similar. The inclusion $M_k(L)(X) \to \prod_{j=1}^k \prod_{i=1}^k L(X)$ is $(2n-1)$-connected for $X = S^n$, so in the limit $\pi_* Q(M_k(L)) \cong M_k(\pi_* Q(L))$ and the definition of a matrix FSP is reasonable.

In particular $\pi_0 Q(M_k(L)) = M_k(\pi_0 Q(L))$. Let $\widehat{GL}_k(L)$ be the pullback in the cartesian square

$$\begin{array}{ccc} \widehat{GL}_k(L) & \longrightarrow & Q(M_k(L)) \\ \downarrow & & \downarrow \\ GL_k(\pi_0 Q(L)) & \longrightarrow & M_k(\pi_0 Q(L)) \end{array}$$

Thus $\widehat{GL}_k(L)$ is the union of the path components of $Q(M_k(L))$ consisting of matrices invertible up to homotopy.

Then $\widehat{GL}_k(L)$ is a topological monoid, and block sum of matrices again defines a product on

$$\coprod_{k \geq 0} B\widehat{GL}_k(L)$$

making it a topological monoid. The *algebraic K-theory space* of L is its group completion

$$K(L) = \Omega B\Big(\coprod_{k \geq 0} B\widehat{GL}_k(L)\Big) \simeq \mathbb{Z} \times B\widehat{GL}(L)^+\,.$$

Using Segal's Γ-spaces again it is possible to make $K(L)$ the underlying space of a spectrum.

When $L = \tilde{A}$ this $K(\tilde{A})$ agrees with Quillen's $K(A)$. When $L = \tilde{M}$ this $K(\tilde{M})$ agrees with Waldhausen's $A(BM)$.

4. Topological Hochschild homology

Next we define the topological Hochschild homology of an FSP.

The prespectrum L^S was built from the sequence of spaces $F(S^n, L(S^n)) = \Omega^n L(S^n)$ with structure maps induced by the right assembly maps $L(S^n) \wedge S^1 \to L(S^{n+1})$. We can think of this diagram as indexed on the subcategory $N \subset I$ of objects $\mathbf{n}$ and the standard inclusions $\{1, \dots, n\} \subset \{1, \dots, m\}$ for $0 \leq n \leq m$.

The extra structure in an FSP ensures that the functor $G \colon \mathbf{n} \mapsto F(S^n, L(S^n))$ admits an extension over $N \to I$. The added morphisms in I are generated by the permutations $\sigma \colon \mathbf{n} \to \mathbf{n}$, which act on $G(\mathbf{n})$ by conjugation on the mapping space. Here σ acts on $S^n = S^1 \wedge \cdots \wedge S^1$ by permuting factors, and on $L(S^n)$ by the induced map $L(\sigma)$.

Then more generally we may define functors $G_q(L)\colon I^{q+1} \to \mathrm{Top}_*$ by

$$G_q(L, n_0, \dots, n_q) = F(S^{n_0} \wedge \dots \wedge S^{n_q}, L(S^{n_0}) \wedge \dots \wedge L(S^{n_q}))\,.$$

(To simplify typesetting we now denote the objects in I as n in place of $\mathbf{n}$.) On a suitable level of prespectra $G_q(L)$ is a model for the $(q+1)$-fold smash product of L^S with itself. On the level of components, $\pi_0 G_q(L)$ is the $(q+1)$-fold tensor product of $\pi_0 Q(L)$ with itself. Hence we will take $G_q(L)$ as the q-simplices in the cyclic construction for the topological Hochschild homology of L.

Furthermore there are face and degeneracy maps

$$\begin{aligned} &d_i\colon G_q(L, n_0, \dots, n_q) \to G_{q-1}(L, n_0, \dots, n_i + n_{i+1}, \dots, n_q) \qquad \text{for } 0 \le i < q \\ &d_q\colon G_q(L, n_0, \dots, n_q) \to G_{q-1}(L, n_q + n_0, \dots, n_{q-1}) \\ &s_j\colon G_q(L, n_0, \dots, n_q) \to G_{q+1}(L, n_0, \dots, n_j, 0, n_{j+1}, \dots, n_q)\,. \end{aligned}$$

For instance the two face maps $d_0, d_1\colon G_1(L) \to G_0(L)$ map

$$F(S^{n_0} \wedge S^{n_1}, L(S^{n_0}) \wedge L(S^{n_1}))$$

to

$$F(S^{n_0+n_1}, L(S^{n_0+n_1}))$$

and

$$F(S^{n_1+n_0}, L(S^{n_1+n_0}))$$

respectively. Explicitly d_0 identifies $S^{n_0} \wedge S^{n_1}$ with $S^{n_0+n_1}$ through concatenation, and maps $L(S^{n_0}) \wedge L(S^{n_1})$ to $L(S^{n_0+n_1})$ by the product map. On the other hand d_1 identifies $S^{n_0} \wedge S^{n_1}$ with $S^{n_1+n_0}$ through the twist map followed by concatenation, and maps $L(S^{n_0}) \wedge L(S^{n_1})$ to $L(S^{n_1+n_0})$ by the twist map followed by the product map.

The degeneracy map $s_0\colon G_0(L) \to G_1(L)$ uses the unit, and maps

$$F(S^{n_0}, L(S^{n_0})) \to F(S^{n_0} \wedge S^0, L(S^{n_0}) \wedge L(S^0))$$

by taking a map f to $f \wedge \eta_{S^0}$. The other simplicial structure maps are similar. In every case they model the simplicial structure maps in the cyclic complex defining Hochschild homology.

Hence we can form a simplicial space $THH(L)_\bullet$ by setting

$$THH(L)_q = \operatorname*{hocolim}_{(n_i)\in I^{q+1}} G_q(L) = \operatorname*{hocolim}_{(n_i)\in I^{q+1}} F(S^{n_0} \wedge \dots \wedge S^{n_q}, L(S^{n_0}) \wedge \dots \wedge L(S^{n_q}))\,.$$

The face and degeneracy maps above induce the face and degeneracy maps of $THH(L)_\bullet$ by passage to the homotopy colimit.

The *topological Hochschild homology space* $THH(L)$ of an FSP L is defined as the geometric realization of $THH(L)_\bullet$. Its homotopy groups $\pi_i THH(L) = THH_i(L)$ are the topological Hochschild homology groups of F.

By Segal's Γ-spaces again it is possible to recover $THH(L)$ as the underlying space of a spectrum.

For a ring A or topological monoid M we write $THH(A) = THH(\tilde{A})$ and $THH(BM) = THH(\tilde{M})$. In the ring case Bökstedt made the following computations:

$$THH_*(\mathbb{Z}/p) = \begin{cases} \mathbb{Z}/p & \text{for } * \text{ even,} \\ 0 & \text{otherwise,} \end{cases}$$

and

$$THH_*(\mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{for } * = 0, \\ \mathbb{Z}/i & \text{for } * = 2i-1, \\ 0 & \text{otherwise.} \end{cases}$$

In the monoid case $THH(BM) \simeq Q(\Lambda BM_+)$, where $\Lambda X = F(S^1_+, X)$ denotes the free loop space.

5. The Bökstedt trace map

We need to define maps

$$B\widehat{GL}_k(L) \xrightarrow{I} N^{cy}\widehat{GL}_k(L) \xrightarrow{S} THH(M_k(L)) \xrightarrow{\mathrm{Tr}^{(k)}} THH(L)\,.$$

The difficulty with defining I is that in the algebraic case our I_q used $g_0 = (g_1 \dots g_q)^{-1}$ and strict inverses are not available in the grouplike topological monoid $\widehat{GL}_k(L)$. Nonetheless a grouplike topological monoid G admits a functorial simplicial resolution by free monoids $G'_\bullet$, which is equivalent to a simplicial free group $G''_\bullet$, where a strict inverse is available. This produces a weak map I, which gives a section to the natural projection $\pi\colon N^{cy}\widehat{GL}_k(L) \to B\widehat{GL}_k(L)$

The map S is easily defined by including $\widehat{GL}_k(L)$ into $M_k(L)$ and mapping $(g_0, g_1, \dots, g_q)$ to the smash product $g_0 \wedge g_1 \wedge \cdots \wedge g_q$. Here each g_i is a map $g_i\colon S^{n_i} \to M_k(L)(S^{n_i})$.

To define the multitrace map $\mathrm{Tr}^{(k)}$ one may either prove that inclusion of 1×1 matrices defines a homotopy equivalence $THH(L) \to THH(M_k(L))$ and use a homotopy inverse, or work with "nonunital FSPs" to restrict to a subspace of $THH(M_k(L))$ where the multitrace formula from the algebraic case may be used.

Having overcome these obstacles, a combined weak map

$$\coprod_{k\geq 0} B\widehat{GL}_k(L) \to THH(L)$$

extends over the group completion ($THH(L)$ is already group complete) to define the *Bökstedt trace map*

$$\mathrm{tr}\colon K(L) \to THH(L)$$

for an FSP L. This is a natural weak transformation of functors.

This is a better invariant of K-theory. As an example, for every prime p the first p-torsion in $THH_i(\mathbb{Z})$ appears in degree $i = 2p-1$, and in fact the trace map $K_{2p-1}(\mathbb{Z}) \to THH_{2p-1}(\mathbb{Z}) = \mathbb{Z}/p$ is surjective. For $p = 2$ it is the generator in

$K_3(\mathbb{Z}) \cong \mathbb{Z}/48$ not coming from $\pi_3 Q(S^0) \cong \mathbb{Z}/24$ which is detected by $THH_3(\mathbb{Z}) = \mathbb{Z}/2$. For odd primes p there is a generator of a direct cyclic summand of infinite order in $K_{2p-1}(\mathbb{Z})$ which maps to $THH_{2p-1}(\mathbb{Z}) = \mathbb{Z}/p$. Likewise the trace map $A(X) \to Q(\Lambda X_+)$ (writing $X = BM$) is a useful invariant of A-theory.

The Bökstedt trace map can be lifted to the spectrum level by means of Γ-spaces.

There are other categorical models for the K-theory and THH of an FSP due to Dundas and McCarthy, for which the Bökstedt trace map is a natural transformation (in fact a forgetful map). Hence the apparent technical difficulties with describing the trace map as a direct map in the present terms is only a superficial problem.

6. The circle action

Recall that a simplicial object in a category $\mathcal{C}$ is a functor $\Delta^{op} \to \mathcal{C}$ where Δ has objects the finite ordered sets $[n] = \{0, 1, \dots, n\}$ for integers $n \geq 0$, and morphisms the order-preserving functions. The cyclic construction $Z_\bullet A$ defining Hochschild homology is a simplicial abelian group, but in addition to the simplicial structure maps it admits *cyclic maps* $t_q \colon A^{\otimes q+1} \to A^{\otimes q+1}$ given by

$$t_q(a_0 \otimes \cdots \otimes a_q) = a_q \otimes a_0 \otimes \cdots \otimes a_{q-1} \,.$$

Hence $Z_\bullet A$ admits an extension from Δ^{op} to Λ^{op} where Λ is Connes' *cyclic category*. This category has the same objects as Δ, but there are additional morphisms $t_q \colon [q] \to [q]$ of order $q+1$ satisfying certain relations with the face and degeneracy maps. A *cyclic object* in a category $\mathcal{C}$ is a functor $\Lambda^{op} \to \mathcal{C}$.

Likewise $THH(L)_\bullet$ becomes a cyclic space when we adjoin the cyclic action $t_q \colon THH(L)_q \to THH(L)_q$ induced on the homotopy colimit over I^{q+1} by the maps

$$t_q \colon G_q(L, n_0, \dots, n_q) \to G_q(L, n_q, n_0, \dots, n_{q-1})$$

that cyclically permute the factors in the mapping space

$$G_q(L, n_0, \dots, n_q) = F(S^{n_0} \wedge \cdots \wedge S^{n_q}, L(S^{n_0}) \wedge \cdots \wedge L(S^{n_q}))\,.$$

For instance t_0 is the identity in degree 0, while t_1 conjugates a map $f \colon S^{n_0} \wedge S^{n_1} \to L(S^{n_0}) \wedge L(S^{n_1})$ by the twist map in both the source and target.

For a fixed n, the simplicial set $[q] \mapsto \Delta([q], [n])$ has geometric realization the standard n-simplex Δ^n. For the cyclic category, the analogous simplicial set $[q] \mapsto \Lambda([q], [n])$ has geometric realization a space Λ^n, and there is a natural homeomorphism

$$\Lambda^n \cong S^1 \times \Delta^n \,.$$

A cyclic space $Z_\bullet \colon \Lambda^{op} \to \text{Top}$ determines a simplicial space by restriction over $\Delta^{op} \to \Lambda^{op}$. We can form a cyclic realization

$$\|Z_\bullet\| = \coprod_q Z_q \times \Lambda^q / \approx$$

where the identifications $\approx$ take into account the face maps, degeneracy maps and cyclic maps. Alternatively we can form a geometric realization

$$|Z_\bullet| = \coprod_q Z_q \times \Delta^q / \sim$$

where the identifications $\sim$ take into account the face maps and degeneracy maps. Then the natural S^1-action on each Λ^n extends to determine an S^1-action on $\|Z_\bullet\|$, and furthermore there is a natural homeomorphism $|Z_\bullet| \cong \|Z_\bullet\|$. Hence the geometric realization of (the underlying simplicial space of) a cyclic space $Z_\bullet$ admits a natural S^1-action.

Hence $THH(L)$ comes equipped with a natural S^1-action. We shall see below that for every finite (cyclic) subgroup $C \subset S^1$ the Bökstedt trace map can be factored through the C-fixed points of $THH(L)$:

$$\mathrm{tr}\colon K(L) \xrightarrow{\mathrm{tr}_c} THH(L)^C \xrightarrow{F} THH(L)$$

where c is the order of C and F denotes the forgetful map including the fixed points under the action by $C \subset S^1$ into the whole space. To construct the lift tr_c we need a simplicial model for the action by C on $THH(L)$. This can be achieved by the technique of *edgewise subdivision*.

Let $X_\bullet\colon \Delta^{op} \to \mathcal{C}$ be a simplicial object and $c \geq 1$ an integer. Define a functor $\mathrm{sd}_c\colon \Delta \to \Delta$ by c-fold repetition and concatenation:

$$\mathrm{sd}_c([q]) = [c(q+1)-1] \cong [q] \sqcup \cdots \sqcup [q] \qquad (c \text{ summands})$$

Likewise $\mathrm{sd}_c(f) \cong f \sqcup \cdots \sqcup f$ for morphisms f. Then the *c-fold edgewise subdivision* $\mathrm{sd}_c X_\bullet$ of $X_\bullet$ is the composite $X_\bullet \circ \mathrm{sd}_c^{op}$. This is again a simplicial object in $\mathcal{C}$. Note that the q-simplices of $\mathrm{sd}_c X_\bullet$ equal $X_{c(q+1)-1}$.

There is a diagonal imbedding of Δ^q into the c-fold join $\Delta^q * \cdots * \Delta^q \cong \Delta^{c(q+1)-1}$ which induces a natural homeomorphism:

$$D\colon |\,\mathrm{sd}_c X_\bullet| \xrightarrow{\cong} |X_\bullet|$$

This homeomorphism is not induced by a simplicial map.

In the case of a cyclic object $Z.$ the c-fold edgewise subdivision $\mathrm{sd}_c Z_\bullet$ admits a simplicial action by C. In degree q the q-simplices of $\mathrm{sd}_c Z_\bullet$ are $Z_{c(q+1)-1}$ and here the action of $t_{c(q+1)-1}^{q+1}$ is of order c and represents the action by a generator of C.

In the example we are interested in, the cyclic space $THH(L)_\bullet$ has c-fold edgewise subdivision $\mathrm{sd}_c THH(L)_\bullet$ with q-simplices

$$\begin{aligned}\mathrm{sd}_c THH(L)_q &= THH(L)_{c(q+1)-1} = \\ &\qquad \operatorname*{hocolim}_{I^{c(q+1)}} F(S^{n_0} \wedge \cdots \wedge S^{n_{c(q+1)-1}}, L(S^{n_0}) \wedge \cdots \wedge L(S^{n_{c(q+1)-1}}))\,.\end{aligned}$$

The generator of C acts on a map f in the mapping space above by cyclically shifting each factor in the source and target $(q+1)$ places to the right. This action has order c.

The restriction of the S^1-action on $THH(L) = |THH(L)_\bullet|$ to $C \subset S^1$ precisely agrees with the geometric realization of the simplicial action by C on $\mathrm{sd}_c THH(L)_\bullet$, by way of the homeomorphism D.

7. The restriction and Frobenius maps

We now define two families of maps relating the fixed point spaces $THH(L)^C$ for the various finite subgroups $C \subset S^1$, called the restriction and Frobenius maps.

Let $B \subset C \subset S^1$ be two finite subgroups. Then the *Frobenius map*

$$F\colon THH(L)^C \to THH(L)^B$$

is simply the forgetful map that includes the fixed points of C into the fixed points of the smaller group B.

The restriction map is also a morphism

$$R\colon THH(L)^C \to THH(L)^B$$

but it is different from F. For simplicity we describe this map in the absolute case when $B = 1$. The relative cases are similar. Hence we want to define a map

$$R\colon THH(L)^C \to THH(L)\,.$$

We identify $THH(L)^C = |THH(L)_\bullet|^C \cong |\operatorname{sd}_c THH(L)_\bullet|^C = |\operatorname{sd}_c THH(L)_\bullet^C|$ as explained in the previous section. Then in simplicial degree q

$$\operatorname{sd}_c THH(L)_q^C = THH(L)_{c(q+1)-1}^C = \\ \operatorname*{hocolim}_{I^{c(q+1)}} F(S^{n_0} \wedge \dots \wedge S^{n_{c(q+1)-1}}, L(S^{n_0}) \wedge \dots \wedge L(S^{n_{c(q+1)-1}}))^C\,.$$

We may diagonally imbed I^{q+1} into $I^{c(q+1)}$ by c-fold repetition and concatenation. There is then a homotopy equivalence

$$\operatorname*{hocolim}_{I^{q+1}} F((S^{n_0} \wedge \dots \wedge S^{n_q})^{\wedge c}, (L(S^{n_0}) \wedge \dots \wedge L(S^{n_q}))^{\wedge c})^C \xrightarrow{\simeq} \\ \operatorname*{hocolim}_{I^{c(q+1)}} F(S^{n_0} \wedge \dots \wedge S^{n_{c(q+1)-1}}, L(S^{n_0}) \wedge \dots \wedge L(S^{n_{c(q+1)-1}}))^C$$

since the imbedding is cofinal. Here $X^{\wedge c}$ denotes the c-fold smash product of X with itself. The group C cyclically permutes the c factors in the source and target of the first mapping space.

For any group G and G-spaces X, Y there is a natural map $r\colon F(X,Y)^G \to F(X^G, Y^G)$ taking a G-map $f\colon X \to Y$ to its restriction $r(f) = f|X^G\colon X^G \to Y^G$. In our case this map appears as follows:

$$\operatorname*{hocolim}_{I^{q+1}} F((S^{n_0} \wedge \dots \wedge S^{n_q})^{\wedge c}, (L(S^{n_0}) \wedge \dots \wedge L(S^{n_q}))^{\wedge c})^C \xrightarrow{r} \\ \operatorname*{hocolim}_{I^{q+1}} F(\Delta(S^{n_0} \wedge \dots \wedge S^{n_q}), \Delta(L(S^{n_0}) \wedge \dots \wedge L(S^{n_q})))$$

Here $\Delta(X) \subset X^{\wedge c}$ denotes the diagonal subspace left invariant by the cyclic permutation action by C.

Identifying $\Delta(X)$ with X in both places, the target above is precisely

$$\operatorname*{hocolim}_{I^{q+1}} F(S^{n_0} \wedge \dots \wedge S^{n_q}, L(S^{n_0}) \wedge \dots \wedge L(S^{n_q})) = THH(L)_q\,.$$

By composition we obtain a simplicial map $THH(L)^C_\bullet \to THH(L)_\bullet$, whose geometric realization is the *restriction map*

$$R\colon THH(L)^C \to THH(L)\,.$$

The intermediate restriction maps $R\colon THH(L)^C \to THH(L)^B$ are given by restricting suitable C-equivariant maps to their fixed sets for the subgroup of C of order equal to the index of B in C.

Hesselholt has proved that for $L = \tilde{A}$ with A a commutative ring the ring of path components $\pi_0 THH(A)^{C_{p^n}}$ equals the ring $W_{n+1}(A,p)$ of p-typical Witt vectors of length $n+1$ over A. Here C_{p^n} is the cyclic group of order p^n. The R- and F-maps induce the classically named restriction and Frobenius homomorphisms

$$R, F\colon W_{n+1}(A,p) \to W_n(A,p)$$

between the truncated Witt rings. Hence the present terminology. (In the case $A = \mathbb{F}_p$, the ring of Witt vectors equals the p-adic integers $\mathbb{Z}_p$, and the truncated ring $W_n(\mathbb{F}_p,p)$ equals $\mathbb{Z}/p^n$.)

The Frobenius maps are also present in the cyclic construction defining the Hochschild homology space, but the restriction maps are only only known to be present in the topological setting. In particular any extension of this theory to other ground ring spectra than the sphere spectrum should have to construct restriction maps on Hochschild homology in the case when the ground ring spectrum is a usual ring. These would most likely not have a direct algebraic description.

8. Topological cyclic homology

We can now define the topological cyclic homology of an FSP L. Fix a prime p, and consider only the cyclic subgroups $C = C_{p^n}$ of order p^n for $n \geq 0$.

We have the following diagram:

$$\cdots \underset{F}{\overset{R}{\rightrightarrows}} THH(L)^{C_{p^n}} \underset{F}{\overset{R}{\rightrightarrows}} \quad \cdots \underset{F}{\overset{R}{\rightrightarrows}} THH(L)^{C_p} \underset{F}{\overset{R}{\rightrightarrows}} THH(L)$$

The diagram commutes because the restriction and Frobenius maps commute. As we shall see in the next section the Bökstedt trace map from K-theory admits lifts

$$tr_{p^n} : K(L) \to THH(L)^{C_{p^n}}$$

which are compatible under the restriction maps, and compatible up to chosen homotopies under the Frobenius maps. Hence these lifts and chosen homotopies will define a map from $K(L)$ to the homotopy limit of the diagram above, over both the restriction and Frobenius maps.

Hence it is natural to define the *topological cyclic homology* of L at p as

$$TC(L,p) = \operatorname*{holim}_{R,F} THH(L)^{C_{p^n}}$$

where the homotopy limit is formed over the category $\mathbb{I}_p$ indexing the diagram above.

We may also consider the homotopy limit over only the restriction maps

$$TR(L,p) = \operatorname*{holim}_{R} THH(L)^{C_{p^n}}$$

or over only the Frobenius maps

$$TF(L,p) = \operatorname*{holim}_{F} THH(L)^{C_{p^n}} .$$

These homotopy limits are indexed over the natural numbers, and may be realized by converting the maps in the diagram to fibrations and forming the space-level limit.

Since R and F commute there is an induced self-map F of $TR(L,p)$, and an induced self-map R of $TF(L,p)$. The topological cyclic homology TC may then be obtained from the fiber sequences

$$TC(L,p) \to TR(L,p) \xrightarrow{F-1} TR(L,p)$$

and

$$TC(L,p) \to TF(L,p) \xrightarrow{R-1} TF(L,p) .$$

So $TC(L,p)$ can be identified with the homotopy fixed points for the action of F on $TR(L,p)$, or for the action of R on $TF(L,p)$.

There are then short exact sequences

$$0 \to R^1 \lim_{R} \pi_{*+1} THH(L)^{C_{p^n}} \to \pi_* TR(L,p) \to \lim_{R} \pi_* THH(L)^{C_{p^n}} \to 0$$

and

$$0 \to R^1 \lim_{F} \pi_{*+1} THH(L)^{C_{p^n}} \to \pi_* TF(L,p) \to \lim_{F} \pi_* THH(L)^{C_{p^n}} \to 0$$

suitable for computing $\pi_* TR(L,p)$ and $\pi_* TF(L,p)$. With a finite generation hypotheses on $\pi_* L$ the derived R^1 lim-terms vanish, *e.g.* when using homotopy with finite coefficients. Finally there are long exact sequences

$$\dots \xrightarrow{\partial} TC_*(L,p) \xrightarrow{\pi} TR_*(L,p) \xrightarrow{F-1} TR_*(L,p) \xrightarrow{\partial} \dots$$

and

$$\dots \xrightarrow{\partial} TC_*(L,p) \xrightarrow{\pi} TF_*(L,p) \xrightarrow{R-1} TF_*(L,p) \xrightarrow{\partial} \dots$$

for describing $\pi_* TC(L,p)$ in terms of $\pi_* TR(L,p)$ and the Frobenius map, or in terms of $\pi_* TF(L,p)$ and the restriction map.

Similarly we might define $TC(L)$ as the homotopy limit over the R and F maps of the fixed point spaces $THH(L)^C$ as C runs over all the finite subgroups of S^1, rather than just the p-subgroups. However this space carries little or no extra information, as the natural map

$$TC(L) \to TC(L,p)$$

becomes a homotopy equivalence after p-adic completion. However, Goodwillie has defined a more refined version of $TC(L)$, which carries more integral information than the individual $TC(L,p)$ do.

By Hesselholt's π_0-calculations it follows that for $L = \tilde{A}$ with A commutative the spectrum $TR(A,p)$ is connective (the restriction maps are surjective so the R^1 lim-term vanishes) with

$$\pi_0 TR(A,p) = W(A,p)$$

(the ring of p-typical Witt vectors). Hence $TC(A,p)$ is (-2)-connected, and there is an exact sequence

$$TC_0(A,p) \to W(A,p) \xrightarrow{F-1} W(A,p) \to TC_{-1}(A,p) \to 0.$$

The restriction and Frobenius maps, and hence topological cyclic homology may be defined on the spectrum level, again using Γ-spaces.

9. The cyclotomic trace map

It remains to define the cyclotomic trace map $\mathrm{trc}\colon K(L) \to TC(L,p)$.

Let M be a topological monoid and consider the diagonal imbedding

$$M^{q+1} = N^{cy}_q M \to \mathrm{sd}_c\, N^{cy}_q M = N^{cy}_{c(q+1)-1} M = M^{c(q+1)}$$

taking $(m_0, \ldots, m_q)$ to

$$(m_0, \ldots, m_q, m_0, \ldots, m_q, \ldots, m_0, \ldots, m_q)\,.$$

(The sequence $(m_0, \ldots, m_q)$ is repeated c times.) This defines a homeomorphism

$$\Delta_c\colon N^{cy} M = |N^{cy}_\bullet M| \xrightarrow{\cong} |\,\mathrm{sd}_c\, N^{cy}_\bullet M|^C \cong (N^{cy} M)^C\,.$$

The lifted trace map $\mathrm{tr}_c\colon K(L) \to THH(L)^C$ is induced from the composite map

$$\begin{array}{ccccc} & & (N^{cy}\widehat{GL}_k(L))^C & \xrightarrow{S^C} THH(M_k(L))^C \xrightarrow{\mathrm{Tr}(k)^C} & THH(L)^C \\ & & \uparrow{\scriptstyle \Delta_c} & & \\ B\widehat{GL}_k(L) & \xrightarrow{I} & N^{cy}\widehat{GL}_k(L) & & \end{array}$$

by means of group completion over k.

We now have the following fundamental observations:

(1) Let $B \subset C$ be finite subgroups of S^1, of order b and c. The composite

$$K(L) \xrightarrow{\mathrm{tr}_c} THH(L)^C \xrightarrow{R} THH(L)^B$$

equals the map tr_b.

(2) There is a natural explicit homotopy from the composite

$$K(L) \xrightarrow{\mathrm{tr}_c} THH(L)^C \xrightarrow{F} THH(L)^B$$

to the map tr_b.

Hence even if R and F differ, they become homotopic when precomposed with the lifted trace map tr_c.

Fixing a prime p, the maps tr_{p^n} now define a map

$$\mathrm{tr}_R \colon K(F) \to \operatorname*{holim}_R THH(L)^{C_{p^n}} = TR(L,p)\,.$$

The explicit homotopies determine a homotopy from the composite

$$K(F) \xrightarrow{\mathrm{tr}_R} TR(L,p) \xrightarrow{F} TR(L,p)$$

to tr_R, which in turn determines a map

$$\mathrm{trc} \colon K(L) \to \mathrm{hofib}(F - 1 \colon TR(L,p) \to TR(L,p)) = TC(L,p)\,.$$

This is the *cyclotomic trace map* at p.

A similar construction defines a map $\mathrm{trc} \colon K(L) \to TC(L)$, which agrees with the map above after p-adic completion.

This map trc is a much better invariant of K-theory than the previous trace maps. By a theorem of McCarthy the diagram

$$\begin{array}{ccc} K(A_1) & \xrightarrow{trc} & TC(A_1,p) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f} \\ K(A_2) & \xrightarrow{trc} & TC(A_2,p) \end{array}$$

becomes homotopy cartesian after p-adic completion when $f \colon A_1 \to A_2$ is a map of simplicial rings which induces a surjection $\pi_0 A_1 \to \pi_0 A_2$ with nilpotent kernel.

Using this result, Hesselholt and Madsen have proved that the cyclotomic trace map

$$\mathrm{trc} \colon K(A) \to TC(A,p)$$

induces an equivalence on connective covers after p-adic completion when A is a semi-simple k-algebra over a perfect field k of positive characteristic p, or when A is a finite-dimensional algebra over the ring $W(k,p)$ of p-typical Witt vectors over such a field k.

McCarthy's theorem has been extended by Dundas to apply to the diagram induced by a map $f \colon L_1 \to L_2$ of FSPs inducing a surjection $\pi_0 Q(L_1) \to \pi_0 Q(L_2)$ with nilpotent kernel.

There are involutions on K-theory and topological cyclic homology, related to the involution $A \mapsto (A^{-1})^t$ on matrices or to turning pseudo-isotopies upside down. The explicit homotopy $F \circ \mathrm{tr}_c \simeq \mathrm{tr}_b$ is not involutive, so technical difficulties remain with the definition of an involutive cyclotomic trace map.

This ends our introduction to the cyclotomic trace map.