

# TOPOLOGICAL HOCHSCHILD HOMOLOGY OF THE PRIME FIELDS AND THE INTEGERS

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## 1. HOCHSCHILD HOMOLOGY

Let  $k$  be a commutative ring. The category of  $k$ -modules is tensored over sets, meaning that for each  $k$ -module  $M$  and set  $T$  we can form

$$M \odot T = \bigoplus_{t \in T} M$$

so that

$$\text{Mod}_k(M \odot T, N) \cong \{T \rightarrow \text{Mod}_k(M, N)\}.$$

Likewise, the category of commutative  $k$ -algebras is tensored over sets, meaning that for each commutative  $k$ -algebra  $A$  and set  $T$  we can form

$$A \odot T = \bigotimes_{t \in T} A$$

so that

$$\text{CAlg}_k(A \odot T, B) \cong \{T \rightarrow \text{CAlg}_k(A, B)\}.$$

If  $T_\bullet: [q] \mapsto T_q$  is a simplicial set, we can prolong the tensored structure, so that

$$M \odot T_\bullet: [q] \mapsto M \odot T_q$$

is a simplicial  $k$ -module, and

$$A \odot T_\bullet: [q] \mapsto A \odot T_q$$

is a simplicial commutative  $k$ -algebra. We can then form the associated normalized chain complexes, and consider their homology. In the  $k$ -module case we obtain  $H_*^{cell}(|T_\bullet|; M)$ . In the commutative  $k$ -algebra case we focus on the case  $T_\bullet = S^1 = \Delta^1 / \partial\Delta^1$ .

Let  $\Delta^1$  be the simplicial 1-simplex, with  $q$ -simplices

$$\Delta_q^1 = \{\alpha: [q] \rightarrow [1]\}$$

the order-preserving functions  $\alpha: \{0, \dots, q\} \rightarrow \{0, 1\}$ . There are  $q+2$  of these. Let  $0 \leq s \leq q+1$  correspond to  $\alpha_s$  given by

$$\alpha_s(i) = \begin{cases} 0 & \text{for } 0 \leq i < s, \\ 1 & \text{for } s \leq i \leq q. \end{cases}$$

Then

$$A \odot \Delta_q^1 \cong A \odot \{0, 1, \dots, q, q+1\} \cong A \otimes A^{\otimes q} \otimes A.$$

The  $i$ -th face map  $d_i: \Delta_q^1 \rightarrow \Delta_{q-1}^1$  induces

$$d_i(x_0 \otimes x_1 \otimes \dots \otimes x_q \otimes x_{q+1}) = x_0 \otimes \dots \otimes x_i x_{i+1} \otimes \dots \otimes x_{q+1}$$

for  $0 \leq i \leq q$ . The  $j$ -th degeneracy map  $s_j: \Delta_q^1 \rightarrow \Delta_{q+1}^1$  induces

$$s_j(x_0 \otimes x_1 \otimes \dots \otimes x_q \otimes x_{q+1}) = x_0 \otimes \dots \otimes x_j \otimes 1 \otimes x_{j+1} \otimes \dots \otimes x_{q+1}.$$

Let  $\partial\Delta^1$  be the simplicial subset with  $q$ -simplices

$$\partial\Delta^1 = \{\alpha_0, \alpha_{q+1}\}$$

the constant functions  $\alpha: \{0, \dots, q\} \rightarrow \{0, 1\}$ . Then

$$A \odot \partial\Delta_q^1 \cong A \odot \{0, q+1\} \cong A \otimes A.$$

The pushout square

$$\begin{array}{ccc} \partial\Delta^1 & \longrightarrow & \Delta^1 \\ \downarrow & & \downarrow \\ * & \longrightarrow & S^1 \end{array}$$

of simplicial sets induces a pushout square

$$\begin{array}{ccc} A \otimes A & \longrightarrow & A \odot \Delta^1 \\ \downarrow & & \downarrow \\ A & \longrightarrow & A \odot S^1 \end{array}$$

of commutative  $k$ -algebras. Hence

$$HH(A)_\bullet := A \odot S^1 \cong A \otimes_{A \otimes A} (A \odot \Delta^1)$$

is a simplicial commutative  $k$ -algebra under  $A$ , i.e., a simplicial commutative  $A$ -algebra, with

$$HH(A)_\bullet: [q] \mapsto A \otimes_{A \otimes A} (A \odot \Delta_q^1) \cong A \otimes A^{\otimes q}.$$

The  $i$ -th face map is given by

$$d_i(x_0 \otimes x_1 \otimes \cdots \otimes x_q) = \begin{cases} x_0 x_1 \otimes x_2 \otimes \cdots \otimes x_q & \text{for } i = 0, \\ x_0 \otimes \cdots \otimes x_i x_{i+1} \otimes \cdots \otimes x_q & \text{for } 0 < i < q, \\ x_q x_0 \otimes x_1 \otimes \cdots \otimes x_{q-1} & \text{for } i = q. \end{cases}$$

The  $j$ -th degeneracy map is given by

$$s_j(x_0 \otimes x_1 \otimes \cdots \otimes x_q) = x_0 \otimes \cdots \otimes x_j \otimes 1 \otimes x_{j+1} \otimes \cdots \otimes x_q.$$

The associated chain complex is the Hochschild complex  $(C(A), b)$  with

$$C(A)_q = HH(A)_q = A \otimes A^{\otimes q}$$

and  $b = \sum_{i=0}^q (-1)^i d_i: HH(A)_q \rightarrow HH(A)_{q-1}$ . Its homology groups

$$HH_*(A) = H_*(C(A), b)$$

are the Hochschild homology groups of  $A$ . This is a graded commutative  $A$ -algebra, with multiplication given by the shuffle product. If  $A$  is flat over  $k$ , then  $A \odot \Delta^1$  is a flat  $A \odot \partial\Delta^1 \cong A \otimes A$ -module resolution of  $A \odot * \cong A$ , and

$$HH_*(A) = \mathrm{Tor}_*^{A \otimes A}(A, A).$$

*Example 1.1.* Let  $A = E(x) = k\{1, x\}$  be the exterior algebra on one generator, with  $x^2 = 0$ . The normalized Hochschild complex  $(NC(A)_*, b)$  is given in degree  $q$  by

$$NC(A)_q = A \otimes \bar{A}^{\otimes q} \cong A\{1 \otimes x^{\otimes q}\}$$

where  $\bar{A} = \mathrm{cok}(k \rightarrow A) = k\{x\}$ . Suppose that  $A$  is graded and  $x$  is in an odd degree. The face maps  $d_i$  for  $0 < i < q$  are zero because  $x^2 = 0$ , and  $d_0 + (-1)^q d_q = 0$  because the cyclic permutation of  $x^{\otimes q}$  has sign  $(-1)^{q-1}$ . Hence  $b = 0$  and

$$HH_q(E(x)) \cong E(x)\{1 \otimes x^{\otimes q}\}$$

for each  $q \geq 0$ . We write  $\sigma x = 1 \otimes x$  for the  $E(x)$ -module generator of  $HH_1(E(x))$ , and more generally write

$$\gamma_q(\sigma x) = 1 \otimes x \otimes \cdots \otimes x,$$

with  $q$  copies of  $x$ , for the  $E(x)$ -module generator of  $HH_q(E(x))$ . The shuffle product satisfies  $\gamma_i(\sigma x) \cdot \gamma_j(\sigma x) = (i, j)\gamma_{i+j}(\sigma x)$ , so we have a divided power algebra:

$$HH_*(E(x)) \cong E(x) \otimes \Gamma(\sigma x).$$

## 2. DE RHAM FORMS

In low dimensions,

$$\dots \xrightarrow{b} C(A)_2 = A \otimes A^{\otimes 2} \xrightarrow{b} C(A)_1 = A \otimes A \xrightarrow{b} C(A)_0 = A \rightarrow 0.$$

with  $b(x_0 \otimes x_1) = x_0x_1 - x_1x_0$ , and  $b(x_0 \otimes x_1 \otimes x_2) = x_0x_1 \otimes x_2 - x_0 \otimes x_1x_2 + x_2x_0 \otimes x_1$ .

The Kähler differentials  $\Omega_A$  of  $A$  over  $k$  is the  $A$ -module generated by symbols  $dx$  for  $x \in A$ , subject to the relations

$$d(xy) = xdy + ydx$$

for  $x, y \in A$ . In other words, it is the cokernel of

$$b: A \otimes A^{\otimes 2} \longrightarrow A \otimes A,$$

with  $x_0 \otimes x_1$  mapping to  $x_0dx_1$ . Hence  $HH_0(A) \cong A$  and  $HH_1(A) \cong \Omega_A$  for commutative  $k$ -algebras  $A$ .

The Kähler  $q$ -forms of  $A$  is the exterior power

$$\Omega_A^q = \Omega_A \wedge_A \cdots \wedge_A \Omega_A = \bigwedge_A^q \Omega_A,$$

i.e., an  $A$ -module generated by symbols

$$dx_1 \wedge \cdots \wedge dx_q$$

for  $x_1, \dots, x_q \in A$ , subject to suitable relations.

*Example 2.1.* If  $k = \mathbb{R}$  and  $A = C^\infty(M)$  is the algebra of smooth functions on a differentiable  $n$ -manifold  $M$ , then  $\Omega_A$  is the  $C^\infty(M)$ -module of differential 1-forms on  $M$ , locally of the form  $\sum_{i=1}^n f_i dx_i$ , and  $\Omega_A^q$  is the  $C^\infty(M)$ -module of differential  $q$ -forms on  $M$ , locally of the form  $\sum_I f_I dx_I$ , where  $I = (i_1, \dots, i_q)$  with  $1 \leq i_1 < \cdots < i_q \leq n$ .

The isomorphism  $\Omega_A \cong HH_1(A)$  extends, using the graded commutative  $A$ -algebra structure on  $HH_*(A)$ , to an  $A$ -algebra homomorphism

$$\phi: \Omega_A^* \longrightarrow HH_*(A).$$

**Theorem 2.2** (Hochschild–Kostant–Rosenberg). *If  $A$  is a smooth  $k$ -algebra, then  $\phi$  is an isomorphism.*

*Example 2.3.* If  $A = k[x]$ , then  $\Omega_A = A\{dx\}$  and

$$HH_*(k[x]) \cong k[x] \otimes E(dx)$$

where  $E(dx) = k\{1, dx\}$  is the exterior algebra on  $dx$ . Here  $dx \in \Omega_A$  corresponds to the class of  $\sigma x = 1 \otimes x$  in  $HH_1(A)$ .

## 3. DE RHAM COHOMOLOGY AND THE CIRCLE ACTION

The differential  $d: A \rightarrow \Omega_A$ , mapping  $x$  to  $dx$ , extends to a  $k$ -linear derivation

$$d: \Omega_A^q \longrightarrow \Omega_A^{q+1}$$

with  $d \circ d = 0$ . In the case  $A = C^\infty(M)$ , this is the exterior (de Rham) differential. Connes defined an operator

$$B: C_q(A) \longrightarrow C_{q+1}(A)$$

that induces a derivation

$$B: HH_q(A) \longrightarrow HH_{q+1}(A),$$

and the exterior differential  $d$  and Connes'  $B$ -operator are compatible under  $\phi$ :

$$\begin{array}{ccc} \Omega_A^q & \xrightarrow{d} & \Omega_A^{q+1} \\ \phi \downarrow & & \downarrow \phi \\ HH_q(A) & \xrightarrow{B} & HH_{q+1}(A) \end{array}$$

Hence there is a natural homomorphism

$$H_q^{dR}(A) = H_q(\Omega_A^*, d) \xrightarrow{\phi} H_q(HH_*(A), B)$$

from the de Rham homology of  $A$  to the homology of Hochschild homology with respect to the  $B$ -operator. (This is close to Connes' periodic and cyclic homology.)

The simplicial object  $HH(A)_\bullet = A \odot S^1: [q] \mapsto A \otimes A^{\otimes q}$  is a cyclic object, in the sense that the cyclic permutations

$$t_q: A \otimes A^{\otimes q} \longrightarrow A \otimes A^{\otimes q}$$

given by

$$t_q(x_0 \otimes x_1 \otimes \cdots \otimes x_q) = x_q \otimes x_0 \otimes \cdots \otimes x_{q-1}$$

satisfy  $t_q^{q+1} = \text{id}$ , and are suitably compatible with the face and degeneracy maps.

For any cyclic set the topological realization of the underlying simplicial set has a natural circle action. In the case of

$$HH(A) := |HH(A)_\bullet| = |A \odot S^1|$$

the action

$$\rho: HH(A) \wedge S_+^1 \longrightarrow HH(A)$$

is base-point preserving. The  $S^1$ -orbit of a 0-simplex  $x \in A = HH(A)_0$  is given by the 1-simplex

$$t_1 s_0(x) = 1 \otimes x \in HH(A)_1$$

which corresponds to a closed loop at  $x$  in  $HH(A)$ . The  $S^1$ -orbit of a  $q$ -simplex traces through  $q+1$  different  $q+1$ -simplices, in a similar manner.

The topological realization  $HH(A)$  of the simplicial abelian group underlying  $HH(A)_\bullet$  has homotopy groups given by the homology of the (normalized) Hochschild complex  $NC_*(A)$ :

$$\pi_*(HH(A)) \cong HH_*(A).$$

The circle action  $\rho$  induces a homomorphism

$$\sigma: \pi_q(HH(A)) \longrightarrow \pi_{q+1}(HH(A)),$$

i.e., an operator  $\sigma: HH_q(A) \rightarrow HH_{q+1}(A)$ . For example, if  $x \in A = C(A)_0 \cong HH_0(A)$  we deduce that  $\sigma x \in HH_1(A)$  is the class of  $1 \otimes x$  in  $C(A)_1$ , corresponding to  $dx \in \Omega_A$ . In fact,  $\sigma$  is equal to Connes  $B$ -operator.

#### 4. TOPOLOGICAL HOCHSCHILD HOMOLOGY

Let  $(Sp, \wedge, S, \gamma, F)$  be one of the closed symmetric monoidal categories of structured ring spectra. Examples include symmetric spectra and orthogonal spectra. Let  $A$  be a strictly commutative ring spectrum, i.e., a commutative monoid in  $Sp$ , with multiplication  $\mu: A \wedge A \rightarrow A$  and unit  $\eta: S \rightarrow A$ . We refer to  $A$  as a commutative  $S$ -algebra. The category  $\text{CAlg}_S$  is tensored over sets, by letting

$$A \odot T = \bigwedge_{t \in T} A$$

so that

$$\text{CAlg}_S(A \odot T, B) \cong \{T \mapsto \text{CAlg}_S(A, B)\}.$$

Let

$$THH(A)_\bullet := A \odot S^1: [q] \mapsto A \odot S_+^1 \cong A \wedge A \wedge \cdots \wedge A$$

be the simplicial commutative  $S$ -algebra obtained by prolongation, with  $1+q$  copies of  $A$  on the right hand side. The topological realization

$$THH(A) := \coprod_{q \geq 0} THH(A)_q \wedge \Delta_+^q / \sim$$

defines the topological Hochschild homology spectrum of  $A$ , which is a commutative  $S$ -algebra under  $A = A \odot S_0^1$ , i.e., a commutative  $A$ -algebra. The topological Hochschild homology groups of  $A$  are the (spectrum level) homotopy groups

$$THH_q(A) = \pi_q THH(A).$$

Taken together,  $\pi_*THH(A)$  is a graded commutative  $\pi_*(A)$ -algebra.

The simplicial object  $THH(A)_\bullet$  is a cyclic object, with additional structure maps

$$t_q = \gamma: A^{\wedge q} \wedge A \longrightarrow A \wedge A^{\wedge q}.$$

Its geometric realization has a circle action

$$\rho: THH(A) \wedge S_+^1 \longrightarrow THH(A).$$

Let  $s \in \pi_1^S(S_+^1)$  be a generator with Hurewicz image the fundamental class  $\sigma \in H_1(S_+^1)$ , such that  $s^2 = \eta s$ . The group action induces a pairing

$$\rho_*: \pi_q THH(A) \otimes \pi_1^S(S_+^1) \longrightarrow \pi_{q+1} THH(A)$$

and we define the homotopical  $s$ -operator  $s: THH_q(A) \rightarrow THH_{q+1}(A)$  by  $sx = \rho_*(x \otimes s)$ . Passing to homology, there is also a pairing

$$\rho_*: H_q THH(A) \otimes H_1(S_+^1) \longrightarrow H_{q+1} THH(A)$$

and a homological  $\sigma$ -operator  $\sigma: H_q THH(A) \rightarrow H_{q+1} THH(A)$  given by  $\sigma x = \rho_*(x \otimes \sigma)$ .

## 5. DYER–LASHOF OPERATIONS

The circle action has a right adjoint

$$\tilde{\rho}: THH(A) \longrightarrow F(S_+^1, THH(A)) \xleftarrow{\simeq} THH(A) \wedge DS_+^1,$$

where  $DS_+^1 = F(S_+^1, S)$  is the functional (Spanier–Whitehead) dual of  $S_+^1$ . These are maps of commutative  $S$ -algebras. Passing to homology, we obtain an algebra homomorphism

$$\tilde{\rho}_*: H_* THH(A) \longrightarrow H_* F(S_+^1, THH(A)) \cong H_* THH(A) \otimes H_* DS_+^1.$$

Here  $H_* DS_+^1 \cong \tilde{H}^{-*}(S_+^1) = E(\iota)$ , with  $\iota \in H^1(S_+^1)$  dual to  $\sigma \in H_1(S_+^1)$ , and

$$\tilde{\rho}_*(x) = x \otimes 1 + \sigma x \otimes \iota.$$

**Proposition 5.1.**  $\sigma(xy) = x \cdot \sigma y + (-1)^{|y|} \sigma x \cdot y$ .

*Proof.* From  $\tilde{\rho}_*(xy) = \tilde{\rho}_*(x)\tilde{\rho}_*(y)$  and  $\iota^2 = 0$  we obtain

$$\sigma(xy) \otimes \iota = x \cdot \sigma y \otimes \iota + (-1)^{|y|} \sigma x \cdot y \otimes \iota.$$

□

Let  $p$  be a prime and let  $H_*(X) = H_*(X; \mathbb{Z}/p)$  denote mod  $p$  homology. For commutative  $S$ -algebras  $A$  there is a natural Bockstein homomorphism

$$\beta: H_q(A) \longrightarrow H_{q-1}(A),$$

satisfying  $\beta(xy) = \beta(x)y + (-1)^{|x|} x\beta(y)$ . In particular,  $\beta(x^p) = 0$ . Furthermore, there are natural Dyer–Lashof operations

$$Q^k: H_q(A) \longrightarrow H_{q+(2p-2)k}(A)$$

satisfying the Cartan formula

$$Q^k(xy) = \sum_{i+j=k} Q^i(x)Q^j(y).$$

These operations generalize the  $p$ -th power homomorphism, and in particular

$$Q^k(x) = x^p$$

if  $|x| = 2k$ .

**Proposition 5.2** (Bökstedt/Angeltveit-R.).  $Q^k(\sigma x) = \sigma Q^k(x)$ .

*Proof.* By naturality,  $Q^k(\tilde{\rho}_*(x)) = \tilde{\rho}_*(Q^k(x))$ , so

$$Q^k(x \otimes 1 + \sigma x \otimes \iota) = Q^k(x) \otimes 1 + \sigma Q^k(x) \otimes \iota.$$

Here  $Q^j(1) = 0$  and  $Q^j(\iota) = 0$  for  $j \neq 0$ , and  $Q^0(1) = 1$  and  $Q^0(\iota) = \iota$ , all in  $H_*(DS_+^1) \cong \tilde{H}^{-*}(S_+^1) \cong H^{-*}(S^1)$ . Hence

$$Q^k(\sigma x) \otimes \iota = \sigma Q^k(x) \otimes \iota.$$

□

## 6. THE DUAL STEENROD ALGEBRA

Let  $p$  be an odd prime, and let  $H = H\mathbb{Z}/p$  be the mod  $p$  Eilenberg–Mac Lane (ring) spectrum, with  $n$ -th space a  $K(\mathbb{Z}/p, n)$ . Write  $H_*(X) = \pi_*(H \wedge X)$  for mod  $p$  homology.

For  $n = 1$ ,

$$H^*(K(\mathbb{Z}/p, 1)) = E(x) \otimes P(y)$$

with  $|x| = 1$ ,  $|y| = 2$ ,  $\beta(x) = y$ , and

$$H_*(K(\mathbb{Z}/p, 1)) = \mathbb{Z}/p\{\alpha_n \mid n \geq 0\}$$

with  $\alpha_n$  dual to  $x^\epsilon y^m$  for  $n = 2m + \epsilon$ ,  $m \geq 0$ ,  $\epsilon \in \{0, 1\}$ . Here  $\alpha_1 \cdot \alpha_{2m} = \alpha_{2m+1}$ , and  $\alpha_{2i} \cdot \alpha_{2j} = (i, j)\alpha_{2(i+j)}$ , so

$$H_*(K(\mathbb{Z}/p, 1)) = E(\alpha_1) \otimes \Gamma(\alpha_2).$$

For  $n = 2$  we use the map  $K(\mathbb{Z}, 2) \rightarrow K(\mathbb{Z}/p, 2)$ , where

$$H^*(K(\mathbb{Z}, 2)) = P(y)$$

with  $|y| = 2$ , and

$$H_*(K(\mathbb{Z}, 2)) = \mathbb{Z}/p\{\beta_m \mid m \geq 0\}$$

with  $\beta_m$  dual to  $y^m$ . Here  $\beta_i \cdot \beta_j = (i, j)\beta_{i+j}$ , so

$$H_*(K(\mathbb{Z}, 2)) = \Gamma(\beta_1).$$

**Definition 6.1.** For  $i \geq 0$  let  $\tau_i \in H_*(H)$ , with  $|\tau_i| = 2p^i - 1$ , be the image of  $\alpha_{2p^i} = \gamma_{p^i}(\alpha_2)$  under the colimit structure map

$$H_{*+1}(K(\mathbb{Z}/p, 1)) \longrightarrow H_*(H).$$

For  $i \geq 1$  let  $\xi_i \in H_*(H)$ , with  $|\xi_i| = 2p^i - 2$ , be the image of  $\beta_{p^i} = \gamma_{p^i}(\beta_1)$  under the colimit structure map

$$H_{*+2}(K(\mathbb{Z}, 2)) \longrightarrow H_{*+2}(K(\mathbb{Z}/p, 2)) \longrightarrow H_*(H).$$

**Theorem 6.2** (Milnor).

$$\mathcal{A}_* = H_*(H) \cong E(\tau_i \mid i \geq 0) \otimes P(\xi_i \mid i \geq 1)$$

The coproduct  $\psi: \mathcal{A}_* \rightarrow \mathcal{A}_* \otimes \mathcal{A}_*$ , induced by  $1 \wedge \eta \wedge 1: H \wedge H \rightarrow H \wedge H \wedge H$ , is given by

$$\psi(\tau_k) = \tau_k \otimes 1 + \sum_{i+j=k} \xi_i^{p^j} \otimes \tau_j$$

and

$$\psi(\xi_k) = \sum_{i+j=k} \xi_i^{p^j} \otimes \xi_j,$$

where  $\xi_0 = 1$ .

The involution  $\chi: \mathcal{A}_* \rightarrow \mathcal{A}_*$ , induced by the symmetry  $\gamma: H \wedge H \rightarrow H \wedge H$ , can be recursively calculated using

$$\phi(1 \otimes \chi)\psi = \eta\epsilon.$$

Let  $\bar{\tau}_i = \chi(\tau_i)$  and  $\bar{\xi}_i = \chi(\xi_i)$  be the conjugate generators of the dual Steenrod algebra.

$$\mathcal{A}_* \cong E(\bar{\tau}_i \mid i \geq 0) \otimes P(\bar{\xi}_i \mid i \geq 1).$$

with

$$\psi(\bar{\tau}_k) = 1 \otimes \bar{\tau}_k + \sum_{i+j=k} \bar{\tau}_i \otimes \bar{\xi}_j^{p^i}$$

and

$$\psi(\bar{\xi}_k) = \sum_{i+j=k} \bar{\xi}_i \otimes \bar{\xi}_j^{p^i}.$$

**Proposition 6.3** (Kristensen/Steinberger). *The action of the Bockstein and Dyer–Lashof operations on  $H_*(H) = \mathcal{A}_*$  is known. In particular,  $\beta(\bar{\tau}_k) = (\pm)\xi_k$  and*

$$Q^{p^k}(\bar{\tau}_k) = \bar{\tau}_{k+1}$$

for each  $k \geq 0$ .

Let  $H\mathbb{Z}$  be the integral Eilenberg–Mac Lane (ring) spectrum. The map  $H\mathbb{Z} \rightarrow H$  induces an injection in homology:

**Theorem 6.4.**

$$H_*(H\mathbb{Z}) \cong E(\bar{\tau}_i \mid i \geq 1) \otimes P(\bar{\xi}_i \mid i \geq 1).$$

## 7. BÖKSTEDT’S CALCULATIONS

For each commutative ring  $R$  we write  $THH(R)$  for  $THH(A)$ , where  $A = HR$  is the associated Eilenberg–Mac Lane ring spectrum.

**Theorem 7.1** (Bökstedt). (a)

$$H_*THH(\mathbb{Z}/p) \cong \mathcal{A}_* \otimes P(\sigma\bar{\tau}_0).$$

(b)

$$\pi_*THH(\mathbb{Z}/p) = P(\mu_0) = \mathbb{Z}/p[\mu_0]$$

with  $|\mu_0| = 2$ .

(c)

$$\pi_iTHH(\mathbb{Z}/p) \cong \begin{cases} \mathbb{Z}/p & \text{for } i = 2n \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 7.2** (Bökstedt). (a)

$$H_*THH(\mathbb{Z}) \cong H_*(H\mathbb{Z}) \otimes E(\sigma\bar{\xi}_1) \otimes P(\sigma\bar{\tau}_1).$$

(b)

$$\pi_*(THH(\mathbb{Z}); \mathbb{Z}/p) = E(\lambda_1) \otimes P(\mu_1) = \Lambda(\lambda_1) \otimes \mathbb{Z}/p[\mu_1]$$

with  $|\lambda_1| = 2p - 1$  and  $|\mu_1| = 2p$ .

(c)

$$\pi_iTHH(\mathbb{Z}) \cong \begin{cases} \mathbb{Z} & \text{for } i = 0, \\ \mathbb{Z}/n & \text{for } i = 2n - 1 > 0, \\ 0 & \text{otherwise.} \end{cases}$$

We apply homology to the skeleton filtration

$$A = THH(A)^{(0)} \subset \dots \subset THH(A)^{(s-1)} \subset THH(A)^{(s)} \subset \dots \subset THH(A)$$

with quotients

$$THH(A)^{(s)}/THH(A)^{(s-1)} \cong A \wedge \bar{A}^{\wedge s} \wedge \Delta^s / \partial\Delta^s.$$

Here  $\bar{A}$  denotes the cofiber of  $\eta: S \rightarrow A$ . The resulting Bökstedt spectral sequence has

$$E_{s,*}^1 = H_{s+*}(A \wedge \bar{A}^{\wedge s} \wedge \Delta^s / \partial\Delta^s) \cong H_*(A) \otimes H_*(\bar{A})^{\otimes s} = NC(H_*(A))_s$$

and  $d_s^1: E_{s,*}^1 \rightarrow E_{s-1,*}^1$  agrees with the normalized Hochschild boundary  $b$ . Hence

$$E_{s,*}^2 \cong HH_s(H_*(A)) \implies H_{s+*}THH(A).$$

There is a coproduct

$$\psi: THH(A) \longrightarrow THH(A) \wedge_A THH(A)$$

that makes the spectral sequence an  $H_*(A)$ -Hopf algebra spectral sequence, as long as a flatness hypothesis is satisfied, see Angeltveit–R.

*Proof of Theorem 7.1.* In the case  $R = \mathbb{Z}/p$  we have  $A = H$  and  $H_*(A) = \mathcal{A}_*$ . Hence

$$E_{s,*}^2 \cong HH_s(\mathcal{A}_*) \implies H_{s+*}THH(\mathbb{Z}/p).$$

Let  $p$  be odd. From

$$\mathcal{A}_* \cong E(\bar{\tau}_k \mid k \geq 0) \otimes P(\bar{\xi}_k \mid k \geq 1)$$

and the Künneth theorem we obtain

$$\begin{aligned} HH_*(\mathcal{A}_*) &\cong \bigotimes_{k \geq 0} HH_*(E(\bar{\tau}_k)) \otimes \bigotimes_{k \geq 1} HH_*(P(\bar{\xi}_k)) \\ &\cong \bigotimes_{k \geq 0} E(\bar{\tau}_k) \otimes \Gamma(\sigma\bar{\tau}_k) \otimes \bigotimes_{k \geq 1} P(\bar{\xi}_k) \otimes E(\sigma\bar{\xi}_k) \\ &\cong \mathcal{A}_* \otimes \Gamma(\sigma\bar{\tau}_k \mid k \geq 0) \otimes E(\sigma\bar{\xi}_k \mid k \geq 1). \end{aligned}$$

Here  $\mathcal{A}_* = E_{0,*}^2 = HH_0(\mathcal{A}_*)$ . The  $E^2$ -term is generated as a graded commutative  $\mathcal{A}_*$ -algebra by the divided powers

$$\gamma_{p^i}(\sigma\bar{\tau}_k)$$

for  $i \geq 0$  and  $k \geq 0$ , in bidegree  $p^i(1, 2p^k - 1)$ , and the exterior classes

$$\sigma\bar{\xi}_k$$

for  $k \geq 1$ , in bidegree  $(1, 2p^k - 2)$ .

We claim that there are differentials

$$(1) \quad d^{p-1}(\gamma_p(\sigma\bar{\tau}_k)) \doteq \sigma\bar{\xi}_{k+1}$$

for each  $k \geq 0$ . Here  $\doteq$  means equality up to multiplication by a unit in  $\mathbb{Z}/p$ .

Using the Hopf algebra structure, differential (1) propagates to differentials

$$d^{p-1}(\gamma_m(\sigma\bar{\tau}_k)) \doteq \gamma_{m-p}(\sigma\bar{\tau}_k)\sigma\bar{\xi}_{k+1}$$

for each  $m \geq p$  and  $k \geq 0$ . By the Künneth theorem, this leaves

$$E_{*,*}^p \cong \mathcal{A}_* \otimes P_p(\sigma\bar{\tau}_k \mid k \geq 0).$$

Here

$$P_p(\sigma x) = \mathbb{Z}/p[\sigma x]/(\sigma x)^p \cong \mathbb{Z}/p\{\gamma_m(\sigma x) \mid 0 \leq m < p\} \subset \Gamma(\sigma x)$$

is the truncated polynomial algebra on  $\sigma x$  of height  $p$ . The  $E^p$ -term is generated over  $\mathcal{A}_*$  by classes in filtration  $s = 1$ , hence there is no room for any further differentials, and  $E^p = E^\infty$ .

Consider the composite

$$H_*(A) \longrightarrow H_*THH(A) \xrightarrow{\sigma} H_{*+1}THH(A).$$

From  $Q^{p^k}(\bar{\tau}_k) = \bar{\tau}_{k+1}$ , and  $|\sigma\bar{\tau}_k| = 2p^k$ , we deduce

$$\sigma\bar{\tau}_{k+1} = \sigma Q^{p^k}(\bar{\tau}_k) = Q^{p^k}(\sigma\bar{\tau}_k) = (\sigma\bar{\tau}_k)^p$$

in  $H_*THH(\mathbb{Z}/p)$ . Hence  $P_p(\sigma\bar{\tau}_k \mid k \geq 0)$  is the associated graded of  $P(\sigma\bar{\tau}_0)$ , and

$$H_*THH(\mathbb{Z}/p) \cong \mathcal{A}_* \otimes P(\sigma\bar{\tau}_0)$$

as  $\mathcal{A}_*$ -algebra.

It follows that  $THH(\mathbb{Z}/p) \simeq \bigvee_{m \geq 0} \Sigma^{2m} H$ , there is a unique class

$$\mu_0 \in \pi_2 THH(\mathbb{Z}/p)$$

with Hurewicz image  $\sigma\bar{\tau}_0$ , and

$$\pi_* THH(\mathbb{Z}/p) \cong P(\mu_0) = \mathbb{Z}/p[\mu_0].$$

It remains to establish differential (1). Here is a quick argument due to Ausoni. From  $\beta(\bar{\tau}_{k+1}) = (\pm)\bar{\xi}_{k+1}$  we deduce

$$\sigma\bar{\xi}_{k+1} = (\pm)\beta(\sigma\bar{\tau}_{k+1}) = \beta((\sigma\bar{\tau}_k)^p) = 0$$

in  $H_*THH(\mathbb{Z}/p)$ . Hence  $\sigma\bar{\xi}_{k+1} \in E_{1,2p^{k+1}-2}^2$  must be a boundary, i.e., the target of a differential. By induction on  $k$ , the only possible source of a differential is  $\gamma_p(\sigma\bar{\tau}_k) \in E_{p,p(2p^k-1)}^2$ .  $\square$



*Proof of Theorem 7.2.* In the case  $R = \mathbb{Z}$  we have  $A = H\mathbb{Z}$ . Hence

$$E_{s,*}^2 \cong HH_s(H_*(H\mathbb{Z})) \implies H_{s+*}THH(\mathbb{Z}).$$

Let  $p$  be odd. From

$$H_*(H\mathbb{Z}) \cong E(\bar{\tau}_k \mid k \geq 1) \otimes P(\bar{\xi}_k \mid k \geq 1)$$

we obtain

$$\begin{aligned} HH_*(H_*(H\mathbb{Z})) &\cong \bigotimes_{k \geq 1} HH_*(E(\bar{\tau}_k)) \otimes \bigotimes_{k \geq 1} HH_*(P(\bar{\xi}_k)) \\ &\cong \bigotimes_{k \geq 1} E(\bar{\tau}_k) \otimes \Gamma(\sigma\bar{\tau}_k) \otimes \bigotimes_{k \geq 1} P(\bar{\xi}_k) \otimes E(\sigma\bar{\xi}_k) \\ &\cong H_*(H\mathbb{Z}) \otimes \Gamma(\sigma\bar{\tau}_k \mid k \geq 1) \otimes E(\sigma\bar{\xi}_k \mid k \geq 1). \end{aligned}$$

The  $E^2$ -term is generated as a graded commutative  $\mathcal{A}_*$ -algebra by the divided powers

$$\gamma_{p^i}(\sigma\bar{\tau}_k)$$

for  $i \geq 0$  and  $k \geq 1$ , in bidegree  $p^i(1, 2p^k - 1)$ , and the exterior classes

$$\sigma\bar{\xi}_k$$

for  $k \geq 1$ , in bidegree  $(1, 2p^k - 2)$ .

By naturality with respect to  $H\mathbb{Z} \rightarrow H$  there are differentials

$$d^{p-1}(\gamma_p(\sigma\bar{\tau}_k)) \doteq \sigma\bar{\xi}_{k+1}$$

for each  $k \geq 1$ . Using the Hopf algebra structure, they propagate to differentials

$$d^{p-1}(\gamma_m(\sigma\bar{\tau}_k)) \doteq \gamma_{m-p}(\sigma\bar{\tau}_k)\sigma\bar{\xi}_{k+1}$$

for each  $m \geq p$  and  $k \geq 1$ . By the Künneth theorem, this leaves

$$E_{*,*}^p \cong H_*(H\mathbb{Z}) \otimes P_p(\sigma\bar{\tau}_k \mid k \geq 1) \otimes E(\sigma\bar{\xi}_1).$$

The  $E^p$ -term is generated over  $H_*(H\mathbb{Z})$  by classes in filtration  $s = 1$ , hence there is no room for any further differentials, and  $E^p = E^\infty$ .

From  $Q^{p^k}(\bar{\tau}_k) = \bar{\tau}_{k+1}$ , or by naturality, we deduce

$$\sigma\bar{\tau}_{k+1} = (\sigma\bar{\tau}_k)^p$$

in  $H_*THH(\mathbb{Z})$ . Hence  $P_p(\sigma\bar{\tau}_k \mid k \geq 1)$  is the associated graded of  $P(\sigma\bar{\tau}_1)$ . Furthermore,  $\sigma\bar{\xi}_1$  is in odd total degree, so  $(\sigma\bar{\xi}_1)^2 = 0$  in  $H_*THH(\mathbb{Z})$ . Hence

$$H_*THH(\mathbb{Z}) \cong H_*(\mathbb{Z}) \otimes P(\sigma\bar{\tau}_1) \otimes E(\sigma\bar{\xi}_1)$$

as  $H_*(H\mathbb{Z})$ -algebra.

It follows that  $THH(\mathbb{Z}) \wedge S/p$  is a generalized Eilenberg–Mac Lane spectrum, there are unique classes

$$\lambda_1 \in \pi_{2p-1}(THH(\mathbb{Z}); \mathbb{Z}/p) \quad \text{and} \quad \mu_1 \in \pi_{2p}(THH(\mathbb{Z}); \mathbb{Z}/p)$$

with Hurewicz images  $\sigma\bar{\xi}_1 \wedge 1$  and  $\sigma\bar{\tau}_1 \wedge 1 + \sigma\bar{\xi}_1 \wedge \epsilon_0$ , respectively. Here  $H_*(S/p) = E(\epsilon_0)$ . Hence

$$\pi_*(THH(\mathbb{Z}); \mathbb{Z}/p) \cong E(\lambda_1) \otimes P(\mu_1) = \Lambda(\lambda_1) \otimes \mathbb{Z}/p[\mu_1].$$

The integral structure of  $\pi_*THH(\mathbb{Z})$  follows by Bockstein spectral sequence arguments.  $\square$

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