ON TRACE MAPS

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October 15th 1998

1. The trace map to Hochschild homology

Let R be a ring, always associative and unital. We can think of R as a $\mathbb{Z}$-algebra.

1.1 Algebraic K-theory. The algebraic K-groups of the ring R are the homotopy groups $K_*(R) = \pi_* K(R)$ of a space $K(R)$, defined as

$$K(R) = \Omega B \left(\coprod_P B \operatorname{Aut}_R(P) \right)$$

where P ranges through the isomorphism classes of finitely generated projective R-modules [Quillen]. Restricting attention to the finitely generated free R-modules, we have

$$K^f(R) = \Omega B \left(\coprod_{n \geq 0} BGL_n(R) \right) .$$

In either case the coproduct is a topological monoid with respect to direct sum of modules and automorphisms of such. For a topological monoid M, the bar construction BM is the simplicial space with q-simplices

$$[q] \mapsto M \times \cdots \times M$$

(q factors). Denoting a typical q-simplex by $[m_1|\dots|m_q]$, the face maps are given by

$$d_i([m_1|\dots|m_q]) = \begin{cases} [m_2|\dots|m_q] & \text{for } i = 0, \\ [m_1|\dots|m_i m_{i+1}|\dots|m_q] & \text{for } 0 < i < q \text{ and} \\ [m_1|\dots|m_{q-1}] & \text{for } i = q, \end{cases}$$

and the degeneracy maps by

$$s_j([m_1|\dots|m_q]) = [m_1|\dots|m_j|1|m_{j+1}|\dots|m_q] .$$

The inclusion of the 1-skeleton $BM^{(1)} \cong \Sigma M \to BM$ has an adjoint map $\iota \colon M \to \Omega BM$, which is a group completion when $\pi = \pi_0(M)$ is central in the Pontryagin algebra $H_*(M)$. Then there is an isomorphism [Segal–McDuff]

$$H_*(M)[\pi^{-1}] \xrightarrow{\cong} H_*(\Omega BM) .$$

When the monoid $\pi_0(M)$ is in fact a group, we say that M is grouplike. In this case the group completion map ι is a homotopy equivalence, hence does not alter the homotopy type of M.

Returning to algebraic K-theory, there is a homotopy equivalence

$$K(R) \simeq K_0(R) \times BGL(R)^+$$

where $BGL(R)^+$ denotes Quillen's plus construction applied to

$$BGL(R) = \operatorname{colim}_n BGL_n(R)\,.$$

The connected components of $K(R)$ and $K^f(R)$ are homotopy equivalent, hence their positive-dimensional homotopy groups are isomorphic.

1.2 Hochschild homology. The Hochschild homology groups of the ring R are the homology groups $HH_*(R) = H_*(Z_*(R))$ of a chain complex $Z_*(R)$ called the Hochschild complex or the cyclic complex of R. This is the natural homology theory for associative and unital algebras, in our case over the ground ring $\mathbb{Z}$. Alternatively the Hochschild homology groups can be thought of as the homotopy groups of a space $HH(R)$, which we now define.

Let $HH(R)_\bullet$ be the simplicial abelian group

$$[q] \mapsto HH(R)_q = R \otimes_{\mathbb{Z}} R \otimes_{\mathbb{Z}} \cdots \otimes_{\mathbb{Z}} R$$

$(q+1$ factors). The face maps are given by

$$d_i(r_0 \otimes r_1 \otimes \cdots \otimes r_q) = \begin{cases} r_0 r_1 \otimes \cdots \otimes r_q & \text{for } i = 0, \\ r_0 \otimes \cdots \otimes r_i r_{i+1} \otimes \cdots \otimes r_q & \text{for } 0 < i < q \text{ and} \\ r_q r_0 \otimes r_1 \otimes \cdots \otimes r_{q-1} & \text{for } i = q, \end{cases}$$

and the degeneracy maps by

$$s_j(r_0 \otimes r_1 \otimes \cdots \otimes r_q) = r_0 \otimes \cdots \otimes r_j \otimes 1 \otimes r_{j+1} \otimes \cdots \otimes r_q\,.$$

The cyclic complex $Z_*(R)$ is the associated chain complex, with $Z_q(R) = HH(R)_q$ and boundary maps

$$\partial_q = \sum_{i=0}^{q} (-1)^i d_i\,.$$

For any simplicial abelian group $A_\bullet$ with associated chain complex C_* there is a natural isomorphism [Moore]

$$\pi_*(|A_\bullet|) \cong H_*(C_*)\,,$$

and the geometric realization $|A_\bullet|$ is a product of Eilenberg–Mac Lane spaces. Hence $HH_*(R) \cong \pi_*(HH(R)) \cong H_*(Z_*(R))$, where we write $HH(R)$ for the geometric realization of the simplicial abelian group $HH(R)_\bullet$.

1.3 Morita invariance. Let $M_n(R)$ be the ring of $n \times n$ matrices with entries in R. There are homotopy equivalences

$$K(R) \simeq K(M_n(R))$$
$$HH(R) \simeq HH(M_n(R))$$

known as Morita invariance for algebraic K-theory and Hochschild homology, respectively. These are symptoms of the fact that both algebraic K-theory and Hochschild homology for a ring R can be defined in terms of the category $\mathcal{P}(R)$ of finitely generated projective R-modules, and there is an equivalence of categories $\mathcal{P}(R) \simeq \mathcal{P}(M_n(R))$. We shall return to this categorical viewpoint later.

In the case of Hochschild homology, the homotopy equivalence is induced in one direction by the ring inclusion $R \cong M_1(R) \to M_n(R)$ taking $r \in R$ to the $n \times n$ matrix with r in the upper left corner and 0's elsewhere. Its homotopy inverse is induced by the multi-trace map

$$HH(M_n(R))_q = M_n(R) \otimes_{\mathbb{Z}} M_n(R) \otimes_{\mathbb{Z}} \cdots \otimes_{\mathbb{Z}} M_n(R) \xrightarrow{mtr} R \otimes_{\mathbb{Z}} R \otimes_{\mathbb{Z}} \cdots \otimes_{\mathbb{Z}} R = HH(R)_q$$

taking a tensor product of $n \times n$ matrices to the trace of their external tensor product:

$$(a^0_{ij}) \otimes (a^1_{ij}) \otimes \cdots \otimes (a^q_{ij}) \mapsto \sum_{i_0,\dots,i_q} a^0_{i_0 i_1} \otimes a^1_{i_1 i_2} \otimes \dots a^q_{i_q i_0} .$$

1.4 Dennis trace map. The Dennis trace map is a natural transformation

$$K^f(R) \xrightarrow{trd} HH(R)$$

from algebraic K-theory to Hochschild homology. It is obtained by group completing the following composite map:

$$\coprod_{n \geq 0} BGL_n(R) \xrightarrow{SI} \coprod_{n \geq 0} HH(M_n(R)) \xrightarrow{\coprod mtr} HH(R) .$$

In simplicial degree q the left hand map is given by

$$SI \colon [g_1|\dots|g_q] \mapsto g_0 \otimes g_1 \otimes \cdots \otimes g_q$$

where $g_0 = (g_1 \dots g_q)^{-1}$. On the left each g_i lies in $GL_n(R)$, while on the right each g_i is considered as an element in $M_n(R)$ by means of the inclusion $S \colon GL_n(R) \subset M_n(R)$. The right hand map is the coproduct of the multitrace maps inducing the equivalences $HH(M_n(R)) \simeq HH(R)$.

Applying group completion we get a map

$$K^f(R) = \Omega B \left(\coprod_{n \geq 0} BGL_n(R) \right) \to \Omega B \left(HH(R) \right) .$$

The Hochschild homology space is a simplicial abelian group, hence grouplike, so the group completion map $\iota \colon HH(R) \to \Omega B \left(HH(R) \right)$ is a homotopy equivalence. Composing with a homotopy inverse we get the Dennis trace map

$$trd \colon K^f(R) \to HH(R) .$$

Unfortunately, the induced map on homotopy $K_*(R) \to HH_*(R)$ is not very interesting. When $R = \mathbb{Z}$ we have $HH_i(\mathbb{Z}) = 0$ for $i > 0$, so the Dennis trace map does not detect anything of $K_i(\mathbb{Z})$ in positive dimensions.

2. Cyclic structure

Connes noted that the simplicial abelian group $HH(R)_\bullet$ has further structure: it is a cyclic object in the category of abelian groups. This determines an action by the circle S^1 on $HH(R)$, and was used by Connes to define the cyclic homology $HC(R)$, periodic homology $HP(R)$ and negative cyclic homology $HN(R) = HC^-(R)$ of the ring R. These are closely related to the S^1-homotopy orbits $HH(R)_{hS^1}$, the S^1-Tate construction $\hat{\mathbb{H}}(S^1, HH(R))$ and the S^1-homotopy fixed points $HH(R)^{hS^1}$ of the S^1-space $HH(R)$, respectively.

2.1 Cyclic sets. Let Δ be the simplex category, with objects the ordered sets

$$[q] = \{0 < 1 < \cdots < q\}$$

for all integers $q \geq 0$, and morphisms $\Delta([p], [q])$ the set of order preserving maps $f\colon [p] \to [q]$. We can alternatively think of $[q]$ as the category $\{0 \to 1 \to \cdots \to q\}$ and $\Delta([p], [q])$ as the set of functors from $[p]$ to $[q]$. The face and degeneracy maps $\delta_i\colon [q-1] \to [q]$ and $\sigma_j\colon [q+1] \to [q]$ for $0 \leq i, j \leq q$ generate the morphisms in Δ, subject to the so-called simplicial identities (which we omit to write down).

A simplicial object $X_\bullet$ in a category $\mathcal{C}$ is a functor $X\colon \Delta^{op} \to \mathcal{C}$ from the opposite category of Δ to $\mathcal{C}$. With $\mathcal{C} = \mathcal{E}ns$ the category of sets (French: ensemble), this is called a simplicial set. With $\mathcal{C} = \mathcal{A}b$ the category of abelian groups, this is a simplicial abelian group. For each $q \geq 0$ the q-simplices X_q in X equal the object $X([q])$ in $\mathcal{C}$. For each $n \geq 0$ there is a represented simplicial set $\Delta^n_\bullet$ with q-simplices $\Delta^n_q = \Delta([q], [n])$, i.e., Δ^n is the functor $[q] \mapsto \Delta([q], [n])$ from Δ^{op} to $\mathcal{E}ns$. Its geometric realization is the standard n-simplex also denoted Δ^n.

Connes' cyclic category Λ is an extension of Δ, adjoining for each $q \geq 0$ an automorphism $\tau_q\colon [q] \to [q]$ of order $(q+1)$, i.e., $\tau_q^{q+1} = 1$. There is an inclusion $\Delta \to \Lambda$ which is the identity on objects. The face and degeneracy maps together with the cyclic maps τ_q generate the morphisms of Λ, subject to the simplicial identities, the relation $\tau_q^{q+1} = 1$, and some additional identities relating τ_q to the face and degeneracy maps (which we again omit to write down).

A cyclic object in a category $\mathcal{C}$ is a functor $X\colon \Lambda^{op} \to \mathcal{C}$ from the opposite category of Λ to $\mathcal{C}$. With $\mathcal{C} = \mathcal{E}ns$ the category of sets, this is called a cyclic set. With $\mathcal{C} = \mathcal{A}b$ the category of abelian groups, this is a cyclic abelian group (not to be confused with the other meaning of this expression).

The simplicial abelian group $HH(R)_\bullet$ is a functor $\Delta^{op} \to \mathcal{A}b$. It admits an extension over Λ^{op}, given by taking $\tau_q\colon [q] \to [q]$ to the homomorphism

$$t_q\colon HH(R)_q \to HH(R)_q$$

of order $(q+1)$ given by cyclically permuting the tensor factors:

$$t_q(r_0 \otimes r_1 \otimes \cdots \otimes r_q) = r_q \otimes r_0 \otimes \cdots \otimes r_{q-1}\,.$$

Hence $HH(R)_\bullet$ is a cyclic abelian group. We assert that this defines an S^1-action on $HH(R)$.

For each $n \geq 0$ there is a represented cyclic set $\Lambda^n_\bullet$ defined by the functor $[q] \mapsto \Lambda([q], [n])$ from Λ^{op} to $\mathcal{E}ns$. Restricting to Δ^{op}, we obtain the underlying simplicial set with q-simplices the set of morphisms $[q] \to [n]$ in Λ. By inspection, its geometric realization has the form:

$$\Lambda^n = |\Lambda^n_\bullet| \cong \Delta^n \times S^1 .$$

Hence there is natural S^1-action on Λ^n given by multiplying in the right hand factor of $\Delta^n \times S^1$ using the group structure on S^1.

Let $X\colon \Lambda^{op} \to \mathcal{E}ns$ be any cyclic set. The geometric realization of its underlying simplicial set is

$$|X| = \coprod_{q\geq 0} X_q \times \Delta^q / \sim_\Delta \cong \coprod_{q\geq 0} X_q \times \Lambda^q / \sim_\Lambda$$

where $\sim_\Delta$ refers to identifications $(f^*(x), \xi) \sim (x, f_*(\xi))$ for $x \in X_q$, $\xi \in \Delta^p$ and all morphisms $f \in \Delta([p],[q])$, while $\sim_\Lambda$ refers to the same identifications for $x \in X_q$, $\xi \in \Lambda^p$ and f in the larger set of morphisms $\Lambda([p],[q])$. The two identification spaces are homeomorphic for categorical reasons. Each space

$$X_q \times \Lambda^q \cong X_q \times \Delta^q \times S^1$$

has an S^1-action by multiplication in the last factor, and these actions are compatible under the identifications $\sim_\Lambda$.

Hence there is a natural S^1-action on the geometric realization of the underlying simplicial set of any cyclic set. Likewise these is an S^1-action on the geometric realization of the underlying simplicial abelian group of any cyclic abelian group, and this applies to $HH(R)$.

2.2 Edgewise subdivision. The preceding description of the S^1-action on cyclic sets can be made much more explicit and precise, if we are willing to restrict the group action to a finite cyclic subgroup $C_r \subset S^1$ of order r.

Let $sd_r\colon \Lambda \to \Lambda$ be the functor given on objects by

$$sd_r([q]) = [r(q+1)-1] \cong [q] \coprod \cdots \coprod [q]$$

(r disjoint summands), and on a morphism $f\colon [p] \to [q]$ by

$$sd_r(f) = f \coprod \cdots \coprod f\colon [r(p+1)-1] \to [r(q+1)-1] .$$

Precomposition by sd_r defines an action on cyclic objects in any category, taking $X\colon \Lambda^{op} \to \mathcal{C}$ to $sd_r(X) = X \circ sd_r\colon \Lambda^{op} \to \mathcal{C}$. We call $sd_r(X)$ the r-fold edgewise subdivision of X.

We note three facts:

(1) There is a (non-simplicial) homeomorphism

$$|sd_r(X)| \xrightarrow{\cong} |X|$$

taking a q-simplex $\{x\} \times \Delta^q$ in $|sd_r(X)|$ with $x \in sd_r(X)_q = X_{r(q+1)-1}$ into the $r(q+1)-1$-simplex $\{x\} \times \Delta^{r(q+1)-1}$ in $|X|$ by a suitable diagonal embedding

$$\Delta^q \to \Delta^q * \cdots * \Delta^q \cong \Delta^{r(q+1)-1} .$$

(2) There is a simplicial C_r-action on $sd_r(X)_\bullet$, given on q-simplices $sd_r(X)_q = X_{r(q+1)-1}$ by the action induced by $\tau_{r(q+1)-1}^{q+1}$, which has order r.

(3) The induced C_r-action on the geometric realization $|sd_r(X)|$ is compatible with the restriction of the natural S^1-action on $|X|$ to $C_r \subset S^1$, via the homeomorphism $|sd_r(X)| \cong |X|$.

This allows us the describe the restriction of the S^1-action on $HH(R)$ to each finite subgroup $C_r \subset S^1$ in a simplicial manner, by replacing $HH(R)$ with its r-fold edgewise subdivision $sd_r(HH(R))$, and using the simplicial C_r-action given by the map $t_{r(q+1)-1}^{q+1}$ cyclically shifting the $r(q+1)$ tensor factors in $sd_r(HH(R))_q$ by $(q+1)$ positions to the right. A similar description will apply for related objects similar to $HH(R)$.

2.3 The Goodwillie trace map. The negative cyclic homology $HC^-(R)$ is defined as the homology of a bicomplex related to the homotopy fixed point space

$$HH(R)^{hS^1} = \mathrm{Map}(ES^1_+, HH(R))^{S^1}$$

for the S^1-action on $HH(R)$. There is a natural map $HC^-(R) \to HH(R)$ related to the forgetful map

$$\mathrm{Map}(ES^1_+, HH(R))^{S^1} \subset \mathrm{Map}(ES^1_+, HH(R)) \simeq HH(R)$$

and Goodwillie constructed a natural factorization

$$K^f(R) \xrightarrow{trg} HC^-(R) \to HH(R)$$

of the Dennis trace map $trd\colon K^f(R) \to HH(R)$ through negative cyclic homology.

This map takes the S^1-action on $HH(R)$ into account, and turns out to carry much more precise information about rational algebraic K-theory than the Dennis trace map:

Consider a nilpotent extension $\phi\colon R \to S$ of algebras over $\mathbb{Q}$. This means we have a surjective ring homomorphism ϕ of $\mathbb{Q}$-algebras whose kernel $I = \ker(\phi)$ is a nilpotent ideal in S, i.e., $I^n = 0$ for some n. Then the Goodwillie trace map $trg\colon K(R) \to HC^-(R)$ induces a rational homotopy equivalence

$$\mathrm{hofib}(\phi\colon K(R) \to K(S)) \xrightarrow{trg} \mathrm{hofib}(\phi\colon HC^-(R) \to HC^-(S)),$$

i.e., the rationalized relative algebraic K-groups $K_*(R \to S) \otimes \mathbb{Q}$ and the relative negative cyclic homology groups $HC^-_*(R \to S)$ are isomorphic. Equivalently the commutative square diagram

$$\begin{array}{ccc} K(R) & \xrightarrow{\phi} & K(S) \\ \downarrow{\scriptstyle trg} & & \downarrow{\scriptstyle trg} \\ HC^-(R) & \xrightarrow{\phi} & HC^-(S) \end{array}$$

becomes homotopy cartesian after rationalization.