ALGEBRAIC K-THEORY,
SOME APPLICATIONS AND COMPUTATIONS

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Topological K-theory

We begin by recalling the topological K-theory of Atiyah and Hirzebruch, to motivate the definition of algebraic K-theory.

Let X be a compact topological space, and consider a vector bundle $p\colon E \to X$ over X. This is a family of complex, finite dimensional vector spaces $E_x = p^{-1}(x)$ for $x \in X$ which is locally trivial, *i.e.* X can be covered by open $U \subset X$ where we have fibrewise linear trivializations $\theta\colon p^{-1}(U) \to U \times V$. Here $V = E_x$ for some $x \in U$, and θ is a homeomorphism commuting with the projections to U.

$$\begin{tikzcd} p^{-1}(U) \arrow[r, "\theta", "\cong"'] \arrow[dr, "p"'] & U \times V \arrow[d, "\mathrm{proj}"] \\ & U \end{tikzcd}$$

We can embed E in $\mathbb{C}^\infty$ since X is compact, *e.g.* map $f\colon E \hookrightarrow X \times \mathbb{C}^\infty$ in a manner respecting the projections to X

$$\begin{tikzcd} E \arrow[r, hook] \arrow[dr, "p"'] & X \times \mathbb{C}^\infty \arrow[d, "\mathrm{proj}"] \\ & X \end{tikzcd}$$

We think of E as describing a family of subspaces $f(E_x) \subset \mathbb{C}^\infty$ for all $x \in X$. This amounts to a map $X \to \mathrm{Grass}(\mathbb{C}^\infty)$ where $\mathrm{Grass}(\mathbb{C}^\infty)$ is the space of finite dimensional sub vector spaces of $\mathbb{C}^\infty$. It is the disjoint union of the subspaces $\mathrm{Grass}_n(\mathbb{C}^\infty)$ of n-dimensional sub vector spaces of $\mathbb{C}^\infty$, over all $n \geq 0$.

We get a bijection

$$\begin{tikzcd} \mathrm{Vect}(X) = \{\text{isomorphism classes of vector bundles over } X\} \arrow[d, "\cong"] \\ {[X, \coprod_{n\geq 0} \mathrm{Grass}_n(\mathbb{C}^\infty)]} = \{\text{homotopy classes of maps } X \to \coprod_{n\geq 0} \mathrm{Grass}_n(\mathbb{C}^\infty)\} \end{tikzcd}$$

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We think of the disjoint union as indexed over the isomorphism classes of finite dimensional sub vector spaces of $\mathbb{C}^\infty$.

There is a fibration sequence

$$GL(\mathbb{C}^n) \to \mathrm{Frames}_n(\mathbb{C}^\infty) \to \mathrm{Grass}_n(\mathbb{C}^\infty)$$

with fiber $GL(\mathbb{C}^n)$, where $\mathrm{Frames}_n(\mathbb{C}^\infty)$ is the space of n-frames in $\mathbb{C}^\infty$, *i.e.* n linearly independent vectors $(v_1, \dots, v_n)$ in $\mathbb{C}^\infty$. The projection map is given by

$$(v_1, \dots, v_n) \longmapsto \mathrm{lin}\{v_1, \dots, v_n\}$$

taking a frame to the sub vector space it spans.

$\mathrm{Frames}_n(\mathbb{C}^\infty)$ is (weakly) contractible, so by the Puppe sequence there is a homotopy equivalence $\Omega\,\mathrm{Grass}_n(\mathbb{C}^\infty) \simeq GL(\mathbb{C}^n)$. Here ΩX denotes the loop space of X. We write $\mathrm{Grass}_n(\mathbb{C}^\infty) \simeq BGL(\mathbb{C}^n)$, where $B(-)$ denotes a delooping of G, thought of as the inverse operation to $\Omega(-)$, when one exists.

$$\mathrm{Vect}(X) \cong [X, \coprod_{\mathbb{C}^n} BGL(\mathbb{C}^n)].$$

More generally, when M is a topological monoid (semigroup), we can naturally construct a sequence

$$M \longrightarrow EM \longrightarrow BM$$

with $EM \simeq *$. When $M = G$ is a topological group, this is a principal fibration sequence, and $\Omega BG \simeq G$. In general we only get a map $M \to \Omega BM$. Now ΩBG is a group up to homotopy, because reversing orientation of loops provides an inverse. So the map $M \to \Omega BM$ provides a map from a monoid to a group, up to homotopy.

$\mathrm{Vect}(X)$ is an abelian monoid under Whitney sum of vector bundles. $E \oplus F \to X$ has fibres $(E \oplus F)_x = E_x \oplus F_x$. On the representing space, this pairing is induced by a map

$$\coprod_{\mathbb{C}^n} BGL(\mathbb{C}^n) \times \coprod_{\mathbb{C}^n} BGL(\mathbb{C}^n) \xrightarrow{\oplus} \coprod_{\mathbb{C}^n} BGL(\mathbb{C}^n)$$

making $\coprod_{\mathbb{C}^n} BGL(\mathbb{C}^n)$ a topological monoid.

We desire to consider formal or virtual vector bundles, *i.e.* expressions involving signed sums of vector bundles. This idea was used by Grothendieck to prove a Riemann–Roch theorem, and is necessary in index theory to describe invariant characteristic classes by formulae in vector bundles.

One way to achieve this is the "group completion" map $M \to \Omega BM$.

Definition.

$$K(X) = [X, \Omega B(\coprod_{\mathbb{C}^n} BGL(\mathbb{C}^n))]$$

is the initial abelian group receiving a monoid map from $\mathrm{Vect}(X)$, called the *topological K-theory* of the space X.

One can prove that

$$\Omega B(\coprod_{\mathbb{C}^n} BGL(\mathbb{C}^n)) \simeq \mathbb{Z} \times \mathrm{colim}_n\, BGL(\mathbb{C}^n) = \mathbb{Z} \times BGL(\mathbb{C}^\infty) \simeq \mathbb{Z} \times BU.$$

Here $U = \mathrm{colim}_n\, U(n)$ is the infinite unitary group, with $U(n) \subset GL(\mathbb{C}^n)$ the maximal compact subgroup. $GL(\mathbb{C}^n)/U(n) \cong \mathbb{R}^N$ for some N, so the inclusion $U(n) \hookrightarrow GL(\mathbb{C}^n)$ is a homotopy equivalence.

We now say that the space $\mathbb{Z} \times BU$ represents $K(X)$, in view of the definition above.

Algebraic K-theory

Next we want to consider the analog of $K(X)$ for arithmetic bundles, with applications to number theory, and to a sort of manifold bundles, for use in geometric topology.

The space X played the role of a test space or dummy variable in the previous section, and we choose to focus our interest on the representing space instead

$$K = \Omega B(\coprod_{\mathbb{C}^n} BGL(\mathbb{C}^n)) \simeq \mathbb{Z} \times BU.$$

Let R be an associative ring with unit. We let the representing space for algebraic K-theory be

$$K(R) = \Omega B(\coprod_P B\,\mathrm{Aut}(P))$$

where P runs over the isomorphism classes av finitely generated projective R-modules, and $\mathrm{Aut}(P)$ is the group of R-isomorphisms $P \to P$. The projective R-modules are precisely the direct summands of free R-modules, and serve to generalize the sub vector spaces of $\mathbb{C}^\infty$ used in defining topological K-theory.

$K(R)$ is by definition a loop space, so its homotopy groups $\pi_i K(R)$ are groups for $i \geq 0$. These were defined by Quillen to be the (higher) algebraic K-groups of the ring R.

Definition. The ith *algebraic K-group* of the ring R is

$$K_i(R) = \pi_i K(R) = [S^i, K(R)] = [S^i, \Omega B(\coprod_P B\,\mathrm{Aut}(P))],$$

where S^i is the i-sphere.

Number theory

Algebraic K-theory is linked to algebraic number theory by the following formulas.

When $R = \mathcal{O}_F$ is the ring of integers in a number field F ($\mathbb{Q} \subseteq F \subset \mathbb{C}$)

$$K_0(\mathcal{O}_F) \cong \mathbb{Z} \oplus \mathrm{Cl}(F) \qquad \text{and} \qquad K_1(\mathcal{O}_F) \cong \mathcal{O}_F^\times$$

where $\mathrm{Cl}(F)$ is the ideal class group of F, *i.e.* the group of fractional ideals ($\mathcal{O}_F$-modules $m \subset F$ satisfying $b \cdot m \subseteq \mathcal{O}_F$ for some $b \neq 0$ in $\mathcal{O}_F$), modulo the subgroup of principal fractional ideals ($\mathcal{O}_F$-modules $m \subset F$ of the form $a/b \cdot \mathcal{O}_F$ for some $a/b \neq 0$ in F).

This group $\mathrm{Cl}(F)$ is always finite, and its order is called the class number of F.

More generally, Quillen proved that $K_i(\mathcal{O}_F)$ is finitely generated for all i, and Borel computed the rank of $K_i(\mathcal{O}_F) \otimes \mathbb{Q}$ for all i in terms of the number of real and complex embeddings of the number field F. In the principal example $F = \mathbb{Q}$ with $\mathcal{O}_F = \mathbb{Z}$, we have

$$K_i(\mathbb{Z}) \otimes \mathbb{Q} \cong \begin{cases} \mathbb{Q} & \text{if } i = 0 \text{ or } i \equiv 1 \mod 4,\ i \neq 1, \\ 0 & \text{otherwise.} \end{cases}$$

There are conjectures by Lichtenbaum and Quillen, extended by Dwyer and Friedlander, on the precise values of the groups $K_i(\mathcal{O}_F)$ for any number field F. In the case $\mathcal{O}_F = \mathbb{Z}$ the order of the torsion in $K_i(\mathbb{Z})$ can be expressed in terms of numerators and denominators of Bernoulli numbers, and generally by values of zeta functions at negative integers.

When $F = \mathbb{Q}(\xi_p)$ with p a prime and ξ_p a primitive pth root of unity, the order of $\mathrm{Cl}(\mathbb{Q}(\xi_p))$ is denoted h_p.

p is called a regular prime if $p \not| h_p$, and Kummer proved Fermat's last theorem for such prime exponents, *i.e.* that there are no positive integers x, y and z with $x^p + y^p = z^p$.

h_p factors as $h_p^+ \cdot h_p^-$, where h_p^+ is the class number of the maximal real subfield of $\mathbb{Q}(\xi_p)$, which is $\mathbb{Q}(\xi_p + \xi_p^{-1})$, and h_p^- is called the relative class number.

A classical unresolved conjecture of Kummer and Vandiver asserts that $p \not| h_p^+$ for all p, *i.e.* that for irregular primes p, even if p divides h_p it only divides the relative part h_p^-. This conjecture is known to hold for $p < 125.000$, but this is not considered sufficient to believe the conjecture on statistical grounds.

The Galois group $G = \mathrm{Gal}(\mathbb{Q}(\xi_p)/\mathbb{Q}) \cong \mathbb{Z}/(p-1)$ acts on $\mathrm{Cl}(\mathbb{Q}(\xi_p))$, and there are $(p-1)$ natural idempotents in $\mathbb{Z}_p[G]$ acting on the p-Sylow subgroup A of $\mathrm{Cl}(\mathbb{Q}(\xi_p))$, inducing a splitting

$$A \cong \sum_{0 \leq i < p-1} A^{[i]}$$

of the p-part of the ideal class group into certain eigenspaces. Here $\mathbb{Z}_p$ denotes the p-adic integers. $A^{[0]} = A^{[1]} = 0$ in general.

A refined version of the Kummer–Vandiver conjecture then asserts that all the even eigenspaces are trivial, *i.e.* $A^{[i]} = 0$ for i even. This was shown by Iwasawa to imply that the odd eigenspaces are cyclic groups, of specific orders.

Versions of the follwing theorem have been proved by Bloch–Kato, Kurihara, Banaszak–Gajda and Kolster.

Theorem. *The Kummer–Vandiver conjecture holds for p if there is no p-torsion in $K_{4i}(\mathbb{Z})$ for every $i \geq 1$.*

For p odd, the Kummer–Vandiver conjecture holds for p if there is no p-torsion in $K_4(\mathbb{Z}(\xi_p + \xi_p^{-1}))$.

The Lichtenbaum–Quillen conjecture asserts in particular that $K_{4i}(\mathbb{Z}) = 0$ for all $i > 0$, and hence implies the Kummer–Vandiver conjecture for all primes p.

The theorem $K_4(\mathbb{Z}) = 0$ then suffices to prove that the eigenspace $A^{[p-3]}$ is trivial and that $A^{[3]}$ is cyclic, for all p. This gives relatively little information on the Kummer–Vandiver conjecture for a general prime p, however.

Differentiable manifolds

By the work of Waldhausen there are applications of algebraic K-theory to the study of the topological group of diffeomorphisms of a smooth manifold.

Let X be a topological space. We view the loop space ΩX as a group up to homotopy under multiplication by composition of loops. We can form a ring up to homotopy

$$Q(\Omega X_+) = \operatorname{colim}_n \Omega^n(S^n \wedge \Omega X_+) = \operatorname{colim}_n \operatorname{Map}(S^n, S^n \wedge \Omega X_+)$$

with addition from loop sum in Ω^n, and product from the group up to homotopy ΩX. We think of this as a kind of group ring, and there is a natural map

$$Q(\Omega X_+) \longrightarrow \mathbb{Z}[\pi_1(X)]$$

to the genuine group ring on the fundamental group of X.

We cannot quite use our definition of algebraic K-theory to make sense of $K(Q(\Omega X_+))$, but there is an alternative definition making this precise, due to Waldhausen. It is called the algebraic K-theory of topological spaces, or A-theory

$$A(X) = \text{“}K(Q(\Omega X_+))\text{”} \xrightarrow{l} K(\mathbb{Z}[\pi_1(X)]),$$

and comes equipped with a “linearization” map to algebraic K-theory. The induced map on homotopy groups is a natural transformation from $\pi_i A(X)$ to $K_i(\mathbb{Z}[\pi_1(X)])$.

By work of Farell and Hsiang, this linearization map is a rational equivalence, *i.e.* induces an isomorphism on $\pi_i(-) \otimes \mathbb{Q}$. So for a disc D^n

$$\pi_i A(D^n) \otimes \mathbb{Q} \cong \begin{cases} \mathbb{Q} & \text{if } i = 0 \text{ or } i \equiv 1 \mod 4,\ i \neq 1, \\ 0 & \text{otherwise.} \end{cases}$$

Waldhausen proved the following splitting theorem when M is a smooth manifold

$$A(M) \simeq Q(M_+) \times \mathrm{Wh}^{\mathrm{Diff}}(M),$$

where $\Omega^2 \mathrm{Wh}^{\mathrm{Diff}}(M) \simeq \mathcal{P}(M)$ is the stable smooth pseudoisotopy (concordance) space.

The pseudoisotopy space $P(M)$ is the space of diffeomorphisms of $M \times I$ leaving the subspace of the boundary $M \times \{0\} \cup \partial M \times I$ fixed. Here $I = [0, 1]$.

There is a stabilization map $P(M) \to \mathcal{P}(M)$ which induces an isomorphism on homotopy groups in a stable range, *e.g.* on π_i when $i \ll \dim(M)/3$.

For example $\pi_i P(D^n) \otimes \mathbb{Q} \cong \mathbb{Q}$ when $i \equiv 3 \mod 4$ and $i \ll n/3$. To see this, note that in this range $\pi_i P(D^n) \cong \pi_i \mathcal{P}(D^n) \cong \pi_{i+2} \mathrm{Wh}^{\mathrm{Diff}}(D^n) \cong \pi_{i+2} A(D^n)$, which rationally is the same as $K_{i+2}(\mathbb{Z})$ as given by Borel's calculation.

There is a map

$$P(M) \to \mathrm{Diff}(M, \partial M)$$

to the topological group of diffeomorphisms of M leaving the boundary fixed, given by restricting a pseudoisotopy to $(M \times \{1\}, \partial M \times \{1\})$. With suitable models for these spaces, this is a fibration with fiber the space of diffeormorphisms of $M \times I$ relative to the boundary.

Consider the case $M = D^n$. We get a fibration sequence

$$\mathrm{Diff}(D^n \times I, \partial) \to P(D^n) \to \mathrm{Diff}(D^n, \partial),$$

and a long exact sequence of homotopy groups

$$\pi_i \,\mathrm{Diff}(D^n \times I, \partial) \to \pi_i P(D^n) \to \pi_i \,\mathrm{Diff}(D^n, \partial) \xrightarrow{\partial} \pi_{i-1} \,\mathrm{Diff}(D^n \times I, \partial).$$

Here ∂ denotes the boundary of the relevant manifold, or the connecting homomorphism, and $D^n \times I \cong D^{n+1}$.

In particular either $\pi_i \,\mathrm{Diff}(D^n, \partial) \otimes \mathbb{Q}$ or $\pi_i \,\mathrm{Diff}(D^n \times I, \partial) \otimes \mathbb{Q}$ must be nontrivial when $i \equiv 3 \mod 4$. Further work yields :

Theorem (Farrell–Hsiang–Jahren). *Let $i \ll n/3$ be in the pseudoisotopy stable range. Then*

$$\pi_i \,\mathrm{Diff}(D^n, \partial) \otimes \mathbb{Q} \cong \begin{cases} \mathbb{Q} & \textit{when } n \textit{ is odd and } i \equiv 3 \mod 4, \\ 0 & \textit{otherwise.} \end{cases}$$

$$\pi_i \,\mathrm{Diff}(S^n) \otimes \mathbb{Q} \cong \begin{cases} \mathbb{Q} \oplus \mathbb{Q} & \textit{when } n \textit{ is odd and } i \equiv 3 \mod 4, \\ \mathbb{Q} & \textit{when } n \textit{ is even and } i \equiv 3 \mod 4, \\ 0 & \textit{otherwise.} \end{cases}$$

More generally one can use algebraic K-theory to get information about

$$\pi_i \,\mathrm{Diff}(M, \partial M) \otimes \mathbb{Q}$$

when M is *e.g.* a smooth manifold with contractible universal cover, and $\dim(M)$ large with respect to i.

Computations

We now turn to the question of determining torsion information about $K_i(R)$.

Theorem.

$$K_*(\mathbb{Z}) \cong (\mathbb{Z}, \mathbb{Z}/2, \mathbb{Z}/2, \mathbb{Z}/48, 0, \mathbb{Z} \oplus \textit{torsion}, \dots).$$

Here the complete calculation of $K_2(\mathbb{Z})$ is due to Milnor, $K_3(\mathbb{Z})$ to Lee and Sczcarba, $K_4(\mathbb{Z})$ to the author, and the rational information on $K_5(\mathbb{Z})$ is due to Borel. It should be noted that Lee and Sczcarba had determined $K_4(\mathbb{Z})$ up to 2– and 3–torsion.

The algebraic K-groups $K_i(\mathbb{Z})$ are, as we have seen, the homotopy groups of the loop space $K(\mathbb{Z})$. In fact $K(\mathbb{Z})$ is an infinite loop space, in the sense that we can construct spaces $B^n K(\mathbb{Z})$ for every $n \geq 0$ with $\Omega^n B^n K(\mathbb{Z}) \simeq K(\mathbb{Z})$. This sequence of spaces $\{B^n K(\mathbb{Z})\}$ assembles into a spectrum in the sense of algebraic topology, which we denote $\mathbf{K}(\mathbb{Z})$. We think of $\mathbf{K}(\mathbb{Z})$ as a space with additional structure analogous to a topological abelian group up to homotopy.

We can now construct a rank filtration of spectra

$$* \simeq F_0\mathbf{K}(R) \to F_1\mathbf{K}(R) \to \dots \to F_k\mathbf{K}(R) \to \dots \to \mathbf{K}(R)$$

for quite general rings R, where $F_k\mathbf{K}(R)$ is a subspectrum of $\mathbf{K}(R)$ built using only R-modules of rank less than or equal to k.

Next we recover the filtration subquotient spectra as

$$F_k\mathbf{K}(R)/F_{k-1}\mathbf{K}(R) \simeq EGL_k(R)_+ \wedge_{GL_k(R)} \mathbf{D}(R^k)$$

where $\mathbf{D}(R^k)$ is a suspension spectrum with $GL_k(R)$-action called a stable building. This is combinatorially defined in terms of cubical diagrams of R-modules.

From the model for the stable building we obtain a spectral sequence

$$E^1_{**} \Longrightarrow H^{spec}_*(F_k\mathbf{K}(R)/F_{k-1}\mathbf{K}(R))$$

going from an E^1-term given by group homology of certain parabolic subgroups of $GL_k(R)$, with coefficients twisted by permutation representations. The target is the spectrum homology of the kth subquotient spectrum of the rank filtration.

In the case $R = \mathbb{Z}$ we can compute these spectral sequences for $k \leq 3$, and resolve the extension problems, to find

$$H^{spec}_*(F_3\mathbf{K}(\mathbb{Z})) \cong (\mathbb{Z}, 0, 0, \mathbb{Z}/2, 0, \mathbb{Z} \oplus X, \dots).$$

Here X is a group of order dividing four. These computations rely on Soulé's determination of the group homology of $SL_3(\mathbb{Z})$.

Further we can prove that for R a principal ideal domain, the stable building $\mathbf{D}(R^k)$ is at least $(k-2)$-connected, and at least $(k-1)$-connected when $k \geq 2$. Finally we can show that $\mathbf{D}(R^4)$ is at least 4-connected.

Since the reduced Borel construction $EGL_k(R)_+ \wedge_{GL_k(R)} (-)$ preserves connectivity, we find $F_3\mathbf{K}(\mathbb{Z}) \to \mathbf{K}(\mathbb{Z})$ is at least 4–connected. Hence

$$H^{spec}_*(\mathbf{K}(\mathbb{Z})) \cong (\mathbb{Z}, 0, 0, \mathbb{Z}/2, 0, \mathbb{Z}\oplus ?, \dots).$$

Now there is an Atiyah–Hirzebruch spectral sequence for stable homotopy theory, viewed as a generalized homology theory :

$$E^2_{s,t} = H^{spec}_s(\mathbf{K}(\mathbb{Z}); \pi^S_t) \Longrightarrow \pi_{s+t}\mathbf{K}(\mathbb{Z}) = K_{s+t}(\mathbb{Z}).$$

It appears as follows in low degrees.

6	$\mathbb{Z}/2$	0	0	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/2\oplus ?$
5	0	0	0	0	0	0
4	0	0	0	0	0	0
3	$\mathbb{Z}/24$	0	0	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/24\oplus ?$
2	$\mathbb{Z}/2$	0	0	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/2\oplus ?$
1	$\mathbb{Z}/2$	0	0	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/2\oplus ?$
0	$\mathbb{Z}$	0	0	$\mathbb{Z}/2$	0	$\mathbb{Z}\oplus ?$
	0	1	2	3	4	5

By comparison with a model for étale K-theory at two, considered by Bökstedt, we can prove that there is a nontrivial extension in $K_3(\mathbb{Z})$

$$\mathbb{Z}/24 \to \mathbb{Z}/48 \to \mathbb{Z}/2$$

and that the first nontrivial differential is the surjection

$$d^2_{5,0} \colon E^2_{5,0} \cong \mathbb{Z} \oplus ? \to E^2_{3,1} \cong \mathbb{Z}/2.$$

This proves the theorem

$$K_*(\mathbb{Z}) \cong (\mathbb{Z}, \mathbb{Z}/2, \mathbb{Z}/2, \mathbb{Z}/48, 0, \mathbb{Z} \oplus \text{torsion}, \dots).$$