UNSTABLE MODULES AND SULLIVAN'S CONJECTURE

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Steenrod operations. We consider mod p cohomology, and let $q = 2p-2$. Steenrod constructed natural transformations (operations)

$$Sq^i \colon H^n(X;\mathbb{F}_2) \to H^{n+i}(X;\mathbb{F}_2)$$

(Steenrod squares) for $p = 2$, and

$$P^i \colon H^n(X;\mathbb{F}_p) \to H^{n+qi}(X;\mathbb{F}_p)$$

(reduced powers) for odd primes p. Together with the mod p Bockstein operation $\beta = \beta_p$ associated to the short exact sequence $0 \to \mathbb{F}_p \to \mathbb{Z}/p^2 \to \mathbb{F}_p \to 0$, these generate all the stable cohomology operations in mod p cohomology, i.e., all those operations that commute with suspensions.

We have $Sq^0 = 1$ and $Sq^1 = \beta$ when $p = 2$, while $P^0 = 1$ for p odd.

The Steenrod algebra A. The mod 2 Steenrod algebra $A = A(2)$ is the (associative unital) graded $\mathbb{F}_2$-algebra generated by elements Sq^i for $i > 0$, modulo the two-sided ideal generated by the Adem relations

$$Sq^a Sq^b = \sum_{i=0}^{[a/2]} \binom{b-i-1}{a-2i} Sq^{a+b-i} Sq^i$$

for all $a, b > 0$ such that $a < 2b$. Here Sq^0 is equal to the unit 1. The generator Sq^i has degree i.

For odd primes p, the mod p Steenrod algebra $A = A(p)$ is the (associative unital) $\mathbb{F}_p$-algebra generated by elements β and P^i for $i > 0$, modulo the two-sided ideal generated by $\beta^2 = 0$ and the Adem relations

$$P^a P^b = \sum_{i=0}^{[a/p]} (-1)^{a+i} \binom{(p-1)(b-i)-1}{a-pi} P^{a+b-i} P^i$$

for all $a, b > 0$ such that $a < pb$, and

$$P^a \beta P^b = \sum_{i=0}^{[a/p]} (-1)^{a+i} \binom{(p-1)(b-i)}{a-pi} \beta P^{a+b-i} P^i + \sum_{i=0}^{(a-1)/p} (-1)^{a+i-1} \binom{(p-1)(b-i)-1}{a-pi-1} P^{a+b-i} \beta P^i$$

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for all $a, b > 0$ such that $a \leq pb$. Here P^0 is equal to the unit 1. The generator β (Bockstein) has degree 1, and the generator P^i has degree $qi = 2i(p-1)$.

The classes Sq^{2^s} for $s \geq 0$ generate $A(2)$ as an algebra, and the classes β and P^{p^s} for $s \geq 0$ generate $A(p)$ as an algebra.

Let $p = 2$. For a sequence of natural numbers $I = (i_1, \dots, i_n)$ let

$$Sq^I = Sq^{i_1} \dots Sq^{i_n} \in A(2)\,.$$

The sequence I is *admissible* if $i_s \geq 2i_{s+1}$ for all $s \geq 1$. The set of *admissible monomials* Sq^I form a basis for $A(2)$ as a vector space.

Let p be odd. For a sequence of integers $I = (\epsilon_0, i_1, \dots, i_n, \epsilon_n)$ where the ϵ_s are 0 or 1 and the i_s are positive, let

$$P^I = \beta^{\epsilon_0} P^{i_1} \beta^{\epsilon_1} \dots P^{i_n} \beta^{\epsilon_n} \in A(p)\,.$$

The sequence I is *admissible* if $i_s \geq pi_{s+1} + \epsilon_s$ for all $s \geq 1$. The *admissible monomials* P^I form a basis for $A(p)$ as a vector space.

The mod p cohomology $H^*(X; \mathbb{F}_p)$ of any space or spectrum X is a graded left $A(p)$-module, with Sq^I (resp. P^I) acting by the composite of Steenrod's operations with the same name. Hereafter we write $H^*(X)$ for $H^*(X; \mathbb{F}_p)$ and A for $A(p)$.

The category $\mathcal{U}$ of unstable A-modules. The mod p cohomology of a space X satisfies a further instability condition. For $p = 2$ it is $Sq^i(x) = 0$ when $i > \deg(x)$, for p odd it is $\beta^\epsilon P^i(x) = 0$ when $\epsilon + 2i > \deg(x)$.

Definition. A graded A-module M is *unstable* if it satisfies the instability condition above. Let the category $\mathcal{U}$ of unstable A-modules be the full subcategory of the category of graded A-modules, with objects the unstable modules.

The cohomology of a space is an unstable A-module. The cohomology of a spectrum is generally not unstable.

The Hopf algebra structure. The cohomology of a space X is a graded commutative unital $\mathbb{F}_p$-algebra, with respect to the cup product $xy = x \cup y$. The product and A-module structure are related: For any $x, y \in H^*(X)$ we have the Cartan formulas:

$$Sq^k(xy) = \sum_{i+j=k} Sq^i(x) Sq^j(y)$$

for $p = 2$ and

$$P^k(xy) = \sum_{i+j=k} P^i(x) P^j(y)$$

$$\beta(xy) = \beta(x)y + (-1)^{\deg(x)} x\beta(y)$$

for p odd. Milnor showed that the coproduct homomorphism $\Delta\colon A \to A \otimes A$ defined by

$$\Delta(Sq^k) = \sum_{i+j=k} Sq^i \otimes Sq^j$$

for $p = 2$ and

$$\Delta(P^k) = \sum_{i+j=k} P^i \otimes P^j$$
$$\Delta(\beta) = \beta \otimes 1 + 1 \otimes \beta$$

makes A a Hopf algebra. This lets us define an A-module structure on the tensor product of two A-modules (over $\mathbb{F}_p$), and the Cartan formulas assert that the cup product

$$H^*(X) \otimes H^*(X) \xrightarrow{\cup} H^*(X)$$

is A-linear. Thus the cohomology $H^*(X)$ of any space is an A-algebra.

The category $\mathcal{K}$ of unstable A-algebras.

The mod p cohomology of a space satisfies a further instability condition. For $p = 2$ it is

$$Sq^i(x) = x^2 \qquad \text{for } i = \deg(x),$$

and for p odd it is

$$P^i(x) = x^p \qquad \text{for } 2i = \deg(x).$$

Definition. A commutative unital graded A-algebra K is *unstable* if it satisfies the instability condition above. Let the category $\mathcal{K}$ of unstable A-algebras be the full subcategory of the category of graded A-algebras, with objects the unstable algebras.

The cohomology of a space is an unstable A-algebra.

A basic example. Let $V = (\mathbb{Z}/p)^d$ be an elementary abelian p-group, i.e., a finite dimensional $\mathbb{F}_p$-vector space. The mod p cohomology of the classifying space BV is also the group cohomology of V, and denoted $H^*(V)$.

By the Künneth theorem, $H^*(V) \cong H^*(\mathbb{Z}/p)^{\otimes d}$.

When $p = 2$, $H^*(\mathbb{Z}/2) \cong \mathbb{Z}/2[x]$ is polynomial on a generator x of degree 1. The A-module structure is given by

$$Sq^i(x^n) = \binom{n}{i} x^{n+i}\,.$$

In general $H^*(V) \cong S(V^*)$ is the symmetric algebra on the dual of V in degree 1.

When p is odd, $H^*(\mathbb{Z}/p) \cong \mathbb{F}_p[x, y]/(x^2 = 0)$ is the tensor product of an exterior algebra on a generator x of degree 1 and a polynomial algebra on a generator y of degree 2. The A-module structure is given by $\beta(x) = y$ and

$$P^i(y^n) = \binom{n}{i} y^{n+i(p-1)}\,.$$

In general $H^*(V) \cong E(V^*) \otimes S(V^*)$ is the exterior algebra on the dual of V in degree 1, tensored with the symmetric algebra on the dual of V in degree 2.

An A-algebra homomorphism from $H^*(V)$ is determined by its behavior in degree 1. Thus the map

$$\mathrm{Hom}(V, W) \to \mathrm{Hom}_{\mathcal{K}}(H^*(W), H^*(V))$$

taking $f\colon V \to W$ to f^* is a bijection.

More generally a map of spaces $f\colon X \to Y$ induces a function

$$f^*\colon\ \mathrm{Hom}_{\mathcal{K}}(H^*(Y), H^*(X))\,.$$

When is this set in bijection with $[X, Y]$?

Free objects $F(n)$ in $\mathcal{U}$. The category $\mathcal{U}$ is an abelian category. (The morphism sets are abelian groups, and morphisms have well-behaved kernels and cokernels.) We shall now see that it has enough projectives. We can therefore do classical homological algebra in $\mathcal{U}$, i.e., define $\mathrm{Ext}^s_{\mathcal{U}}(M,N)$.

Let $M = (M^n)_n$ be an unstable A-module. The functor $M \mapsto M^n$ from $\mathcal{U}$ to the category $\mathcal{E}$ of $\mathbb{F}_p$-vector spaces, is co-representable. Hence there is, up to isomorphism, a unique unstable A-module $F(n)$ such that there is a natural isomorphism

$$\mathrm{Hom}_{\mathcal{U}}(F(n), M) \cong M^n .$$

Since $M \mapsto M^n$ is exact, it follows that $F(n)$ is projective in the abelian category $\mathcal{U}$.

The module $F(n)$ can be explicitly constructed. For $p = 2$ let $I = (i_1, \dots, i_n)$ be an admissible sequence. The *excess* of I is the sum

$$e(I) = \sum_s (i_s - 2i_{s+1}) = i_1 - i_2 - \cdots - i_n .$$

Given n the sub-vector space of A generated by the Sq^I with $e(I) > n$ is a left ideal. Similar definitions apply for odd p.

Proposition 1.6.2. *The unstable A-module $F(n)$ is the cyclic A-module on one generator ι_n in degree n, modulo the ideal of admissible monomials with $e(I) > n$. For $p = 2$ this is*

$$F(n) = \Sigma^n A / \mathbb{F}_2\{Sq^I \mid e(I) > n\} .$$

The admissible monomials with $e(I) \le n$ form a vector space basis for $F(n)$. Thus $F(n)$ is of finite type, i.e., finite dimensional in each degree.

For example, $F(0) = \mathbb{F}_p$, while $F(1) = \mathbb{F}_2\{x, x^2, x^4, x^8, \dots\}$ for $p = 2$, and $F(1) = \mathbb{F}_p\{x, y, y^p, y^{p^2}, \dots\}$ for p odd.

The unstable module $F(1)^{\otimes n}$ has dimension 1 in degree n. Thus there is a nontrivial map $F(n) \to F(1)^{\otimes n}$. If $p = 2$ the image of ι_n in $F(1)^{\otimes n}$ is Σ_n-invariant. Hence there is a map of unstable A-modules

$$F(n) \to (F(1)^{\otimes n})^{\Sigma_n} .$$

This map is an isomorphism.

A representability lemma. To construct injective modules in $\mathcal{U}$, we shall represent exact contravariant functors $\mathcal{U} \to \mathcal{E}$.

Lemma 2.2.1. *A (contravariant) functor $R \colon \mathcal{U}^{op} \to \mathcal{E}$ is representable if and only if it is right exact and takes direct sums to products.*

An unstable representing module $B(R)$ must satisfy

$$R(M) \cong \mathrm{Hom}_{\mathcal{U}}(M, B(R))$$

for all M. With $M = F(n)$ this says $R(F(n)) \cong \mathrm{Hom}_{\mathcal{U}}(F(n), B(R)) \cong B(R)^n$, so evaluating R on the $F(n)$ determined the degree n part of the representing module $B(R)$.

Brown–Gitler modules.

The functor $H_n\colon \mathcal{U} \to \mathcal{E}$ given by

$$H_n(M) = \mathrm{Hom}_{\mathcal{E}}(M^n, \mathbb{F}_p) = M^{n*}$$

is representable, by the lemma above. The notation is such that $H_n(H^*(X)) = H_n(X)$ for spaces X of finite type. This functor is right exact and takes direct sums to products.

Definition. The *nth Brown–Gitler module* $J(n)$ is the representing unstable A-module for the functor H_n. There is a natural isomorphism

$$H_n(M) \cong \mathrm{Hom}_{\mathcal{U}}(M, J(n))\,.$$

As H_n is also left exact, this module $J(n)$ is in fact injective in the abelian category $\mathcal{U}$. Any unstable A-module injects into a product of $J(n)$'s, so $\mathcal{U}$ has enough injectives.

We have $J(n)^m \cong \mathrm{Hom}_{\mathcal{U}}(F(m), J(n)) \cong F(m)^{n*}$, which determines the groups $J(n)^m$. Since $F(m)$ is $(m-1)$-connected, it follows that $J(n)$ is concentrated in degrees $0 \le * \le n$. Thus $J(n)$ is finite.

For example, $J(0) = \mathbb{F}_p$ in degree 0 is injective in $\mathcal{U}$.

Carlsson modules. Carlsson constructed certain unstable A-modules $K(i)$ as sequential limits of the $J(n)$'s:

$$K(i) = \lim_s(\cdots \to J(2^s i) \to J(2^{s-1} i) \to \cdots \to J(i)).$$

These are also injective in $\mathcal{U}$ for general reasons.

Carlsson showed that for $p = 2$ the unstable A-module $H^*(\mathbb{Z}/p)$ is a direct summand of $K(1)$, and thus injective. Miller extended this to odd p, and Lannes and Zarati extended the injectivity assertion to $H^*(V)$ for general $V = (\mathbb{Z}/p)^d$.

$\mathcal{U}$-injectivity of $H^*(V) \otimes J(n)$.

Theorem 3.1.1 (Carlsson, Miller, Lannes, Zarati). *Let V be an elementary abelian p-group. Then the unstable A-module $H^*(V) \otimes J(n)$ is injective in $\mathcal{U}$ for all n.*

Lannes' functor T_V. Let L be an unstable A-module of finite type. The following is a consequence of Freyd's adjoint functor theorem.

Theorem 3.2.1 (Lannes). *The functor $M \mapsto L \otimes M$ from $\mathcal{U}$ to itself has a left adjoint denoted $N \mapsto (N : L)_{\mathcal{U}}$. There is a natural isomorphism*

$$\mathrm{Hom}_{\mathcal{U}}((N : L)_{\mathcal{U}}, M) \cong \mathrm{Hom}_{\mathcal{U}}(N, L \otimes M)\,.$$

We call $N \mapsto (N : L)_{\mathcal{U}}$ the *division by L* functor.

Definition. Lannes' functor T_V is the division by $H^*(V)$ functor:

$$T_V(N) = (N : H^*(V))_{\mathcal{U}}\,.$$

We write $T = T_{\mathbb{F}_p}$, so for $V = (\mathbb{Z}/p)^d$ we have $T_V = T^d$ (the d-fold composition). There is an adjunction

$$\mathrm{Hom}_{\mathcal{U}}(T_V(N), M) \cong \mathrm{Hom}_{\mathcal{U}}(N, H^*(V) \otimes M)\,.$$

Theorem 3.2.2. *The functor T_V is exact.*

Proof. This is implied by the $\mathcal{U}$-injectivity of $H^*(V) \otimes J(n)$. □

First computations.

Proposition 3.3.2. *The functor T_V commutes with colimits.*

This holds for any left adjoint.

Proposition 3.3.2. *The functor T_V commutes with suspensions.*

Sketch proof. The suspension functor Σ in $\mathcal{U}$ has a right adjoint, which commutes with tensor product with $H^*(V)$. □

An unstable A-module M is *locally finite* if each element $x \in M$ is contained in a finite A-submodule. Thus M is a colimit of finite A-modules.

The split injection $\mathbb{F}_p \to H^*(V)$ induces a split surjection $T_V(M) \to M$.

Proposition 3.3.6. *Let M be a locally finite unstable A-module. Then $T_V(M) \cong M$.*

Proof. $T_V(\mathbb{F}_p) \cong \mathbb{F}_p$ since $H^0(V) \cong \mathbb{F}_p$. By exactness $T_V(M) \cong M$ for any finite M, by induction over the dimension of M. By passage to colimits, the same holds for any locally finite M. □

For example, $T_V(H^*(X)) \cong H^*(X)$ for any finite dimensional CW-complex X.

T_V and tensor products. Lannes' functor T_V commutes with tensor products.

Theorem 3.5.1 (Lannes). *There is a natural isomorphism*

$$T_V(M \otimes N) \cong T_V(M) \otimes T_V(N)$$

for unstable A-modules M and N.

The map arises by a chain of adjunctions from the product $H^*(V) \otimes H^*(V) \to H^*(V)$.

T_V and unstable algebras. If K is an unstable A-algebra, so is $T_V(K)$. The product is given by

$$T_V(K) \otimes T_V(K) \cong T_V(K \otimes K) \to T_V(K)$$

using the isomorphism of 3.5.1 and the product $K \otimes K \to K$.

Theorem 3.8.1 (Lannes). *Let K, L be unstable A-algebras. The unstable A-module $T_V(K)$ is in a natural way an unstable A-algebra, and there is a natural isomorphism*

$$\mathrm{Hom}_K(T_V(K), L) \cong \mathrm{Hom}_{\mathcal{K}}(K, H^*(V) \otimes L)\,.$$

In fact there exists a division functor $K \mapsto (K : H^*(V))_{\mathcal{K}}$ also in $\mathcal{K}$, which equals T_V.

Thus $T_V \colon \mathcal{K} \to \mathcal{K}$ is also a left adjoint in the category of unstable A-algebras. Since T_V preserves injections of unstable A-algebras, viewed as unstable A-modules, it follows that $H^*(V)$ is categorically injective in $\mathcal{K}$.

But $\mathcal{K}$ is not an abelian category (not even additive), so we cannot do ordinary homological algebra in $\mathcal{K}$. Instead one uses simplicial resolutions and comonad-derived functors.

There are then adjunctions

$$\mathrm{Ext}^s_{\mathcal{K}}(T_V(K), L) \cong \mathrm{Ext}^s_{\mathcal{K}}(K, H^*(V) \otimes L)$$

and similar formulas for the derived functors of suitable groups of derivations, rather than homomorphisms.

Cosimplicial spaces. A cosimplicial space $X^\bullet$ is a (covariant) functor $X\colon \Delta \to$ Spaces, where we by spaces typically mean simplicial sets. It is a sequence of spaces $[q] \mapsto X^q$, together with coface and codegeneracy maps.

The cosimplicial space $\Delta^\bullet$ is given by $[q] \mapsto \Delta^q$, with the usual coface and codegeneracy maps. The *totalization* $\mathrm{Tot}\, X^\bullet$ of a cosimplicial space is the mapping space

$$\mathrm{Tot}\, X^\bullet = \mathrm{Map}(\Delta^\bullet, X^\bullet)\,.$$

Its p-simplices are the set of maps (= natural transformations) $\Delta^p \times \Delta^\bullet \to X^\bullet$.

Restricting a map from $\Delta^\bullet$ to the s-skeleton $(\Delta^q)^{(s)} \subseteq \Delta^q$ in each codegree q, yields a map

$$\mathrm{Tot}\, X^\bullet \to \mathrm{Tot}_s\, X^\bullet$$

to the *sth partial totalization.*

Bousfield–Kan R-completion. Let R be a commutative unital ring, Y a simplicial set (= space). Let $R(Y)$ be the simplicial set which in degree q is the free R-module $R\{Y_q\}$ on the set of q-simplices in Y. There is a monad (= triple) (R, μ, η), with product $\mu\colon R(R(Y)) \to R(Y)$ and unit $\eta\colon Y \to R(Y)$. There is a cosimplicial space $[q] \mapsto R^{q+1}(Y)$ denoted $R^\bullet(Y)$, with coaugmentation η:

$$Y \longrightarrow R(Y) \rightrightarrows R(R(Y)) \rightrightarrows \cdots$$

Then $R_\infty(Y) = \mathrm{Tot}\, R^\bullet(Y)$, and the coaugmentation defines the completion map $Y \to R_\infty(Y)$.

Bousfield–Kan $\mathbb{F}_p$-completion. The map $Y \to \mathbb{F}_{p\infty}(Y)$ is Bousfield localization with respect to $H_*(-;\mathbb{F}_p)$ for virtually nilpotent spaces, i.e., connected spaces Y for which a subgroup of finite index in $\pi_1(Y)$ acts nilpotently on $\pi_*(Y)$. This includes all connected spaces with finite fundamental group. We often write $Y^\wedge_p = \mathbb{F}_{p\infty}(Y)$.

Bousfield–Kan homotopy spectral sequence. Let $X^\bullet$ be a fibrant and pointed cosimplicial space. Then $[s] \mapsto \pi_t(X^s)$ defines a cosimplicial group. Its cohomotopy $\pi^s\pi_t(X^\bullet)$ is the sth cohomology group of the associated cochain complex.

Get a pointed tower of principal fibrations

$$\mathrm{Tot}\, X^\bullet \to \ldots \to \mathrm{Tot}_s\, X^\bullet \to \mathrm{Tot}_{s-1}\, X^\bullet \to \ldots \to \mathrm{Tot}_0\, X^\bullet\,.$$

The associated spectral sequence has E_2-term

$$E_2^{s,t} = \pi^s\pi_t(X^\bullet)$$

for $t \geq s \geq 0$ and converges (under mild conditions on $X^\bullet$) to $\pi_{t-s}\, \mathrm{Tot}\, X^\bullet$.

Homotopy of $\mathrm{Map}(X, Y_p^\wedge)$**.** Let X and Y be connected spaces, with $H^*(X)$ and $H^*(Y)$ of finite type.

Proposition 8.4.1. *Take the trivial map as base point in* $\mathrm{Map}(X, \mathbb{F}_{p\infty}(Y))$. *Then*

$$\pi^s\pi_t(\mathrm{Map}(X, \mathbb{F}_p^\bullet(Y))) \cong \mathrm{Ext}^s_{\mathcal{K}}(H^*(Y), \Sigma^t(H^*(X))^+)$$

for $t \geq 1$, $s \geq 0$, *and*

$$\pi^0\pi_0(\mathrm{Map}(X, \mathbb{F}_p^\bullet(Y))) \cong \mathrm{Hom}_{\mathcal{K}}(H^*(Y), H^*(X)).$$

The Ext-groups are formed in the sense of homotopical algebra in the (non-abelian) category $\mathcal{K}$ of unstable A-algebras, defined using resolutions generated by a comonad (= cotriple). This is the comonad generated by the forgetful functor $\mathcal{K} \to \mathcal{U}$ and its left adjoint $\mathcal{U} \to \mathcal{K}$.

The p-adic completion $\mathbb{F}_{p\infty}(Y)$ is the totalization of a cosimplicial space which is a generalized Eilenberg–Mac Lane space in each codegree. Hence its cohomology is accessible through the calculations of Cartan and Serre. This gives the E_2-term above.

Similar formulas apply for other choices of base point, replacing groups of homomorphisms with groups of derivations, and Ext-groups with the derived functors of the derivations.

Miller's conjecture.

((Use the ($s = t$)-line in the spectral sequence. Lannes' vanishing results.))

Theorem 8.1.1 (Lannes). *Let* Y *be a connected nilpotent space such that* $H^*(Y)$ *is of finite type and* $\pi_1(Y)$ *is finite. Then the natural map* $[f] \mapsto f^*$ *induces a bijection*

$$[BV, Y] \xrightarrow{\cong} \mathrm{Hom}_{\mathcal{K}}(H^*(Y), H^*(V)).$$

Remark. When Y is connected and finite dimensional, there is only the trivial unstable algebra homomorphism $H^*(Y) \to H^*(V)$. For (when $p = 2$) $Sq_0\colon x \mapsto x^2$ acts injectively on $H^*(V)$, and nilpotently on $H^*(Y)$ in positive degrees.

Miller's theorem.

Theorem 8.6.1. *Let* Y *be a connected nilpotent space such that* $H^*(Y)$ *is of finite type and* $\pi_1(Y)$ *is finite. Then the canonical map* $H^*(Y) \to TH^*(Y)$ *is an isomorphism if and only if the space of based maps* $\mathrm{Map}_*(B\mathbb{Z}/p, Y)$ *is contractible.*

Proof sketch. Evaluation at the base point of $B\mathbb{Z}/p$ defines a map $\mathrm{Map}(B\mathbb{Z}/p, Y) \to Y$, and a map of spectral sequences with E_2-terms:

$$\mathrm{Ext}^s_{\mathcal{K}}(H^*(Y), H^*(\mathbb{Z}/p) \otimes H^*(S^t)) \to \mathrm{Ext}^s_{\mathcal{K}}(H^*(Y), H^*(S^t)).$$

((Relate $H^*(\mathbb{Z}/p) \otimes H^*(S^t)$ to $\Sigma^t(H^*(\mathbb{Z}/p))^+$.))

The left term is isomorphic to $\mathrm{Ext}^s_{\mathcal{K}}(TH^*(Y), H^*(S^t))$ and the map is induced by the inclusion $H^*(Y) \to TH^*(Y)$. If this is an isomorphism the spectral sequences are isomorphic from the E_2-terms onwards. This proves that $\mathrm{Map}_*(B\mathbb{Z}/p, Y)$ is contractible after p-completion. An arithmetic square argument yields the integral result. □

Remark. Recall that $H^*(Y) \to TH^*(Y)$ is an isomorphism whenever $H^*(Y)$ is locally finite, e.g., when Y is finite dimensional.

The Sullivan conjecture. Let G be a group and X a G-space. The homotopy fixed point space $X^{hG} = \mathrm{Map}(EG, X)^G$ is the space of G-equivariant maps $EG \to X$. There is a canonical map

$$\gamma\colon X^G \to X^{hG}$$

taking a fixed point of X to the constant map from EG to it.

When the G-action is trivial, this is the "constant maps" section $\gamma\colon X \to \mathrm{Map}(BG, X)$ in the fibration

$$\mathrm{Map}_*(BG, X) \to \mathrm{Map}(BG, X) \xrightarrow{ev} X\,.$$

Here ev evaluates a map at the base point.

The following was proven for $G = \mathbb{Z}/p$ and trivial G-action by Miller, then for a general $\mathbb{Z}/p$-action by Carlsson, Lannes and Miller (independently), and for general p-groups by Dwyer and Zabrodsky.

Theorem 9.1.1 and 9.1.2. *Let G be a finite p-group, and X a finite G-CW complex. Then the map*

$$(X^G)^\wedge_p \xrightarrow{\gamma} (X^\wedge_p)^{hG}$$

is a homotopy equivalence.

The proof for $G = \mathbb{Z}/p$ is by reduction to Miller's theorem 8.6.1. The proof for general p-groups is by an induction on the order of p, using the existence of a central $C \cong \mathbb{Z}/p$ in any nontrivial p-group G.

Corollary. *Let G be a finite p-group, and X a finite CW complex with trivial G-action. Then the based mapping space $\mathrm{Map}_*(BG, X^\wedge_p)$ is contractible.*

Remark. The finiteness hypothesis on X is necessary. Take for example $X = K(\mathbb{Z}/p, n)$. Then $\pi_0 \mathrm{Map}_*(BG, X^\wedge_p) \cong H^n(BG)$ is typically nonzero.

The cohomology of $\mathrm{Map}(BV, Y^\wedge_p)$.

The evaluation map

$$BV \times \mathrm{Map}(BV, Y) \to Y$$

induces a homomorphism

$$H^*(Y) \to H^*(V) \otimes H^*(\mathrm{Map}(BV, Y)),$$

adjoint to a homomorphism

$$T_V H^*(Y) \to H^*(\mathrm{Map}(BV, Y))\,.$$

It admits a lift over $H^*(\mathrm{Map}(BV, Y^\wedge_p))$.

Theorem 9.7.1 (Lannes). *Let Y be a space such that $H^*(Y)$ and $T_V H^*(Y)$ are of finite type, and suppose $T_V H^*(Y)$ is trivial in degree 1. Then the natural map*

$$T_V H^*(Y) \xrightarrow{\cong} H^*(\mathrm{Map}(BV, X^\wedge_p))$$

is an isomorphism.

This pretty much pins down the meaning of the functor T_V.