MOTIVIC TRUNCATION AND ARITHMETIC DUALITY FOR SOME STRUCTURED RING SPECTRA

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These are informal notes for a talk by the author. Handle with care.

Algebraic K-theory. We wish to understand the category $h\mathcal{C}_R$ of modules over a structured ring spectrum R, up to weak equivalence and subject to suitable finiteness conditions. This is analogous to the study of the category of finitely generated projective modules over a given ring. In the case of a Dedekind domain, such as the ring of integers $\mathcal{O}_F$ in a number field, these consist of sums of finitely generated free modules and an ideal in the ring. In the case of the topological ring of continuous complex functions on a compact Hausdorff space, these consist of the finite-dimensional complex vector bundles over that space.

The classifying space $|h\mathcal{C}_R|$ of such a category contains much arithmetic information about the ring spectrum. More realistically, we will study the group completion of this classifying space, which is obtained as the underlying space $\Omega^\infty K(R)$ of its algebraic K-theory spectrum, and whose homotopy groups are the algebraic K-groups of R. In the case of Dedekind domains, these algebraic K-groups incorporate classical number-theoretic information about the ideal class group $\mathrm{Cl}(F) = \mathrm{Pic}(\mathcal{O}_F)$, the group of units $\mathcal{O}_F^\times$ and of the Brauer group $\mathrm{Br}(\mathcal{O}_F)$. In the case of a ring of complex functions, we obtain the complex topological K-theory of the space.

We shall focus on "brave new" structured ring spectra, alias S-algebras, where S is the sphere spectrum. One motivation for this is Waldhausen's stable parametrized h-cobordism theorem. Given a compact connected smooth manifold M the theorem concerns the space $H(M)$ of all smooth h-cobordisms (M, W, N) with M at one end. It parametrizes, or classifies, h-cobordisms in the same way that Grassmannians classify vector bundles. The h-cobordism space $H(M)$ is closely related via fiber sequences to the group of smooth symmetries of M relative to the boundary, i.e., the diffeomorphism group $\mathrm{Diff}(M)$. The classifying space $B\,\mathrm{Diff}(M)$ of the latter group is the moduli space of smooth manifolds diffeomorphic to M, and these objects are clearly of geometric topological interest.

There is an infinitely stabilized version $H^\infty(M)$ of the h-cobordism space, and the connectivity of the stabilization map $H(M) \to H^\infty(M)$ increases linearly with the dimension of M [Igusa (1988)]. Waldhausen's theorem asserts that there is a fiber sequence

$$H^\infty(M) \to \Omega^\infty\Sigma^\infty M_+ \to \Omega^\infty K(S[\Omega M])$$

where the left hand map is null-homotopic. Here $S[\Omega M] = \Sigma^\infty \Omega M_+$ is the spherical group ring of the loop group of M. Note that its bottom homotopy group $\pi_0 S[\Omega M]$

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is the integral group ring $\mathbb{Z}[\pi_1(M)]$. Waldhausen writes $A(M)$ for $K(S[\Omega M])$. For closed non-positively curved manifolds M, $A(M)$ is assembled from a copy of $A(*) = K(S)$ for each point of M, and a copy of $A(S^1) = K(S[\mathbb{Z}])$ for each closed geodesic in M [Farrell–Jones (1991)]. We are therefore particularly interested in the algebraic K-theory of the sphere spectrum S itself, and of its ring of Laurent polynomials. The arithmetic properties of these ring spectra incorporate information about high-dimensional geometric topology.

The sphere spectrum. The map $S \to H\mathbb{Z}$ of structured ring spectra is 1-connected and a rational equivalence, so it induces a rational equivalence $K(S) \to K(\mathbb{Z})$. From Borel's rational calculation

$$\pi_i K(\mathbb{Z}) \otimes \mathbb{Q} = \begin{cases} \mathbb{Q} & \text{if } i = 0, \text{ or } i = 4k+1 \text{ for } k \geq 1, \\ 0 & \text{otherwise,} \end{cases}$$

which involves symmetric spaces, Lie algebra cohomology and some nontrivial real harmonic analysis, one obtains the same formula for $\pi_i K(S) \otimes \mathbb{Q}$. As a consequence of Waldhausen's theorem,

$$\pi_i \operatorname{Diff}(D^n) \otimes \mathbb{Q} = \begin{cases} \mathbb{Q} & \text{if } i = 4k-1 \text{ for } n \text{ odd and } k \geq 1, \\ 0 & \text{otherwise,} \end{cases}$$

in the stable range where $n \gg i$ [Farrell–Hsiang (1978)]. This is to be contrasted with the topological or piecewise-linear homeomorphism group of D^n, relative to the boundary, which is contractible.

What about the p-primary torsion in $\pi_i K(S)$, for each prime p? In principle, this can be computed using the cyclotomic trace map $trc\colon K(R) \to TC(R)$ from algebraic K-theory to topological cyclic homology [Bökstedt–Hsiang–Madsen (1993)] and the homotopy Cartesian square

$$\begin{array}{ccc} K(S) & \longrightarrow & TC(S) \\ \downarrow & & \downarrow \\ K(\mathbb{Z}) & \longrightarrow & TC(\mathbb{Z}) \end{array}$$

of [Dundas (1997)]. This was done for $p = 2$ in [Rognes (2002)] and for odd regular primes p in [Rognes (2003)], giving the cohomology $H^*(K(S); \mathbb{F}_p)$ as a module over the mod p Steenrod algebra. This is the input for the Adams spectral sequence converging to $\pi_* K(S) \otimes \mathbb{Z}_p$, and these homotopy groups can indeed be computed in a finite range (done in degrees $* \leq 21$ at $p = 2$).

However, these results on $K(S)$ are not particularly conceptual. We would instead like to have an understanding of the p-torsion in $K(S)$ in similar terms to the one we have of $K(\mathbb{Z})$, by way of the proven Lichtenbaum–Quillen conjectures on étale descent for algebraic K-theory.

To simplify the presentation, speaking about Galois descent rather than étale descent, we will pass to the p-adically completed ring spectra S_p and $H\mathbb{Z}_p$. This is not so serious, due to the homotopy Cartesian square

$$\begin{array}{ccc} K(S)_p & \longrightarrow & K(S_p)_p \\ \downarrow & & \downarrow \\ K(\mathbb{Z})_p & \longrightarrow & K(\mathbb{Z}_p)_p \end{array}$$

of p-completed spectra. The conceptual understanding of $K(\mathbb{Z}_p)_p$ then goes in two steps: first by passage to the fraction field $\mathbb{Q}_p$ through (Zariski) localization, and second by motivic truncation from a modified form of algebraic K-theory that satisfies Galois descent, and which is computable in terms of Galois cohomology.

Zariski localization. First, the algebraic K-theory of $\mathbb{Z}_p$ appears in the localization (co-)fiber sequence of spectra

$$K(\mathbb{F}_p) \xrightarrow{i_*} K(\mathbb{Z}_p) \xrightarrow{j^*} K(\mathbb{Q}_p)\,,$$

associated to the algebro-geometric picture

$$\operatorname{Spec}\mathbb{F}_p \xrightarrow{i} \operatorname{Spec}\mathbb{Z}_p \xleftarrow{j} \operatorname{Spec}\mathbb{Q}_p$$

with i a closed and j an open immersion. The localization sequence in algebraic K-theory arises by comparing two notions of weak equivalence on the category of finitely generated $\mathbb{Z}_p$-modules, one being isomorphism and the other being rational isomorphism. (More precisely one works with categories of chain complexes of modules, or of simplicial modules, up to quasi-isomorphism or rational quasi-isomorphism.) At the level of algebraic K-theory the inclusion of the former into the latter induces the right hand map j^*, and the fiber is the algebraic K-theory of the category of finite p-torsion $\mathbb{Z}_p$-modules, which consists of extensions of finite $\mathbb{F}_p$-modules. The dévissage or additivity theorems express how algebraic K-theory treats all such extensions as being split, so that the fiber term is identified with $K(\mathbb{F}_p)$. The left hand map i_* is then the transfer map, which perceives a finite $\mathbb{F}_p$-module as a finitely generated $\mathbb{Z}_p$-module.

More generally, there is such a localization sequence for Dedekind domains. We shall encounter several such localization sequences in the following discussion.

By Quillen's computation, $K(\mathbb{F}_p)_p \simeq H\mathbb{Z}_p$ is well understood. More precisely, there is a class $\delta_1 = d\log p$ (a logarithmic pole) in $\pi_1 K(\mathbb{Q}_p)$ whose image under the connecting map of the localization sequence is $1 \in \pi_0 K(\mathbb{F}_p)$, and in algebraic K-theory with mod p coefficients there is a class $\lambda_1 \in V(0)_{2p-1}K(\mathbb{Z}_p)$ that becomes divisible by v_1 in $V(0)_*K(\mathbb{Q}_p)$, in that $\lambda_1 = v_1\delta_1$.

Here $V(0) = S/p$ denotes the mod p Moore spectrum, defined by the cofiber sequence

$$S \xrightarrow{p} S \to V(0)\,.$$

Its associated homology theory $V(0)_*(X) = \pi_*(X;\mathbb{F}_p)$ is mod p stable homotopy. It is a ring spectrum for p odd, which we now assume. It admits a periodic Adams self-map $v_1 \colon \Sigma^{2p-2}V(0) \to V(0)$, which induces a natural module action by the polynomial ring $P(v_1)$ on each $V(0)_*(X)$.

Galois cohomology. Second, there is a comparison map

$$K(\mathbb{Q}_p)_p \to K^{Gal}(\mathbb{Q}_p)_p$$

from p-completed algebraic K-theory to a "Galois" K-theory that is constructed to satisfy Galois descent, i.e., for each G-Galois extension $F \to E$, the map $K^{Gal}(F)_p \to K^{Gal}(E)_p^{hG}$ is a weak equivalence. The Lichtenbaum–Quillen conjecture (now a theorem) asserts that p-completed algebraic K-theory is close to satisfying Galois descent, in the modified form that for each local or global number field F the map

$$K(F)_p \to K^{Gal}(F)_p$$

induces an injection on π_0 and an isomorphism on homotopy in all higher degrees, i.e., it is a 0-coconnected map.

In general, for a field F of characteristic $\neq p$ let $\bar{F}$ be a separable closure of F. The absolute Galois group $G_F = \mathrm{Gal}(\bar{F}|F)$ is a profinite group acting continuously on $\bar{F}$ through F-algebra automorphisms. By functoriality, it also acts on the p-completed algebraic K-theory $K(\bar{F})_p$, fixing $K(F)_p$. By the theorem of Suslin on the algebraic K-theory of separably closed fields,

$$K(\bar{F})_p \simeq ku_p$$

is weakly equivalent to p-complete connective complex K-theory. So $\pi_* K(\bar{F})_p \cong \pi_* ku_p = \mathbb{Z}_p[u]$. With some care (involving pro-objects), we can form the continuous homotopy fixed point spectrum for this action

$$K^{Gal}(F)_p := K(\bar{F})_p^{hG_F}$$

and there is a Galois descent spectral sequence (a homotopy fixed point spectral sequence)

$$E^2_{s,t} = H^{-s}_{Gal}(F; \mathbb{Z}_p(t/2)) \Longrightarrow \pi_{s+t} K^{Gal}(F)_p$$

concentrated in the second quadrant, in homological indexing. Here $\mathbb{Z}_p(t/2) = \pi_t(ku_p)$ is $\mathbb{Z}_p\{u^{t/2}\}$ for $t \geq 0$ even, and 0 otherwise. By $H^*_{cont}(G; M)$ we mean the continuous group cohomology of G, with coefficients in a suitable G-module M. We write Galois cohomology $H^*_{Gal}(F; M) := H^*_{cont}(G_F; M)$ for the continuous group cohomology of the absolute Galois group G_F.

Thomason proved an asymptotic form of the Lichtenbaum–Quillen conjecture, in his theorem on algebraic K-theory made Bott periodic [Thomason (1985)]. Passing to mod p homotopy and inverting the action by v_1, he proved that for a large class of fields, including local and global number fields, the comparison map above induces an isomorphism

$$v_1^{-1} V(0)_* K(F) \xrightarrow{\cong} v_1^{-1} V(0)_* K^{Gal}(F)\,.$$

The target is better known as the mod p étale K-theory of F. After adjoining a p-th root of unity to F, multiplication by v_1 can be rewritten as multiplication by a power β^{p-1} of a suitable mod p Bott element in degree 2. (Thomason's result also applies to a large class of schemes.)

In the Galois descent spectral sequence with mod p coefficients, v_1 is represented by a generator of $E^2_{0,2p-2} = H^0_{Gal}(F; \mathbb{F}_p(p-1)) = \mathbb{F}_p$, since always G_F acts trivially on $\mathbb{F}_p(p-1) = \mu_p^{\otimes(p-1)}$, by Fermat's little theorem. If F contains a p-th root of unity, then G_F also acts trivially on $\mathbb{F}_p(1) = \mu_p$, and β is represented by a generator of $E^2_{0,2} = H^0_{Gal}(F; \mathbb{F}_p(1)) = \mu_p(F)$.

One can explicitly compute the Galois cohomology of local fields, either through the identifications $H^0_{Gal}(F; \mathbb{G}_m) = F^\times$, $H^1_{Gal}(F; \mathbb{G}_m) = 0$ (Hilbert's Theorem 90) and $H^2_{Gal}(F; \mathbb{G}_m) = \mathrm{Br}(F)$, the Kummer sequence $\mu_p \to \mathbb{G}_m \xrightarrow{(-)^p} \mathbb{G}_m$ and Tate's arithmetic duality, or by the chain of maps $TC(\mathcal{O}_F)_p \leftarrow K(\mathcal{O}_F)_p \to K(F)_p \to K^{Gal}(F)_p$ and the Galois descent spectral sequence. For $F = \mathbb{Q}_p$ and with mod p coefficients, we obtain

$$\begin{aligned} E^2_{**} &= H^{-*}_{Gal}(\mathbb{Q}_p; \mathbb{F}_p(*/2)) \\ &= P(v_1^{\pm 1}) \otimes \big(E(\delta_1, \partial v_1) \oplus \mathbb{F}_p\{t^i\lambda_1 \mid 0 < i < p\}\big) \\ &\Longrightarrow v_1^{-1}V(0)_*K^{Gal}(\mathbb{Q}_p)\,. \end{aligned}$$

Here $P(v_1^{\pm 1})$ denotes the Laurent polynomials over $\mathbb{F}_p$ on one generator v_1, and $E(\delta_1, \partial v_1)$ the exterior algebra over $\mathbb{F}_p$ on two generators $\delta_1 = d\log p$ and ∂v_1. Restricting to the second quadrant, one obtains the Galois descent spectral sequence abutting to $V(0)_*K^{Gal}(\mathbb{Q}_p)$.

A chart for $p = 3$ is displayed in Figure 1 below. The odd rows are omitted. Multiple generators in the same bidegree are separated by colons. The full E^2-term is the free $P(v_1^{\pm 1})$ module on these generators, where v_1 is represented in bidegree $(s, t) = (0, 2p - 2)$.

$\partial v_1 \delta_1$	$\cdot$	$\cdot$	$2p$
$\cdot$	$\partial v_1 : t\lambda_1$	$\cdot$	$2p-2$
$\cdot$	$\delta_1 : t^2\lambda_1$	$\cdot$	2
$\cdot$	$\cdot$	1	0
-2	-1	0	$s\backslash t$

FIGURE 1

For later use, we note that the cohomology cup product satisfies

$$t^i\lambda_1 \cup t^{p-i}\lambda_1 = \partial\lambda_1$$

for $0 < i < p$. Including the integral powers of v_1 we get the more crowded chart of Figure 2 below.

$\partial v_1 \delta_1$	$v_1\delta_1 : v_1 t^2\lambda_1$	$\cdot$	$2p$
$\cdot$	$\partial v_1 : t\lambda_1$	v_1	$2p-2$
$\partial\delta_1$	$\delta_1 : t^2\lambda_1$	$\cdot$	2
$\cdot$	$\partial : v_1^{-1}t\lambda_1$	1	0
-2	-1	0	$s\backslash t$

FIGURE 2

Motivic cohomology. To establish the original strong form of the Lichtenbaum–Quillen conjecture we need motivic cohomology, which is to Galois cohomology as algebraic K-theory is to Galois K-theory.

For a field F let

$$\Delta^q_F = \operatorname{Spec} F[X_0, \dots, X_q]/(X_0 + \cdots + X_q - 1)$$

be the affine q-simplex over $\operatorname{Spec} F$. The association $[q] \mapsto \Delta^q_F$ defines a cosimplicial variety. For each integer k, Bloch's cycle complex $z^k(F, *)$ is the chain complex (or simplicial abelian group) defined by

$$z^k(F, q) = \mathbb{Z}\left\{\begin{matrix}\text{closed subvarieties } W \subset \Delta^q_F \text{ that} \\ \text{meet each face in codimension} \geq k\end{matrix}\right\}.$$

The simplicial face maps are given by intersecting with the faces. Bloch's higher Chow groups are defined as

$$CH^k(F, q) := H_q(z^k(F, *)) .$$

The motivic cohomology of $\operatorname{Spec} F$, as defined by Voevodsky, Hanamura and Levine, can then be expressed as

$$H^n_{mot}(F; \mathbb{Z}(k)) := CH^k(F, 2k - n) .$$

Note that here the coefficient $\mathbb{Z}(k)$ is a symbol, rather than a module over some group. Note also that $z^k(F, q) = 0$ for $k > q$, since there are no varieties of dimension $q - k < 0$. So $CH^k(F, q) = 0$ for $k > q$ and $H^n_{mot}(F, \mathbb{Z}(k)) = 0$ for $k > 2k - n$, or equivalently, for $n > k$.

There is a Bloch–Lichtenbaum spectral sequence (ca. 1995)

$$E^2_{s,t} = H^{-s}_{mot}(F; \mathbb{Z}(t/2)) \Longrightarrow \pi_{s+t} K(F)$$

derived from a filtration of algebraic K-theory by codimension of support.

After p-completion, we get a map of spectral sequences

$$\begin{array}{ccc} E^2_{s,t} = H^{-s}_{mot}(F; \mathbb{Z}_p(t/2)) & \Longrightarrow & \pi_{s+t} K(F)_p \\ \downarrow & & \downarrow \\ E^2_{s,t} = H^{-s}_{Gal}(F; \mathbb{Z}_p(t/2)) & \Longrightarrow & \pi_{s+t} K^{Gal}(F)_p . \end{array}$$

The precise form of the modified Galois descent for algebraic K-theory is the isomorphism

$$H^n_{mot}(F; \mathbb{Z}_p(k)) \cong \begin{cases} H^n_{Gal}(F; \mathbb{Z}_p(k)) & \text{for } n \leq k, \\ 0 & \text{for } n > k \end{cases}$$

for F of characteristic $\neq p$. This is the content of the Milnor- and Bloch–Kato conjectures, apparently now proven by Voevodsky and Rost. We call this the **motivic truncation**, which expresses how to obtain p-adic or mod p motivic cohomology, by only retaining a part of the corresponding Galois cohomology. It is the cohomological consequence of the geometric fact that in Bloch's cycle complex there are no codimension k subvarieties of Δ^{2k-n}_F for $n > k$. Conversely, one can recover mod p Galois cohomology from mod p motivic cohomology by inverting v_1 (a power of the Bott element).

In terms of spectral sequences, the motivic truncation recognizes the E^2-term of the Bloch–Lichtenbaum spectral sequence as the part of the Galois descent spectral sequence E^2-term where $-s \le t/2$, or equivalently $2s + t \ge 0$, i.e., the part that lies on or above the line of slope -2 through the origin.

For local or global number fields $H^n_{Gal}(F; \mathbb{Z}_p(k))$ is concentrated in cohomological degrees $0 \le n \le 2$, except when $p = 2$ and F admits a real embedding. So the Galois descent spectral sequence is concentrated in the three columns $-2 \le s \le 0$ and in the even rows. There is no room for differentials, so $E^2 = E^\infty$. The motivic truncation eliminates the terms below the line $2s + t = 0$, which amounts to the term

$$E^2_{-2,2} = H^2_{Gal}(F; \mathbb{Z}_p(1)) = \mathrm{Hom}(\mathbb{Q}_p/\mathbb{Z}_p, \mathrm{Br}(F))$$

(the p-primary Tate module of the Brauer group of F), which contributes to $\pi_0 K^{Gal}(F)_p$ but not to $\pi_0 K(F)_p$, and to the terms contributing to $\pi_* K^{Gal}(F)_p$ in negative degrees. In particular, the $P(v_1^{\pm 1})$-module generators for the Galois cohomology in Figure 1 above, all of which lie just on or above the line $2s+t = 0$, are also the $P(v_1)$-module generators for the motivic cohomology. This explains why the comparison map is 0-coconnected.

More generally, these constructions can be made to work for a regular scheme X of finite type over a regular noetherian base scheme B of dimension one [Levine (2001)], in place of F. This includes rings of integers in local or global number fields. The motivic truncation phenomenon is now more complicated, in part because the notion of codimension is not so straightforward for general subschemes. Can some of this theory be made to work for suitable S-algebras? The affine q-simplex Δ^q_S does make sense.

Arithmetic duality. A general reference for this section is [Milne (1986)].

The Galois cohomology of a field F is defined as the continuous cohomology of the absolute Galois group G_F, and as such as essentially the singular cohomology of the classifying space BG_F. When F is a local number field, this cohomology has an additional feature over that of generic fields, namely it satisfies a form of Poincaré duality. This was visible already in the chart for $\mathbb{Q}_p$ displayed above. Thus the classifying space BG_F for F a local number field behaves cohomologically like a closed 2-dimensional manifold, i.e., a surface.

More precisely, for each local number field F there is a canonical isomorphism

$$inv\colon \mathrm{Br}(F) = H^2_{Gal}(F; \bar{F}^\times) \xrightarrow{\cong} \mathbb{Q}/\mathbb{Z}$$

that classifies Morita classes of central simple F-algebras by a $\mathbb{Q}/\mathbb{Z}$-valued invariant. Quaternion algebras have invariant $\frac{1}{2}$; the n-th Morava stabilizer group $\mathbb{S}_n$ is the group of units in the maximal order in a central simple $\mathbb{Q}_p$-algebra of invariant $\frac{1}{n}$. Restricting the coefficients to $\mathbb{F}_p(1) = \mu_p \subset \bar{F}^\times$ gives an isomorphism of p-torsion subgroups

$$\mathrm{Br}(F)[p] = H^2_{Gal}(F; \mathbb{F}_p(1)) \xrightarrow{\cong} \mathbb{Z}/p\,.$$

In the example $F = \mathbb{Q}_p$ displayed in Figure 2 above, $H^2_{Gal}(\mathbb{Q}_p; \mathbb{F}_p(1)) \cong \mathbb{F}_p\{\partial\delta_1\}$ and the invariant takes $\partial\delta_1$ to a generator of $\mathbb{Z}/p$.

The isomorphism is fundamental to local class field theory, i.e., the classification of abelian extensions of a local number field F in terms of finite index subgroups of the unit group $F^\times$. In outline, it is obtained as follows: the maximal unramified extension of $\mathbb{Q}_p$ is $\mathbb{Q}_p^{nr} = \mathbb{Q}_p(\mu_{\infty,p})$, obtained by adjoining all roots of unity of order prime to p. The p-adic valuation $\mathbb{Q}_p^\times \to \mathbb{Z}$ extends through $\mathbb{Q}_p^{nr,\times}$. There are group isomorphisms $\mathrm{Gal}(\mathbb{Q}_p^{nr}|\mathbb{Q}_p) \cong \mathrm{Gal}(\bar{\mathbb{F}}_p|\mathbb{F}_p) \cong \hat{\mathbb{Z}}$, and cohomology isomorphisms

$$H^2_{cont}(G_{\mathbb{Q}_p}; \bar{\mathbb{Q}}_p^\times) \leftarrow H^2_{cont}(\mathrm{Gal}(\mathbb{Q}_p^{nr}|\mathbb{Q}_p); \mathbb{Q}_p^{nr,\times}) \to H^2_{cont}(\hat{\mathbb{Z}}; \mathbb{Z}) \cong \mathbb{Q}/\mathbb{Z}\,.$$

The local **arithmetic duality** theorem of Tate and Poitou [Tate (1962)] then asserts that for a local number field F, the cup product

$$H^n_{Gal}(F; \mathbb{F}_p(k)) \otimes H^{2-n}_{Gal}(F; \mathbb{F}_p(1-k)) \xrightarrow{\cup} H^2_{Gal}(F; \mathbb{F}_p(1)) \xrightarrow[\cong]{inv} \mathbb{Z}/p$$

defines a perfect pairing, so that the adjoint map

$$H^{2-n}_{Gal}(F; \mathbb{F}_p(1-k)) \xrightarrow{\cong} H^n_{Gal}(F; \mathbb{F}_p(k))^*$$

(to the dual $\mathbb{F}_p$-vector space) is an isomorphism.

In the example $F = \mathbb{Q}_p$, we can see this duality in Figures 1 and 2 above. The theorem also applies with more general p-torsion coefficients. In particular, $H^n_{Gal}(F; \mathbb{F}_p(k)) = 0$ for all $n \geq 3$.

For comparison, a closed connected m-manifold M has a fundamental class $[M] \in H_m(M; \mathbb{Z}^t)$, where $\mathbb{Z}^t$ means $\pi_1(M)$-twisted coefficients taking orientations into account, evaluation at $[M]$ defines an isomorphism $H^m(M; \mathbb{Z}^t) \cong \mathbb{Z}$, and the adjoint to the cup product $H^n(M; \mathbb{Z}) \otimes H^{m-n}(M; \mathbb{Z}^t) \to H^m(M; \mathbb{Z}^t)$ leads to the Poincaré duality isomorphisms

$$H^{m-n}(M; \mathbb{Z}^t) \cong H_n(M; \mathbb{Z})\,.$$

In the orientable case, $\mathbb{Z}^t = \mathbb{Z}$.

There is a similar Tate duality for the Galois cohomology of a global number field F, relative to all of its localizations F_v, which is more like the Lefschetz duality of a compact 3-manifold relative to two parts of its boundary.

There is also an arithmetic duality theorem for higher local fields. A 0-local field is a finite field, and for $d \geq 1$ a d-local field is a complete discrete valuation field with a $(d-1)$-local residue field. A 1-local field is a non-archimedean local field in the usual sense. Typical 2-local fields are the finite extensions of $\mathbb{F}_p((X))((Y))$ and $\mathbb{Q}_p((X))$. For d-local fields F such that all of the residue fields are of characteristic $\neq p$, other than the finite field,, there is a canonical isomorphism

$$H^{d+1}_{Gal}(F; \mathbb{F}_p(d)) \xrightarrow{\cong} \mathbb{Z}/p$$

and the cup product pairing

$$H^n_{Gal}(F; \mathbb{F}_p(k)) \otimes H^{d+1-n}_{Gal}(F; \mathbb{F}_p(d-k)) \xrightarrow{\cup} H^{d+1}_{Gal}(F; \mathbb{F}_p(d))$$

is a perfect pairing, so that the adjoint map

$$H^{d+1-n}_{Gal}(F; \mathbb{F}_p(d-k)) \xrightarrow{\cong} H^n_{Gal}(F; \mathbb{F}_p(k))^*$$

is an isomorphism [Deninger–Wingberg (1986)]. There is a higher class field theory, which classifies abelian extensions of d-local fields F by norm subgroups of the Milnor K-group $K^M_d(F)$

Chromatic localization. We return to the sphere spectrum S. The above discussion suggests that we ask: Does there exist a notion of motivic cohomology

$$H^n_{mot}(R; M(k))$$

for suitable S-algebras R and coefficients $M(k)$, such that there is a spectral sequence

$$E^2_{s,t} = H^{-s}_{mot}(R; M(t/2)) \Longrightarrow K_{s+t}(R)\,?$$

Presumably it helps to assume that R is a commutative S-algebra, since this is needed in the algebraic setting.

In the algebraic case, with a one-dimensional base B (such as a Dedekind domain), Zariski localization techniques suffice to reduce to the field case, in which case we could compare motivic cohomology with Galois cohomology. However, it is not a commonly held belief that the sphere spectrum S has Krull dimension one. Indeed, the p-local stable homotopy category $\mathcal{S}_{(p)}$, i.e., the homotopy category of modules over the p-local sphere spectrum, contains an infinite descending sequence of colocalizing subcategories

$$\mathcal{S}_{(p)} \supset \cdots \supset \mathcal{L}_n \supset \mathcal{L}_{n-1} \supset \cdots \supset \mathcal{L}_0$$

where $\mathcal{L}_n$ is the homotopy category of $E(n)$-local spectra, which is equivalent to the homotopy category of L_nS-module spectra. It suggests that $S_{(p)}$ has infinite Krull dimension.

Also, if we wish to consider $S_{(p)}[\frac{1}{p}] = H\mathbb{Q}$ to be the fraction field of $S_{(p)}$, then what is the residue field? The Moore spectrum $V(0) = S/p = S_{(p)}/p$ is not a strictly associative S-algebra for any prime p, so it does not have a good category of modules and we cannot talk about its algebraic K-theory.

Instead, we are led to filter the objects of $\mathcal{S}$ by the following tower of p-local chromatic localization functors L_n and their "integral" pullbacks $\tilde{L}_n$:

$$\begin{array}{ccccccccc}
X & \longrightarrow \cdots \longrightarrow & \tilde{L}_nX & \longrightarrow & \tilde{L}_{n-1}X & \longrightarrow \cdots \longrightarrow & X[\frac{1}{p}] \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
X_{(p)} & \longrightarrow \cdots \longrightarrow & L_nX & \longrightarrow & L_{n-1}X & \longrightarrow \cdots \longrightarrow & L_0X
\end{array}$$

where, as usual, $L_nX = L_{E(n)}X$ is the Bousfield localization of X with respect to the Johnson–Wilson spectrum $E(n)$ with $\pi_*E(n) = \mathbb{Z}_{(p)}[v_1, \dots, v_{n-1}, v_n^{\pm 1}]$. For example, $\pi_0 L_1 S = \mathbb{Z}_{(p)}$ but $\pi_0 \tilde{L}_1 S = \mathbb{Z}$. By chromatic convergence [Ravenel (1992)], the map

$$X_{(p)} \xrightarrow{\simeq} \operatorname*{holim}_n L_nX$$

is a weak equivalence for finite cell S-modules X. This suggests, but does not prove, that the induced map

$$K(S_{(p)}) \to \operatorname*{holim}_n K(L_nS)$$

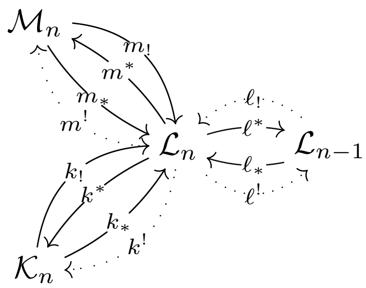
is a weak equivalence.

Following [Hovey–Strickland (1999)], we write M_n for the n-th monochromatic functor, defined by the (co-)fiber sequence

$$M_nX \to L_nX \to L_{n-1}X\,.$$

Let $\mathcal{K}_n$ be the homotopy category of $K(n)$-local spectra, where (also as usual) $K(n)$ is the Morava K-theory spectrum with $\pi_* K(n) = \mathbb{F}_p[v_n^{\pm 1}]$, and let $\mathcal{M}_n$ be the homotopy category of n-monochromatic spectra, i.e., those weakly equivalent to spectra of the form $M_n X$. We now lift the functors L_{n-1}, $L_{K(n)}$ and M_n to land in $\mathcal{L}_{n-1}$, $\mathcal{K}_n$ and $\mathcal{M}_n$, respectively, and write U for the various forgetful functors.

Following Grothendieck, or [Milne (1980)], there are the following functors between these homotopy categories, each left adjoint to the one below it:



Here $k_* = U$, $\ell_* = U$ and $m_! = U$ are forgetful functors, $k^* = L_{K(n)}$ and $\ell^* = L_{n-1}$ are the localization functors left adjoint to k_* and ℓ_*, respectively, $m^* = M_n$ is the acyclization functor right adjoint to $m_!$, and $k_! = UM_nU$ and $m_* = UL_{K(n)}U$.

The composites $k^* m_! \colon \mathcal{M}_n \to \mathcal{K}_n$ and $m^* k_* \colon \mathcal{K}_n \to \mathcal{M}_n$ are inverse equivalences of homotopy categories, given by $L_{K(n)}$ and M_n on the underlying spectra. There is a natural (co-)fiber sequence

$$k_! k^* X \to X \to \ell_* \ell^* X$$

for each $E(n)$-local X, namely the defining sequence for $M_n X$. Here $k_! k^* X \simeq m_! m^* X$ is part of the equivalence just mentioned.

There are no functors $k^!$ ("form subsheaf of sections with support on the image of k") or $m^!$ right adjoint to $k_* = U$ or $m_* = UL_{K(n)}U$, respectively, since the forgetful functor from $\mathcal{K}_n$ to $\mathcal{L}_n$ does not preserve infinite coproducts (they are implicitly $K(n)$-localized in $\mathcal{K}_n$). This is unfamiliar from the algebraic situation, where this subsheaf remains quasi-coherent for closed immersions. It may be possible to fix this by jumping a step ahead, replacing $\mathcal{K}_n$ by a different category of E_n-module spectra and $\mathbb{G}_n$-Galois descent data. This uses that $L_{K(n)} S \to E_n$ is a $K(n)$-local pro-$\mathbb{G}_n$-Galois extension, where $\mathbb{G}_n = \mathbb{S}_n \rtimes \mathrm{Gal}(\mathbb{F}_{p^n} | \mathbb{F}_p)$ is the extended Morava stabilizer group.

There is also no functor $\ell_!$ ("extension by zero") left adjoint to $\ell^* = L_{n-1}$, since the $E(n-1)$-localization functor from $\mathcal{L}_n$ to $\mathcal{L}_{n-1}$ does not preserve infinite products ($(\prod^\infty \mathbb{Z}) \otimes \mathbb{Q} \neq \prod^\infty \mathbb{Q}$). This is familiar from the algebraic case, since extension by zero does not preserve quasi-coherent sheaves for open immersions.

The existence of the functors $k_! = UM_nU$ and $m_! = U$ ("extension by zero") left adjoint to $k^* = L_{K(n)}$ and $m^* = M_n$, respectively, is also unfamiliar from algebra, since there are usually no such functors (within modules/quasi-coherent sheaves) for closed immersions. Again, it may be possible to fix this by replacing $\mathcal{K}_n$ by a similar, but inequivalent, category of Galois descent data.

Finally, there is no (co-)fiber sequence

$$\ell_! \ell^* X \to X \to k_* k^* X \,,$$

in $\mathcal{L}_n$, since the homotopy fiber of $X \to L_{K(n)}X$ does not just depend on $L_{n-1}X$, e.g. for $X = M_nS$. It may make more sense in a category of sheaves, or when restricted to small $E(n)$-local spectra, which are what really matters for the purpose of algebraic K-theory.

Taken together, this is most similar to the (Grothendieck) six functor formalism

$$Sh(Z) \underset{\xleftarrow[\ell^!]{}}{\overset{\xleftarrow{\ell^*}}{\xrightarrow{\ell_*}}} Sh(X) \underset{\xleftarrow[k_*]{}}{\overset{\xleftarrow{k_!}}{\xrightarrow{k^*}}} Sh(U)$$

associated to a diagram

$$Z = \operatorname{Spec} L_{n-1}S \xrightarrow{\ell} X = \operatorname{Spec} L_nS \xleftarrow{k} U = \operatorname{Spec} L_{K(n)}S$$

with ℓ a closed immersion and k an open immersion, where $\operatorname{Spec} L_nS$ is the disjoint union of the images of k and ℓ. Some care should be taken, since by $\operatorname{Spec} L_{K(n)}S$ we do not mean a structure space with quasi-coherent sheaves consisting of all $L_{K(n)}S$-modules, but only the full subcategory of $K(n)$-local S-modules.

However, it appears more familiar to consider the smashing localizations L_n as restrictions to open subsets, as in

$$Sh(Z) \underset{\xleftarrow[k^!]{}}{\overset{\xleftarrow{k^*}}{\xrightarrow{k_*}}} Sh(X) \underset{\xleftarrow[\ell_*]{}}{\overset{\xleftarrow{\ell_!}}{\xrightarrow{\ell^*}}} Sh(U)$$

associated to a diagram

$$Z = \operatorname{Spec} L_{K(n)}S \xrightarrow{k} X = \operatorname{Spec} L_nS \xleftarrow{\ell} U = \operatorname{Spec} L_{n-1}S$$

with k a closed immersion and ℓ an open immersion. In this case, by $\operatorname{Spec} L_{K(n)}S$ we must mean something like a structure space with quasi-coherent sheaves consisting of E_n-modules with $\mathbb{G}_n$-Galois descent data. The details of how this could work remain to be worked out.

In the first case, the analogue of the coherent sheaves on these structure spaces may be played by the category of (small or) dualizable $K(n)$-local spectra over U, the category of small (= dualizable) $E(n)$-local spectra over X and the category of small (= dualizable) $E(n-1)$-local spectra over Z. We may then expect to have a localization sequence

$$K(L_{n-1}S) \xrightarrow{\ell_*} K(L_nS) \xrightarrow{k^*} K^d(L_{K(n)}S)$$

where $K^d(-)$ denotes algebraic K-theory built from dualizable module spectra.

Therefore, to work one's way iteratively from $K(L_0S) = K(\mathbb{Q})$ to $K(S_{(p)})$, or from $K(\tilde{L}_0S) = K(\mathbb{Z}[\frac{1}{p}])$ to $K(S)$, we must understand the algebraic K-theory of $L_{K(n)}S$ to go from $K(L_{n-1}S)$ to $K(L_nS)$.

In the second case, we would expect a localization sequence

$$K^d(L_{K(n)}S) \xrightarrow{k_*} K(L_nS) \xrightarrow{\ell^*} K(L_{n-1}S)$$

where $K^d(-)$ now means algebraic K-theory of the category of $(E_n, \mathbb{G}_n)$-Galois descent data.

Galois descent. As already referred to above, there is a notion of a G-Galois extension $A \to B$ of commutative S-algebras, meaning that natural maps $i\colon A \to B^{hG}$ and $h\colon B \wedge_A B \to \prod_G B$ are weak equivalences [Rognes (2005)], and it is then reasonable to expect that the comparison map

$$K(A) \to K(B)^{hG}$$

is close to an equivalence, after suitable completion. More generally, there is a notion of an E-local G-Galois extension, for a spectrum E, meaning that the maps i and h become weak equivalences after E-localization. In this case it is more reasonable to expect that the comparison map

$$K^d(A) \to K^d(B)^{hG}$$

is close to an equivalence, where now $K^d(-)$ indicates that the finiteness condition imposed on A- and B-modules is appropriate for the E-local category. Concretely, one might consider the dualizable E-local A- and B-modules. This is then the analogue for E-local commutative S-algebras of the Lichtenbaum–Quillen conjectures.

As an application of Morava's change-of-rings theorem, Devinatz–Hopkins show that there is a direct system of $K(n)$-local $\mathbb{G}_n/U_i$-Galois extensions

$$L_{K(n)}S \to E_n^{hU_i}$$

for a sequence of open normal subgroups $U_i \subset \mathbb{G}_n$ of the extended Morava stabilizer group. Here E_n is the Lubin–Tate spectrum with

$$\pi_* E_n = \mathbb{W}\mathbb{F}_{p^n}[[u_1, \dots, u_{n-1}]][u^{\pm 1}]\,,$$

such that $\pi_0 E_n$ represents the universal deformation of the height n Honda formal group law over $\mathbb{F}_{p^n}$. Taken together, these Galois extensions exhibit

$$L_{K(n)}S \to E_n$$

as a $K(n)$-local pro-$\mathbb{G}_n$-Galois extension of commutative S-algebras. By the Galois descent conjecture above, we are then expecting the comparison map

$$K^d(L_{K(n)}S) \to K^d(E_n)^{h\mathbb{G}_n}$$

to be close to an equivalence. Hovey has shown (along the lines of [Hovey–Strickland (1999)] that each dualizable $K(n)$-local E_n-module is a retract of a finite cell E_n-module, i.e., a small E_n-module, so after extending to E_n there is no difference between the natural $K(n)$-local and the global finiteness conditions on E_n-modules. Thus $K^d(E_n) = K(E_n)$.

The task of piecing together $K(S)$ from each $K^d(L_{K(n)}S)$ and the expected localization sequences above is what Zariski hypercohomology is good for. The additional task of replacing $K^d(L_{K(n)}S)$ by $K(E_n)^{h\mathbb{G}_n}$, using Galois descent, is the purpose of étale hypercohomology.

What remains is to understand $K(E_n)$.

Connective covers. We concentrate on the first non-algebraic case, $n = 1$. Let KU be the complex K-theory spectrum, with $\pi_*KU = \mathbb{Z}[u^{\pm 1}]$, $|u| = 2$, let $L = E(1)$ be the Adams summand of its p-localization $KU_{(p)}$, with $\pi_*L = \mathbb{Z}_{(p)}[v_1^{\pm 1}]$, $|v_1| = 2p - 2$, and let KU_p and L_p be their respective p-adic completions. Then $E_1 = KU_p$ is the p-complete complex K-theory spectrum. To simplify the following calculations, we will instead work with the Adams summand L_p, although the results for KU_p are not expected to be qualitatively different. Following [Bökstedt–Madsen (1995)], it is also easy to make the corresponding calculations for the (p-locally) unramified extensions obtained by adjoining root of unity of order prime to p.

There is a discrete valuation on π_*KU, with uniformizer u, valuation ring $\pi_*ku = \mathbb{Z}[u]$ and residue ring $\pi_0 H\mathbb{Z} = \mathbb{Z}$. It is therefore reasonable to view the connective complex K-theory spectrum ku as a commutative valuation S-algebra of KU, with commutative residue S-algebra $H\mathbb{Z}$. Associated to the diagram

$$ku/u = H\mathbb{Z} \xleftarrow{\pi} ku \to KU = u^{-1}ku$$

(π might suggest the zero-th Postnikov section) we can then form the S-algebraic localization sequence

$$K(\mathbb{Z}) \xrightarrow{\pi_*} K(ku) \to K(KU)\,.$$

The author's conjecture that this is indeed a cofiber sequence has been verified by Blumberg–Mandell.

The same proof will show that there is a cofiber sequence

$$K(\mathbb{Z}_p) \xrightarrow{\pi_*} K(\ell_p) \to K(L_p)$$

associated to the diagram of commutative S-algebras

$$\ell_p/v_1 = H\mathbb{Z}_p \xleftarrow{\pi} \ell_p \to L_p = v_1^{-1}\ell_p$$

where ℓ_p is the connective cover of L_p, with $\pi_*\ell_p = \mathbb{Z}_p[v_1]$. It is the p-completion of the connective p-local Adams summand $\ell = BP\langle 1\rangle$.

The advantage of this sequence is that it translates algebraic K-theory questions about the periodic spectrum L_p to questions about the p-complete connective S-algebras ℓ_p and $H\mathbb{Z}_p$. These are precisely the questions that can be answered by the cyclotomic trace map trc to topological cyclic homology. For each connective ring spectrum R with $\pi_0 R_p = \mathbb{F}_p$ or $\pi_0 R_p = \mathbb{Z}_p$, this map sits in a cofiber sequence

$$K(R_p)_p \xrightarrow{trc} TC(R)_p \to \Sigma^{-1} H\mathbb{Z}_p$$

[Hesselholt–Madsen (1997)], so a calculation of $\pi_*TC(R)_p$ is just as good as a calculation of $\pi_*K(R_p)_p$. Here $TC(R)_p \simeq TC(R_p)_p$, but usually $K(R)_p \not\simeq K(R_p)_p$.

In a way we are now back to the beginning, replacing a question about $K(S)$ with a question about topological cyclic homology. However, for spectra like $R = \ell$, the answer about $TC(\ell)$ is much more systematically understood than the answer about $TC(S)$

Just as the formulas for algebraic K-theory and Galois cohomology of number fields were easier to express with mod p coefficients, the formula for the algebraic K-theory of ℓ_p is easier to express with mod (p, v_1) coefficients, also known as

$V(1)$-homotopy. Here $V(1) = S/(p, v_1)$ is the Smith–Toda complex, defined by the cofiber sequence

$$\Sigma^{2p-2} V(0) \xrightarrow{v_1} V(0) \to V(1) \,.$$

It is a homotopy commutative ring spectrum for $p \ge 5$, which we now assume. It admits a periodic self-map $v_2 \colon \Sigma^{2p^2-2} V(1) \to V(1)$, which induces a natural module structure over the polynomial ring $P(v_2)$ on each $V(1)_*(X) = \pi_*(V(1) \wedge X)$.

Direct calculations, starting with topological Hochschild homology and its circle action, and ascending to topological cyclic homology through a tower of cyclic fixed point spectra, leads to the formulas

$$V(1)_* TC(\mathbb{Z}) = E(\partial, \lambda_1) \oplus \mathbb{F}_p\{t^i \lambda_1 \mid 0 < i < p\}$$

of [Bökstedt–Madsen (1992/1995)], and

$$\begin{aligned} V(1)_* TC(\ell) = P(v_2) \otimes E(\partial, \lambda_1, \lambda_2) \\ \oplus P(v_2) \otimes E(\lambda_2) \otimes \mathbb{F}_p\{t^i \lambda_1 \mid 0 < i < p\} \\ \oplus P(v_2) \otimes E(\lambda_1) \otimes \mathbb{F}_p\{t^{ip} \lambda_2 \mid 0 < i < p\} \end{aligned}$$

of [Ausoni–Rognes (2002)]. Here $|v_2| = 2p^2-2$, $|\partial| = -1$, $|\lambda_1| = 2p-1$, $|\lambda_2| = 2p^2-1$ and $|t| = -2$. By the localization sequence, we can then deduce the formula

$$\begin{aligned} V(1)_* TC(L) = P(v_2) \otimes E(\partial, \lambda_1, \delta_2) \\ \oplus P(v_2) \otimes E(\delta_2) \otimes \mathbb{F}_p\{t^i \lambda_1 \mid 0 < i < p\} \\ \oplus P(v_2) \otimes E(\lambda_1) \otimes \mathbb{F}_p\{t^{ip} \lambda_2 \mid 0 < i < p\} \,, \end{aligned}$$

where $\delta_2 = d \log v_1$ (a secondary logarithmic pole) is in degree 1 and satisfies $v_2 \delta_2 = \lambda_2$.

This is then also a formula for $V(1)_* K(L_p)$, except in some low degrees. If we use it to conjecture formulas for mod (p, v_1) motivic cohomology groups of of L_p, fitting in a spectral sequence

$$E^2_{s,t} = H^{-s}_{mot}(L_p; \mathbb{F}_{p^2}(t/2)) \Longrightarrow V(1)_{s+t}(L_p)$$

then we see that $H^n_{mot}(L_p; \mathbb{F}_{p^2}(t/2))$ is indeed $(2p^2-2)$-periodic in t, which partially explains the use of the symbol $\mathbb{F}_{p^2}(t/2)$ for the coefficients, but there are problems with the motivic truncations that we saw for the motivic cohomology of fields.

The fraction field of topological K-theory. The reason is, of course, that L_p is not an S-algebraic field. At the level of coefficients, $\pi_* L_p = \mathbb{Z}_p[u^{\pm 1}]$ has the maximal ideal (p). A first idea might be that the corresponding fraction field $\mathit{ff}(L_p) = p^{-1} L_p$ would have coefficient ring $p^{-1}\mathbb{Z}_p[u^{\pm 1}] = \mathbb{Q}_p[u^{\pm 1}]$. But this would make $p^{-1} L_p$ a rational spectrum, and an algebra over $H\mathbb{Q}_p$. Thus $V(1)_* K(p^{-1} L_p)$ would be a module over $V(1)_* K(\mathbb{Q}_p)$, which is finite and v_2-torsion, so also $V(1)_* K(p^{-1} L_p)$ would be v_2-torsion. This would make it hopeless to recover the v_2-periodicity in $V(1)_* K(L_p)$ from the fraction field.

Something else must be going on. A hint can be obtained from the diagram

$$\ell/p = k(1) \xleftarrow{i} \ell_p \to p^{-1} \ell_p \,.$$

Here ℓ_p is a commutative S-algebra and $\ell/p = k(1)$ is the first connective Morava K-theory. It admits a unique structure as an associative ℓ_p-algebra (following [Lazarev (2001)], and thus is an associative S-algebra, but it is not a commutative S-algebra. It is not even an E_2 ring spectrum.

So ℓ/p must be considered as the associative residue ring of ℓ_p associated to the two-sided ideal (p), rather than the commutative residue ring one usually obtains in commutative algebra. Therefore, to have a localization sequence

$$K(\ell/p) \xrightarrow{i_*} K(\ell_p) \to K(p^{-1}\ell_p)$$

based on the diagram above, we must interpret $p^{-1}\ell_p$ as an associative (also known as non-commutative) localization of ℓ_p. Geometrically, $\operatorname{Spec} p^{-1}\ell_p$ is to be obtained from the commutative structure space $\operatorname{Spec} \ell_p$ by cutting out the closed non-commutative (sub-)space $\operatorname{Spec} \ell/p$.

It remains to decide whether such a non-commutative localization $p^{-1}\ell_p$ can be made to exist within affine S-algebras, or whether a global S-algebro-geometric formalism is required. However, it is clear that however $p^{-1}\ell_p$ is to be constructed, its algebraic K-theory should be given by the localization sequence above.

Similar remarks apply in the periodic setting. In the diagram

$$L/p = K(1) \xleftarrow{i} L_p \to f\!f(L_p) = p^{-1}L_p$$

the first Morava K-theory spectrum $K(1)$ is uniquely an associative L_p-algebra, but it is not commutative. Therefore L_p only has a non-commutative residue ring, and its fraction field $f\!f(L_p) = p^{-1}L_p$ must come from a non-commutative localization of L_p. Then we will have a localization sequence

$$K(L/p) \xrightarrow{i_*} K(L_p) \to K(p^{-1}L_p)\,.$$

All of these maps fit together in the square diagram of associative S-algebras

$$\begin{array}{ccccc}
\mathbb{F}_p & \xleftarrow{\;i\;} & \mathbb{Z}_p & \longrightarrow & \mathbb{Q}_p \\
\uparrow{\scriptstyle\pi} & & \uparrow{\scriptstyle\pi} & & \uparrow{\scriptstyle\pi} \\
\ell/p & \xleftarrow{\;i\;} & \ell_p & \longrightarrow & p^{-1}\ell_p \\
\downarrow & & \downarrow & & \downarrow \\
L/p & \xleftarrow{\;i\;} & L_p & \longrightarrow & p^{-1}L_p
\end{array}$$

where the upper left hand square is a pushout. Geometrically, $\operatorname{Spec} \mathbb{Z}_p$ and $\operatorname{Spec} \ell/p$ sit as two closed subspaces in $\operatorname{Spec} \ell_p$, meeting precisely at $\operatorname{Spec} \mathbb{F}_p$. Cutting out one of the two subspaces yields $\operatorname{Spec} L_p$ and $\operatorname{Spec} p^{-1}\ell_p$; cutting out both yields the generic point of the fraction field, $\operatorname{Spec} p^{-1}L_p$.

At the level of algebraic K-theory, we obtain the following 3×3 diagram of cofiber sequences, which for now must be taken as the definition of the terms $K(p^{-1}\ell_p)$

and $K(p^{-1}L_p)$:

$$\begin{array}{ccccc}
K(\mathbb{F}_p) & \xrightarrow{i_*} & K(\mathbb{Z}_p) & \longrightarrow & K(\mathbb{Q}_p) \\
\downarrow{\scriptstyle \pi_*} & & \downarrow{\scriptstyle \pi_*} & & \downarrow{\scriptstyle \pi_*} \\
K(\ell/p) & \xrightarrow{i_*} & K(\ell_p) & \longrightarrow & K(p^{-1}\ell_p) \\
\downarrow & & \downarrow & & \downarrow \\
K(L/p) & \xrightarrow{i_*} & K(L_p) & \longrightarrow & K(p^{-1}L_p)
\end{array}$$

Again it is possible to make direct calculations with topological cyclic homology. The base case

$$V(1)_*TC(\mathbb{F}_p) = E(\partial, \epsilon_1)$$

of [Hesselholt–Madsen (1997)] has very recently been complemented by joint work of Ausoni–Rognes, where we obtain

$$\begin{aligned}
V(1)_*TC(\ell/p) = P(v_2) \otimes E(\partial, \bar{\epsilon}_1, \lambda_2) \\
&\oplus P(v_2) \otimes E(\delta_2) \otimes \mathbb{F}_p\{t^i v_2 \mid 0 < i < p^2 - p, (i,p) = 1\} \\
&\oplus P(v_2) \otimes E(\bar{\epsilon}_1) \otimes \mathbb{F}_p\{t^{ip}\lambda_2 \mid 0 < i < p\}
\end{aligned}$$

where $|\epsilon_1| = |\bar{\epsilon}_1| = 2p-1$. The transfer map $i_* \colon K(\ell/p) \to K(\ell_p)$ is a module map over the target, which allows us to compute

$$\begin{aligned}
V(1)_*TC(p^{-1}\ell) = P(v_2) \otimes E(\partial, \delta_1, \lambda_2) \\
&\oplus P(v_2) \otimes E(\lambda_2) \otimes \mathbb{F}_p\{t^i \lambda_1 \mid 0 < i < p\} \\
&\oplus P(v_2) \otimes E(\delta_2) \otimes \mathbb{F}_p\{t^i v_2 \delta_1 \mid 0 < i < p^2 - p, (i,p) = 1\} \\
&\oplus P(v_2) \otimes E(\delta_1) \otimes \mathbb{F}_p\{t^{ip}\lambda_2 \mid 0 < i < p\}
\end{aligned}$$

and

$$\begin{aligned}
V(1)_*TC(p^{-1}L) = P(v_2) \otimes E(\partial, \delta_1, \delta_2) \\
&\oplus P(v_2) \otimes E(\delta_2) \otimes \mathbb{F}_p\{t^i \lambda_1 \mid 0 < i < p\} \\
&\oplus P(v_2) \otimes E(\delta_2) \otimes \mathbb{F}_p\{t^i v_2 \delta_1 \mid 0 < i < p^2 - p, (i,p) = 1\} \\
&\oplus P(v_2) \otimes E(\delta_1) \otimes \mathbb{F}_p\{t^{ip}\lambda_2 \mid 0 < i < p\}\,.
\end{aligned}$$

where $\delta_1 = d\log p$ is the primary logarithmic pole in degree 1. Putting the pieces together, we obtain:

Theorem (Ausoni–Rognes). *Let $f\!f(L_p) = p^{-1}L_p$ be the S-algebraic fraction field of the p-complete Adams summand of complex K-theory, for a prime $p \geq 5$. Then there is an exact sequence*

$$\begin{aligned}
0 \to E(\delta_1, \delta_2)\{\Sigma^{-2}\epsilon_1\} \to V(1)_*K(p^{-1}L_p) \\
\xrightarrow{trc} V(1)_*TC(p^{-1}L) \to E(\delta_1, \delta_2)\{\Sigma^{-1}1\} \to 0\,.
\end{aligned}$$

Thus

$$\begin{aligned} V(1)_*K(p^{-1}L_p) \equiv\ & P(v_2)\otimes E(\partial v_2,\delta_1,\delta_2) \\ & \oplus P(v_2)\otimes E(\delta_2)\otimes \mathbb{F}_p\{t^i\lambda_1 \mid 0<i<p\} \\ & \oplus P(v_2)\otimes E(\delta_2)\otimes \mathbb{F}_p\{t^i v_2\delta_1 \mid 0<i<p^2-p, (i,p)=1\} \\ & \oplus P(v_2)\otimes E(\delta_1)\otimes \mathbb{F}_p\{t^{ip}\lambda_2 \mid 0<i<p\} \end{aligned}$$

modulo the kernel of trc. This is a free module over $P(v_2)$ of rank $(2p^2+6)$ and zero Euler characteristic.

We arrange these groups as the hypothetical $E^2 = E^\infty$ term of a motivic spectral sequence

$$E^2_{s,t} = H^{-s}_{mot}(p^{-1}L_p;\mathbb{F}_{p^2}(t/2)) \Longrightarrow V(1)_{s+t}K(p^{-1}L_p)$$

(modulo $\ker(trc)$). It is a free module over $P(v_2)$ on the generators displayed in Figure 3 below, where v_2 is represented in bidegree $(s,t)=(0,2p^2-2)$. The odd rows are omitted, and we cheat a little by drawing the case $p=3$. There is no room for differentials, since the powers of v_2 are infinite cycles.

$\partial v_2\delta_1\delta_2$	·	·	·	$2p^2+2$
·	$\partial v_2\delta_1 : \partial v_2\delta_2 : t\lambda_2\delta_1$	·	·	$2p^2$
·	$t^2\lambda_2\delta_1$	$\partial v_2 : tv_2\delta_1$	·	$2p^2-2$
·	$t^p\lambda_2\delta_1$	$t^2v_2\delta_1$	·	
·	$t^{p+1}\lambda_2\delta_1$	$t^p\lambda_2$	·	
·	$t^{p+2}\lambda_2\delta_1$	$t^{p+1}v_2\delta_1$	·	
·	$t^{2p}\lambda_2\delta_1$	$t^{p+2}v_2\delta_1$	·	
·	$t\lambda_1\delta_2$	$t^{2p}\lambda_2$	·	$2p$
·	$\delta_1\delta_2 : t^2\lambda_1\delta_2$	$t\lambda_1$	·	$2p-2$
·	·	$\delta_1 : \delta_2 : t^2\lambda_1$	·	2
·	·	·	1	0
-3	-2	-1	0	$s\backslash t$

FIGURE 3

Motivic truncation for topological K-theory. At this point, only after passing from ℓ_p via L_p or $p^{-1}\ell_p$ to $p^{-1}L_p$, have we recovered the same motivic truncation pattern, along the line $2s+t=0$ of slope (-2), as in the algebraic case of fields. This confirms the impression that only $p^{-1}L_p$ plays the role of an S-algebraic field.

If we define the Galois cohomology groups

$$H^n_{Gal}(p^{-1}L_p;\mathbb{F}_{p^2}(k)) = v_2^{-1}H^n_{mot}(p^{-1}L_p;\mathbb{F}_{p^2}(k))$$

as the v_2-inverted form of the above motivic cohomology groups, then the calculational result proves that we recover the motivic groups by the same formula as before:

$$H^n_{mot}(p^{-1}L_p;\mathbb{F}_{p^2}(k)) \cong \begin{cases} H^n_{Gal}(p^{-1}L_p;\mathbb{F}_{p^2}(k)) & \text{for } n \le k, \\ 0 & \text{for } n > k. \end{cases}$$

In this sense, the algebraic K-theory of the fraction field of L_p satisfies motivic truncation.

We may reasonably expect that

$$H^n_{Gal}(p^{-1}L_p;\mathbb{F}_{p^2}(k)) = H^n_{cont}(G;\mathbb{F}_{p^2}(k))$$

for a profinite group G and a discrete G-module $\mathbb{F}_{p^2}(k) = \mathbb{F}_{p^2}(1)^{\otimes k}$. It remains to determine in what sense G is the absolute Galois group of $p^{-1}L_p$, and how it naturally acts on $\mathbb{F}_{p^2}(1)$. Suppose that there is a separably closed pro-Galois extension Ω_1 of $p^{-1}L_p$, with Galois group $G = G_{p^{-1}L_p}$ and

$$v_2^{-1}V(1)_*K(\Omega_1) = V(1)_*E_2 = \mathbb{F}_{p^2}[u^{\pm 1}]$$

with $|u| = 2$. Then the homotopy fixed point spectral sequence for $G_{p^{-1}L_p}$ acting on $v_2^{-1}V(1)_*K(\Omega_1)$ would agree with the v_2-inverted motivic spectral sequence.

Since $H^1_{mot}(p^{-1}L_p;\mathbb{F}_{p^2}(t/2))$ is nonzero for all even $t \ge 2$, we can expect that $v_2^{-1}V(1)_*K(\Omega_1)$ is 2-periodic, with $v_2 = u^{p^2-1}$ for a class u in degree 2. There exists a class $b \in V(1)_{2p+2}K(ku)$ with $b^{p-1} = -v_2$ [Ausoni], and complex K-theory will map to Ω_1, so we must expect that $b = \alpha u^{p+1}$ in $v_2^{-1}V(1)_{2p+2}K(\Omega_1)$ for some α with $\alpha^{p-1} = -1$, which implies $\alpha \in \mathbb{F}_{p^2} \setminus \mathbb{F}_p$. This is a second reason to suppose that each group $v_2^{-1}V(1)_tK(\Omega_1)$ equals $\mathbb{F}_{p^2}(t/2)$ for t even, rather than just $\mathbb{F}_p(t/2)$.

By analogy with the consequence $L_{K(1)}K(\bar{F}) = KU_p = E_1$ of Suslin's theorem, we are led to the following motivated guess.

Conjecture. *There is a separable closed pro-Galois extension Ω_1 of the fraction field $p^{-1}L_p$ (and thus of the fraction field $p^{-1}KU_p$), with*

$$L_{K(2)}K(\Omega_1) \simeq E_2\,.$$

In a similar vein, one may speculate that there is a fraction field $p^{-1}E_n$, with separable closure Ω_n, such that $L_{K(n+1)}K(\Omega_n) \simeq E_{n+1}$ for each $n \ge 1$. The p-complete spectrum $e_{n+1} := K(\Omega_n)_p$ should then qualify as the "decent" connective version of E_{n+1}.

Arithmetic duality for topological K-theory. There is an isomorphism

$$H^3_{Gal}(p^{-1}L_p; \mathbb{F}_{p^2}(2)) \cong \mathbb{F}_p\{\partial\delta_1\delta_2\}\,.$$

Figure 3 strongly suggests that the cup product

$$H^n_{Gal}(p^{-1}L_p; \mathbb{F}_{p^2}(k)) \otimes H^{3-n}_{Gal}(p^{-1}L_p; \mathbb{F}_{p^2}(2-k)) \xrightarrow{\cup} H^3_{Gal}(p^{-1}L_p; \mathbb{F}_{p^2}(2))$$

in the Galois cohomology of $p^{-1}L_p$ defines a perfect pairing, so that the adjoint map

$$H^{3-n}_{Gal}(p^{-1}L_p; \mathbb{F}_{p^2}(2-k)) \xrightarrow{\cong} H^n_{Gal}(p^{-1}L_p; \mathbb{F}_{p^2}(k))^*$$

is an isomorphism. The remaining multiplicative relations that need to be established in the Galois cohomology of $p^{-1}L_p$ may follow from those in the Galois cohomology of $\mathbb{Q}_p$ by power operations.

This constitutes arithmetic duality for $p^{-1}L_p$, and emphasizes the arithmetic role of $p^{-1}L_p$ among $K(1)$-local S-algebras, somewhat like that of local number fields among $\mathbb{Q}_p$-algebras. It is not equally present in the v_2-inverted motivic cohomology of ℓ_p or L_p.

Furthermore, the target group of the duality pairing is in the same cohomological degree and with the same twist as that of the duality pairing for a 2-local field. We consider $p^{-1}L_p$ as a complete discretely valued S-algebraic field, with uniformizer v_1, valuation ring $p^{-1}\ell_p$ and residue ring the (1-)local field $p^{-1}\mathbb{Z}_p = \mathbb{Q}_p$. In this sense it is then a "brave new" 2-local field, mixing three different characteristics.

Unramified extensions. For each local number field F containing $\mathbb{Q}_p$, with ring of integers $\mathcal{O}_F$, there exists an associative ℓ_p-algebra $\ell\mathcal{O}_F$, with $\pi_*\ell\mathcal{O}_F = \mathcal{O}_F[v_1]$, by the A_∞ obstruction theory of [Robinson (1989)] and a Hochschild cohomology computation, using e.g. [Larsen–Lindenstrauss (1992)]. This ℓ_p-algebra is not at all unique.

When F is unramified over $\mathbb{Q}_p$, so that $\mathcal{O}_F$ is étale over $\mathbb{Z}_p$, then the E_∞ obstruction theory of [Goerss–Hopkins (2004)] or [Robinson (2003)] can be effectively applied (the André–Quillen- or Γ-cohomology groups vanish), and shows that $\ell\mathcal{O}_F$ is uniquely realized as a commutative ℓ_p-algebra. The maximal unramified extension $\mathbb{Q}_p^{nr} = \mathbb{Q}_p(\mu_{\infty,p})$ is that obtained by adjoining all root of unity of order prime to p. Writing $\mathbb{Z}_p^{nr} = \mathbb{Z}_p(\mu_{\infty,p})$ for its ring of integers $\mathcal{O}_{\mathbb{Q}_p^{nr}}$, we can realize the $\hat{\mathbb{Z}}$-Galois extension $\ell_p \to \ell\mathbb{Z}_p^{nr}$. At the level of fraction fields, it is the maximal unramified extension with respect to the p-adic valuation.

Optimistically, the possible obstructions to existence and uniqueness of $\ell\mathcal{O}_F$ as a commutative ℓ_p-algebra vanish upon localization to $p^{-1}\ell_p$, since in algebraic terms the ramification takes place at (p). Writing $\bar{\mathbb{Z}}_p$ for $\mathcal{O}_{\bar{\mathbb{Q}}_p}$, we will then have that $p^{-1}\ell\bar{\mathbb{Z}}_p$ is a pro-Galois extension of $p^{-1}\ell_p$ with Galois group $G_{\bar{\mathbb{Q}}_p}$. At the level of fraction fields, it is the maximal unramified extension with respect to the discrete valuation with uniformizer v_1. There is an extension

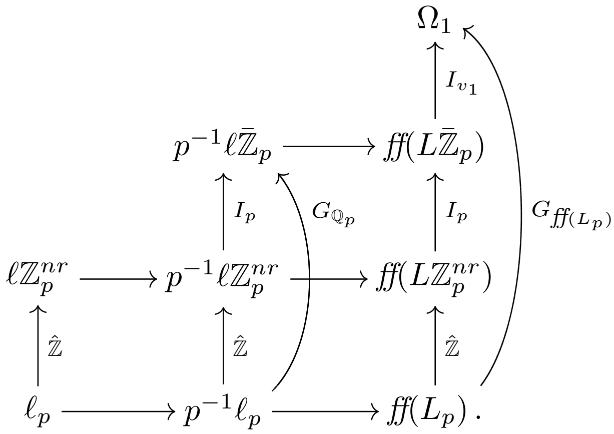
$$I_p \to G_{\mathbb{Q}_p} \to \hat{\mathbb{Z}}$$

where I_p is the inertia group of the p-adic valuation.

We do not expect that $ff(L_p\bar{\mathbb{Z}}_p) = p^{-1}L_p\bar{\mathbb{Z}}_p$ is the full separable closure Ω_1 of $ff(L_p) = p^{-1}L_p$, since the Galois cohomology of $\mathbb{Q}_p$ is smaller than that which we have computed for $p^{-1}L_p$. There is an extension

$$I_{v_1} \to G_{ff(L_p)} \to G_{\mathbb{Q}_p}$$

where I_{v_1} is the inertia group of the discrete valuation with uniformizer v_1.



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