Algebraic K-theory of the fraction field of topological K-theory

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Overview

- Motivation.
- Algebraic K-theory of ℓ.
- The "fraction field" $\mathcal{F}(\ell)$ of ℓ.
- Algebraic K-theory of $\mathcal{F}(\ell)$.
- Speculations.

Algebraic K-theory of $\mathbb{S}$-algebras

We are interested in computing the algebraic K-theory of $\mathbb{S}$-algebras, and more particularly of "easy" $\mathbb{S}$-algebras, like topological K-theory.

Our motivations are

- Waldhausen's A-theory $A(M) \simeq K(\mathbb{S}[\Omega M])$.
- Interpolating from $\mathbb{Z}$ to $\mathbb{S}$ and exploring which structural properties of the algebraic K-theory known for rings also hold when we climb the chromatic tower.
- Study the arithmetic/algebro-geometric properties of $\mathbb{S}$-algebras through their algebraic K-theory, serving as "test" for new concepts.
- Explore the "chromatic red-shift" phenomenon.
- $K(ku)$ represents a form of elliptic cohomology theory derived from a "geometric" construction : the two-vector bundles of Baas, Dundas and Rognes.

Trace methods

The main tool we have so far is the theory of trace maps developed by Waldhausen, Goodwillie, Böktstedt-Hsiang-Madsen, Dundas, Hesselholt, and many others.

If A is a p-completed connective $\mathbb{S}$-algebra, the Bökstedt trace map $\mathrm{tr} : K(A) \to THH(A)$ (topological analogue of the Dennis trace) factors through the cyclotomic trace map to topological cyclic homology

$$K(A) \xrightarrow{\mathrm{trc}} TC(A) \to THH(A)$$

If $\pi_0(A) = \mathbb{Z}_p$ or $\mathbb{F}_p$, then there is a cofibre sequence (Dundas, Hesselholt-Madsen)

$$K(A)_p \xrightarrow{\mathrm{trc}} TC(A) \to \Sigma^{-1} H\mathbb{Z}_p$$

allowing one to evaluate $K(A)_p$ from $TC(A)$.

Connective K-theory

Let p be an odd prime. We consider the commutative $\mathbb{S}$-algebras

- ku, connective complex K-theory (corresponds to the infinite loop space $BU \times \mathbb{Z}$), and ku_p, its p-completion.
- ℓ, the (p-completed) Adams summand of ku_p.

Here $\ell = ku_p^{h\Delta}$ (homotopy fixed points), where $\Delta \cong \mathbb{Z}/(p-1)$ is the torsion subgroup of $\mathbb{Z}_p^*$, acting on ku_p by p-adic Adams operations.

The coefficient rings are

- $ku_* = \mathbb{Z}[u]$ and $ku_p = \mathbb{Z}_p[u]$, with $|u| = 2$ (Bott class),
- $\ell_* = \mathbb{Z}_p[v_1]$, with $|v_1| = 2p - 2$.

The "inclusion" $\ell \to ku_p$ induces $\mathbb{Z}_p[v_1] \subset \mathbb{Z}_p[u]$, $v_1 \mapsto u^{p-1}$. View $\ell \to ku_p$ as a tamely ramified extension with Galois group Δ, analogous to $\mathbb{Z}_p \to \mathbb{Z}_p[\zeta_p]$.

This ramification makes the formulas for $TR(ku)$ more complicated, so in the sequel we concentrate on ℓ.

Finite coefficients and Bott periodicity

The algebraic K-theory of a number field F or of its integers $\mathcal{O}_F$ is much easier to handle if one introduces mod (p) coefficients

$$K_*(F;\mathbb{Z}/p) = V(0)_*K(F) = \pi_*(V(0) \wedge K(F)),$$

where $V(0)$ is the cofibre of $\mathbb{S} \xrightarrow{p} \mathbb{S}$ (notice that $K_*(F;\mathbb{Z}/p)$ is related to $K_*(F)$ by a long exact sequence).

The advantages are that

- divisible subgroups of $K_*(F)$ vanish,
- $V(0)_*K(F)$ is, in high enough degrees, v_1 or "Bott" periodic.

The last fact is accounted for by Suslin's Theorem

$$K(\bar{F}) \simeq_p ku,$$

together with Galois descent, and is the first example of red-shift (explaining why there is no good algebraic definition of algebraic K-theory).

$V(1)$-homotopy

The p-local stable homotopy category features higher forms of periodicity, one for each integer $n \geqslant 0$, referred to as v_n-periodicity.

It is detected by Morava K-theory $K(n)$, with coefficients

$$K(0)_* = \mathbb{Q} \quad \text{and} \quad K(n)_* = \mathbb{F}_p[v_n, v_n^{-1}] \quad \text{if } n \geqslant 1.$$

Here v_n-periodicity has period $|v_n| = 2p^n - 2$.

Because of red-shift, the algebraic K-theory of ℓ or other K-theory spectra is easier to handle by using mod (p, v_1)-coefficients

$$V(1)_* K(\ell) = \pi_*(V(1) \wedge K(F)),$$

where $V(1)$ is the cofibre of $\Sigma^{2p-2} V(0) \xrightarrow{v_1} V(0)$.

Notice that for $p \geqslant 5$, $V(1)$ is a commutative ring spectrum, and has a periodic map $v_2 : \Sigma^{2p^2-2} V(1) \to V(1)$. In particular, for any spectrum X, $V(1)_* X$ is a module over $P(v_2) = \mathbb{F}_p[v_2]$.

- From now on, we assume $p \geqslant 5$.

$V(1)$-homotopy of $THH(\ell)$

Theorem (McCLure-Staffeldt)

There is an isomorphism of $\mathbb{F}_p$-algebras

$$V(1)_* THH(\ell) \cong E(\lambda_1, \lambda_2) \otimes P(\mu_2),$$

where $|\lambda_1| = 2p - 1$, $|\lambda_2| = 2p^2 - 1$ and $|\mu_2| = 2p^2$.

Remark

Angeltveit, Hill and Lawson computed $THH_*(\ell)$ from $V(1)_* THH(\ell)$ by means of Bockstein spectral sequences. The answer shows complicated p- and v_1-torsion patterns.

$V(1)$-homotopy of $K(\ell)$

Theorem (A, Rognes)

There is an isomorphism of $P(v_2)$-modules

$$\begin{aligned} V(1)_*K(\ell) = &P(v_2) \otimes E(\lambda_1, \lambda_2) \\ &\oplus P(v_2) \otimes \mathbb{F}_p\{\partial\lambda_1, \partial v_2, \partial\lambda_2, \partial\lambda_1\lambda_2\} \\ &\oplus P(v_2) \otimes E(\lambda_2) \otimes \mathbb{F}_p\{t^d\lambda_1 \mid 0 < d < p\} \\ &\oplus P(v_2) \otimes E(\lambda_1) \otimes \mathbb{F}_p\{t^{dp}\lambda_2 \mid 0 < d < p\} \\ &\oplus \mathbb{F}_p\{s\} \end{aligned}$$

with $|\partial| = -1$, $|t| = -2$ *and* $|s| = 2p - 3$.

- $V(1)_*K(\ell)$ is almost a free $P(v_2)$-module!

Here we wish to interpret λ_1 and λ_2 as follows: for a suitable fraction field $f : \ell \to \mathcal{F}(\ell)$ we ought to have $f_*(\lambda_1) = v_1\delta_1$ in $V(0)_*K(\mathcal{F}(\ell))$ and $f_*(\lambda_2) = v_2\delta_2$ in $V(1)_*K(\mathcal{F}(\ell))$, with $\delta_1, \delta_2 \in K_1(\mathcal{F}(\ell))$ corresponding to the units p, v_1 in $\mathcal{F}(\ell)$.

Lichtenbaum-Quillen Conjecture for ℓ

A second reason to search for $\mathcal{F}(\ell)$ is to try to understand the above computation of $K(\ell)$ by means of Galois descent, via a generalization of the Lichtenbaum-Quillen Conjectures.

This requires to first pass from ℓ to $\mathcal{F}(\ell)$, because ℓ itself has not enough interesting Galois extensions.

Conjectures (Rognes)

1. *If Ω is a separable closure of $\mathcal{F}(\ell)$, then there is an equivalence*

$$K(\Omega) \simeq_{K(2)} E_2$$

   where E_2 is Morava's E-theory with $E_{2*} = W(\mathbb{F}_{p^2})[[u_1]][u^{\pm 1}]$.

2. *If $\mathcal{F}(\ell) \to B$ is a G-Galois extension, then*

$$V(1)_* K(\mathcal{F}(\ell)) \to V(1)_* K(B)^{hG}$$

   is an isomorphism in high enough degrees.

Periodic K-theory

In view of $\ell_* = \mathbb{Z}_p[v_1]$ and of our understanding of the stable homotopy category (chromatic picture), it seems reasonable that it suffices to "invert" $v_0 = p$ and v_1 in ℓ to obtain $\mathcal{F}(\ell)$.

The $\mathbb{S}$-algebras mentioned above all have a corresponding periodic version :

- KU, with $KU_* = \mathbb{Z}[u^{\pm 1}]$, and KU_p, with $KU_{p*} = \mathbb{Z}_p[u^{\pm 1}]$,
- L, the periodic Adams summand, with $L_* = \mathbb{Z}_p[v_1^{\pm 1}]$.

Thus the canonical algebra map $\ell \to L$ corresponds to inverting v_1 (this will be confirmed later).

What about inverting p ?
The obvious candidate is $\mathcal{F}(\ell) = L[\frac{1}{p}]$, having as coefficients the graded field $\mathbb{Q}_p[v_1^{\pm 1}]$. This would be disappointing since $L[\frac{1}{p}]$ is an $H\mathbb{Q}$-algebra.

- We use algebraic K-theory to test this candidate.

Localization sequences in K-theory

Indeed, we expect $K(\mathcal{F}(\ell))$ to fit in the following diagram of localization sequences in algebraic K-theory

$$\begin{array}{ccccc}
K(\mathbb{F}_p) & \longrightarrow & K(\mathbb{Z}_p) & \longrightarrow & K(\mathbb{Q}_p) \\
\downarrow & & \downarrow & & \downarrow \\
K(\ell/p) & \longrightarrow & K(\ell_p) & \longrightarrow & K(p^{-1}\ell_p) \\
\downarrow & & \downarrow & & \downarrow ? \\
K(L/p) & \longrightarrow & K(L_p) & \longrightarrow & K(\mathcal{F}(\ell)) \,.
\end{array}$$

The top-row is Quillen's localization sequence with respect to $p \in \mathbb{Z}_p$, and the left and middle rows are the localization sequences with respect to $v_1 \in \ell$ and $v_1 \in \ell/p$ conjectured by Rognes and established by Blumberg-Mandell.

The other sequences are potentially localization sequences, and here $K(\mathcal{F}(\ell))$ is actually *defined* as the iterated cofibre of the upper left square. Thus to compute $V(1)_* K(\mathcal{F}(\ell))$ it essentially remains to compute $V(1)_* K(\ell/p)$.

Mod p K-theory

In the above diagram, we defined ℓ/p and L/p as the cofibre of the multiplication-by-p self-maps,

$$\ell \xrightarrow{p} \ell \to \ell/p, \quad L \xrightarrow{p} L \to L/p,$$

with $\ell/p_* = \mathbb{Z}/p[v_1]$ and $L/p_* = \mathbb{Z}/p[v_1, v_1^{-1}]$.
Notice that ℓ/p and L/p admit an $\mathbb{S}$-algebra structure, but not a commutative one.

Main Theorem (A, Rognes)

The $P(v_2)$-module $V(1)_* K(\ell/p)$ is a free $P(v_2)$-module of rank $2p^2 - 2p + 8$, on explicitly given generators.

The $V(1)$-homotopy of $K(\mathcal{F}(\ell))$

Corollary

There is an isomorphism of $P(v_2)$-modules

$$V(1)_* K(\mathcal{F}(\ell)) \cong N \oplus T,$$

where

- *N is a free $P(v_1)$-module of rank $(2p^2 + 6)$,*
- *T is an $\mathbb{F}_p$-module of rank 4 with $v_2 T = 0$.*

Moreover, there are classes $\delta_1, \delta_2 \in V(1)_1 K(\mathcal{F}(\ell))$ with $v_1\delta_1 = \lambda_1$ and $v_2\delta_2 = \lambda_2$.

Remark

Notice that $V(1)_* K(L[\frac{1}{p}])$ is a v_2-torsion module, because it is a module over $V(1)_* K(\mathbb{Q}_p)$, which is v_2-torsion. With the computation of $V(1)_* K(\mathcal{F}(\ell))$ above, this indicates:

- The $H\mathbb{Q}_p$-algebra $L[\frac{1}{p}]$ does not qualify for $\mathcal{F}(\ell)$.

The fraction field

It turns out that the structure of $V(1)_*K(\mathcal{F}(\ell))$ fits very nicely with the Lichtenbaum-Quillen Conjectures for ℓ.

Indeed, we consider a conjectural Galois-descent spectral sequence

$$E^2_{s,t} = H^{-s}_{Gal}(\mathcal{F}(\ell); \mathbb{F}_{p^2}(t/2)) \Rightarrow v_2^{-1}V(1)_{s+t}K(\mathcal{F}(\ell))$$

where the coefficient module is

$$v_2^{-1}V(1)_*K(\Omega) = V(1)_*E_2 = \mathbb{F}_{p^2}[u^{\pm 1}]$$

with action of $G_{\mathcal{F}(\ell)}$ induced by its action on $K(\Omega)$. Working backwards, Rognes conjecturally evaluated $E^2_{s,t}$.

The good surprise is that

- The cohomological dimension is 3,
- $H^*_{Gal}(\mathcal{F}(\ell); \mathbb{F}_{p^2}(*))$ has self-duality, analogous to Tate-Poitou duality for local fields:

$$H^{3-n}_{Gal}(\mathcal{F}(\ell); \mathbb{F}_{p^2}(2-k)) \cong H^n_{Gal}(\mathcal{F}(\ell); \mathbb{F}_{p^2}(k))^*.$$

This feature is not present in the case of ℓ or L. It suggest that $\mathcal{F}(\ell)$ is a form of a 2-local field in mixed characteristic 0, p and v_1.

Summary of the above computations of $K(\mathcal{F}(\ell))$

- Use trace methods to compute $K(\mathbb{Z}_p)$, $K(\ell/p)$ and $K(\ell)$.
- Use iterated cofibre sequences to evaluate $K(L/p)$, $K(L)$ and $K(\mathcal{F}(\ell))$.

A direct trace computation of $K(\mathcal{F}(\ell))$, starting with $THH(\mathcal{F}(\ell))$, would be nicer and would shed light on the nature of $\mathcal{F}(\ell)$.

However, notice that the corresponding sequences

$$THH(\mathbb{Z}_p) \to THH(\ell) \to THH(L)$$

and

$$TC(\mathbb{Z}_p) \to TC(\ell) \to TC(L)$$

are NOT cofibre sequences.
Similarly, we do not expect to have a localization sequence

$$THH(L/p) \to THH(L) \to THH(\mathcal{F}(\ell)),$$

and neither for TC.

The case of local fields

Suppose that K is a complete discrete valuation field with perfect residue field k of odd characteristic p. Let A be the valuation ring. Hesselholt and Madsen define ad hoc relative versions $THH(A|K)$ and $TC(A|K)$ (using a suitable category of modules), which fit in a diagram of horizontal localization cofibre sequences

$$\begin{array}{ccccc} K(k) & \longrightarrow & K(A) & \longrightarrow & K(K) \\ \downarrow & & \downarrow & & \downarrow \\ TC(k) & \longrightarrow & TC(A) & \longrightarrow & TC(A|K) \\ \downarrow & & \downarrow & & \downarrow \\ THH(k) & \longrightarrow & THH(A) & \longrightarrow & THH(A|K). \end{array}$$

They compute $V(0)_* TC(A|K)$, which turns out to be much more manageable then $V(0)_* TC(A)$.

Differentials with logarithmic poles

There is a natural short exact sequence

$$0 \to \Omega^1_A = THH_1(A) \to THH_1(A|K) \to THH_0(k) = k \to 0.$$

It can be compared to the natural short exact sequence

$$0 \to \Omega^1_A \to \omega^1_{(A,M)} \to k \to 0.$$

Here $M = A \cap K^*$ and $\omega^1_{(A,M)} = (\Omega^1_A \oplus A \otimes_{\mathbb{Z}} K^*)/\sim$ is the target of the universal derivation of the log ring (A, M).

Define

- $d : A \to THH_1(A|K)$ as Connes' operator (cyclic action), using the identification $A = THH_0(A|K)$, and
- $\mathrm{dlog} : M \subset K^* = K_1(K) \xrightarrow{\mathrm{tr}} THH_1(A|K)$.

Then $(d, dlog) : (A, M) \to THH_1(A|K)$ is a derivation, and the canonical map

$$\omega^1_{(A,M)} \xrightarrow{\cong} THH_1(A|K)$$

is an isomorphism.

Log differential graded rings

This extends to a natural map of "log differential graded rings"

$$\omega^*_{(A,M)} \longrightarrow THH_*(A|K)$$

and to a natural map of "log Witt complexes"

$$W_\bullet\omega^*_{(A,M)} \longrightarrow TR^\bullet_*(A|K)$$

over (A, M) (left-hand side is universal).

These maps allow H.-M. to organize the computation of $V(0)_* THH(A|K)$ and $V(0)_* TC(A|K)$, and to evaluate $V(0)_* K(K)$.

In particular, they establish an isomorphism of log differential graded rings

$$\omega^*_{(A,M)} \otimes_{\mathbb{Z}} \mathbb{F}_p[\kappa] \xrightarrow{\cong} V(0)_* THH(A|K),$$

where $|\kappa| = 2$ and $d\kappa = \kappa \, \mathrm{dlog}(-p)$.

In the example of $K = \mathbb{Q}_p$, this evaluates as

$$V(0)_* THH(\mathbb{Z}_p \,|\, \mathbb{Q}_p) \cong E(\mathrm{dlog}(p)) \otimes P(\kappa).$$

Returning to $\mathcal{F}(\ell)$

Inspired by the case of local fields, we search for a good construction of $THH(- \mid -)$ in the case of ring spectra, leading to localization cofibre sequences

- $THH(\mathbb{Z}_p) \to THH(\ell) \to THH(\ell \mid L)$
- $THH(\mathbb{F}_p) \to THH(\ell/p) \to THH(\ell/p \mid L/p)$
- $THH(\ell/p \mid L/p) \to THH(\ell \mid L) \to THH(\ell \mid \mathcal{F}(\ell))$

and similarly for $TC(- \mid -)$, such that the localization cofibre sequences for K, TC and THH are compatible with the traces, as above.

Then $TC(\ell \mid \mathcal{F}(\ell))$ would be the desired approximation of $K(\mathcal{F}(\ell))$.

This test-case might lead to a suitable definition of $K(\mathcal{F}(\ell))$ via "log algebraic K-theory".

Modest computational evidence

Some evidence for the existence of "log-THH" is provided by our computations. First, using iterated cofiber sequences, as for $K(\mathcal{F}(\ell))$, we compute

$$V(1)_* THH(\ell \,|\, \mathcal{F}(\ell)) \cong E(\mathrm{dlog}(p), \mathrm{dlog}(v_1)) \otimes P(\kappa)$$

with $|\,\mathrm{dlog}(p)| = |\,\mathrm{dlog}(v_1)| = 1$ and $|\kappa| = 2$.

- This fits very well with Hesselholt-Madsen's result on $V(0)_* THH(\mathbb{Z}_p \,|\, \mathbb{Q}_p) \cong E(\mathrm{dlog}(p)) \otimes P(\kappa)$.
- Because of the simplicity of $V(1)_* THH(\ell \,|\, \mathcal{F}(\ell))$, computing $V(1)_* TC(\ell \,|\, \mathcal{F}(\ell))$ is expected to be easier than $V(1)_* TC(\ell)$.
- In our formula for $V(1)_* K(\mathcal{F}(\ell))$, it allows us to interpret the classes δ_1 and δ_2 as $\mathrm{dlog}(p)$ and $\mathrm{dlog}(v_1)$, via the trace map

$$V(1)_* K(\mathcal{F}(\ell)) \to V(1)_* THH(\ell \,|\, \mathcal{F}(\ell)).$$

Log-étale descent

In context of local fields as above, a finite extension $K \to F$ with ramification index e prime to p defines a "log-étale" extension $(A, M_A) \to (B, M_B)$, where B is the integral closure of A in F, as reflected by the formulas

$$\omega^1_{(B,M_B)/W(k)} \cong B \otimes_A \omega^1_{(A,M_A)/W(k)} \text{ and}$$
$$THH(B \,|\, F) \simeq_p B \wedge_A THH(A \,|\, K).$$

Using the cofibre-definition of log-THH, we established an equivalence

$$THH(ku_p \,|\, KU_p) \simeq_p ku \wedge_\ell THH(\ell \,|\, L)$$

This suggests that the "tamely ramified" extension $\ell \to ku_p$ corresponds to a "log-étale" extension $(\ell \,|\, L) \to (ku_p \,|\, KU_p)$, and that $THH(- \,|\, -)$ ought also to have log-geometric content in this context.