EQUIVARIANT DERIVED ALGEBRAIC GEOMETRY AND K-THEORY

by

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The worst part is over; now get back on that horse and ride. — THE SHINS

These are notes for the Conference on p-adic Geometry and Homotopy Theory, 2–9 August 2009, in Loen, organized by J. Rognes and (to a lesser extent) me. The standard caveats apply here: (1) These notes are very informal, and most proofs are sketched or omitted completely; even when I'm giving details, I'm skipping details. (2) Some of the ideas and results here appear to be new, but many of the foundational results should be ascribed to others. (More on this below.) (3) All errors are mine, and I'm duly ashamed. Really, I am.

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Historical note. — A comment from Andrew Blumberg led me to realize that, in my eagerness to share the ideas, I'd been altogether remiss in giving credit where credit is due in preparing these talks. In an effort to rectify this oversight, let me try to summarize, in a few paragraphs, some of the origins of the various important aspects of this work.

The idea that algebraic geometry might be developed by gluing more general kinds of "rings" goes back to Alexander Grothendieck, Monique Hakim, and Pierre Deligne, who developed and used relative algebraic geometry. (It's interesting to note that Hakim's perspective on relative algebraic geometry plays an important role in some proofs in Luc Illusie's work on the cotangent complex.) This idea was later developed in a homotopical context by Bertrand Toën and Gabriele Vezzosi, inspired by a conversation with Markus Spitzweck, and providing foundations for ideas of Deligne, Alexander Beilinson, Vladimir Drinfeld, Mikhail Kapranov, Ionut Ciocan-Fontaine, and others.

In the context of E_∞ ring spectra, many of the basic algebro-geometric intuitions were employed by Friedhelm Waldhausen, Mike Hopkins, Haynes Miller, Doug Ravenel, Peter May, Maria Basterra, Mike Mandell, and a host of other topologists. In particular, the cotangent complex that plays such a major role in these notes is the one developed in the E_∞ context by Maria Basterra and Mike Mandell. These notions were developed and given thorough ∞-categorical foundations more recently in the work of Jacob Lurie.

Classical Tannaka duality was proposed by Grothendieck, explored by Neantro Saavedra–Rivano, corrected by Deligne and James Milne, and generalized by Torsten Wedhorn. Higher categorical variants were proposed by Toën, and have been the subject of work by Lurie, Spike Francis, Davids Ben-Zvi and Nadler, and Toën's Ph.D student James Wallbridge.

Equivariant homotopy has been developed by John Greenlees, Gaunce Lewis, Peter May, Mark Steinberger, and others, using foundations quite different from the ones suggested here. A conversation with Blumberg has suggested that through his work, a comparison between the foundations proposed here and theirs should be possible.

The first application to K-theory discussed in these notes addresses a conjecture of Gunnar Carlsson. My principal motivation for developing equivariant derived algebraic geometry was to address his conjectures, inspired by a series of conversations with Paul-Arne Østvær and Grace Lyo. The filtration that I use on the K-theory spectrum was developed by Mark Walker, based upon a construction of Dan Grayson, and shown by Andrei Suslin to give rise to the motivic cohomology of Vladimir Voedvodsky, Suslin, and Eric Friedlander.

The second application to K-theory seems rather more mysterious, and I'm unable to place it in a historical context.

1. Recollections on derived algebraic geometry

I recall some of the basic ideas of derived algebraic geometry.

1.1. — (1.1.1) Denote by $\mathscr{S}p$ the ∞-category of spectra. This is a symmetric monoidal ∞-category with respect to the smash product. Denote by $\mathscr{S}p_{\geq 0}$ the full subcategory of *connective spectra*.

(1.1.2) Denote by $E_\infty(\mathscr{S}p)$ the category of E_∞-algebras in this symmetric monoidal category, which will here be called E_∞ *rings*. Denote by $E_\infty(\mathscr{S}p)_{\geq 0}$ the full subcategory of *connective* E_∞ *rings*.

(1.1.3) Denote by $\mathscr{A}ff_{\geq 0} := E_\infty(\mathscr{S}p)^{\mathrm{op}}_{\geq 0}$. The objects of $\mathscr{A}ff_{\geq 0}$ will be called *(derived) affines*; for any connective E_∞ ring A, denote by $\operatorname{Spec} A$ the corresponding object of $\mathscr{A}ff_{\geq 0}$.

(1.1.4) Denote by $\mathscr{A}ff_0$ the category of classical affines, i.e., the opposite category to the category of commutative rings. The Eilenberg-Mac Lane functor defines a functor

$$H : \mathscr{A}ff_0 \longrightarrow \mathscr{A}ff_{\geq 0}.$$

(1.1.5) For any affine $X = \operatorname{Spec} A$, denote by $\mathscr{P}erf(X)$ the ∞-category of compact objects in the ∞-category $\mathscr{M}od(X)$ of A-modules.

(1.1.6) For any affine X and any topology or hypertopology τ on $\mathscr{A}ff_{\geq 0,/X}$, we denote by $\mathscr{S}^\tau(\mathscr{A}ff_{\geq 0,/X})$ the corresponding ∞-topos. The objects of $\mathscr{S}^\tau(\mathscr{A}ff_{\geq 0,/X})$ are presheaves of spaces

$$F : \mathscr{A}ff^{\mathrm{op}}_{\geq 0,/X} \longrightarrow \mathscr{K}an$$

that satisfy descent (or hyperdescent) with respect to τ. Pulling back τ along the Eilenberg-Mac Lane functor H defines a topology τ_0 on $\mathscr{A}ff_0$; there is an induced adjunction

$$H_! : \mathscr{S}^{\tau_0}(\mathscr{A}ff_0) \rightleftarrows \mathscr{S}^\tau(\mathscr{A}ff_{\geq 0}) : H^\star,$$

the left adjoint of which is fully faithful, and which is, significantly, *not* a morphism of ∞-topoi.

(1.1.7) Suppose X an affine, and suppose τ any topology or hypertopology on $\mathscr{A}ff_{\geq 0,/X}$. Attached to any sheaf F of ∞-categories on $\mathscr{A}ff_{\geq 0,/X}$ for τ is a cartesian fibration

$$\Gamma^\tau(F) \longrightarrow \mathscr{S}^\tau(\mathscr{A}ff_{\geq 0,/X}),$$

given by the ∞-categorical analogue of the Grothendieck construction. The fibers of this cartesian fibration over an affine X (more precisely, over the sheafification of the presheaf represented by X) is canonically equivalent to the ∞-category FX.

Flatness and the flat hypertopology. — A first important notion of derived algebraic geometry is the analogue of the notion of *flat families*.

Theorem 1.2. — *The following are equivalent for a morphism* $f : Y = \operatorname{Spec} B \longrightarrow \operatorname{Spec} A = X$ *of affines.*

(3.2.1) As an A-module, B can be written as a filtered colimit of finitely generated free A-modules.

(3.2.2) For any discrete A-module M, the B-module $f^\star M := M \otimes_A B$ *is discrete.*

(3.2.3) The functor

$$f^\star : \mathscr{M}od(X) \longrightarrow \mathscr{M}od(Y)$$

is left t-exact, so that it carries $\mathscr{M}od(X)_{\leq 0}$ *into* $\mathscr{M}od(Y)_{\leq 0}$.

(3.2.4) The following pair of conditions is satisfied.

(3.2.4.1) The induced homomorphism $\operatorname{Spec} \pi_0 B \longrightarrow \operatorname{Spec} \pi_0 A$ *is a flat morphism of ordinary schemes.*

(3.2.4.2) For every integer $j \in \mathbf{Z}$, *the homomorphism*

$$\pi_j A \otimes_{\pi_0 A} \pi_0 B \longrightarrow \pi_j B$$

of $\pi_0 B$*-modules is an isomorphism.*

In this case, the morphism f will be called flat.

Theorem 1.3. — *The following are equivalent for a flat morphism* $f : Y = \operatorname{Spec} B \longrightarrow \operatorname{Spec} A = X$ *of affines.*

(3.3.1) The functor

$$f^\star : \mathscr{M}od(X) \longrightarrow \mathscr{M}od(Y)$$

is conservative, so that for any nonzero A-module M, the B-module $f^\star M := M \otimes_A B$ *is nonzero.*

(3.3.2) The induced morphism $\operatorname{Spec} \pi_0 B \longrightarrow \operatorname{Spec} \pi_0 A$ *is faithfully flat.*

In this case, the morphism f will be called faithfully flat.

1.4. — A simplicial object $V_\bullet$ of $\mathscr{A}\!f\!f_{\geq 0,/U}$ is a *flat hypercovering* of an affine U if for any integer $n \geq 0$, the morphism

$$(\operatorname{sk}_{n-1} V_\bullet)_n \longrightarrow U$$

is faithfully flat.

Along with Čech nerves of covering families

$$\{ U_i \longrightarrow \textstyle\coprod_{j \in I} U_j \}_{i \in I},$$

the flat hypercoverings generate the *flat hypertopology* $\flat$ on the ∞-category $\mathscr{A}\!f\!f_{\geq 0,/X}$ for an affine X. The corresponding *flat ∞-topos* $\mathscr{S}^\flat(\mathscr{A}\!f\!f_{\geq 0,/X})$ is hypercomplete. A presheaf

$$F : \mathscr{A}\!f\!f^{\mathrm{op}}_{\geq 0,/X} \longrightarrow \mathscr{K}an$$

is a *flat hypersheaf over X* if it lies in $\mathscr{S}^\flat(\mathscr{A}\!f\!f_{\geq 0,/X})$, i.e., if the following two conditions are satisfied.

(3.4.1) For any object $U \in \mathscr{A}\!f\!f_{\geq 0,/X}$ and any flat hypercovering $V_\bullet$ of U, the induced morphism

$$FU \longrightarrow \lim FV_\bullet$$

is an equivalence.

(3.4.2) For any object $U = \coprod_{i \in I} U_i \in \mathscr{A}\!f\!f_{\geq 0,/X}$, the induced morphism

$$FU \longrightarrow \textstyle\prod_{i \in I} FU_i$$

is an equivalence.

Theorem 1.5. — *Suppose X an affine. The flat hypertopology $\mathscr{A}\!f\!f_{\geq 0,/X}$ is subcanonical. Moreover, the assignments*

$$\mathscr{M}od : U \longmapsto \mathscr{M}od(U) \qquad \text{and} \qquad \mathscr{P}erf : U \longmapsto \mathscr{P}erf(U)$$

are flat hypersheaves of ∞-categories. In particular, the associated presheaves $\iota\mathscr{M}od$ and $\iota\mathscr{P}erf$ of spaces are flat hypersheaves.

The cotangent complex and the étale topology. — One of the easiest pieces of classical algebraic geometry to transfer to the derived setting is Illusie's *cotangent complex*.

1.6. — Suppose $f : Y = \operatorname{Spec} B \longrightarrow \operatorname{Spec} A = X$ a morphism of affines. For any B-module M, one has an associated *square zero extension $B \oplus M$ of B*. Write $Y_M := \operatorname{Spec}(B \oplus M)$; there is an obvious morphism $Y \longrightarrow Y_M$. Now define the *space of derivations on Y over X with coefficient in M* as the fiber $\operatorname{Der}_X(Y;M)$ of the morphism of spaces

$$\operatorname{Mor}_X(Y_M, Y) \longrightarrow \operatorname{Mor}_X(Y,Y)$$

over the identity map. The result is a functor

$$\operatorname{Der}_X(Y;-) : \mathscr{M}od(T) \longrightarrow \mathscr{S}.$$

Theorem 1.7. — *For any morphism $f : Y = \operatorname{Spec} B \longrightarrow \operatorname{Spec} A = X$ of affines, the functor $\operatorname{Der}_X(Y;-)$ is corepresentable; that is, there exists a B-module $\mathbf{L}_{Y|X}$ and an equivalence*

$$\operatorname{Der}_X(Y;M) \simeq \operatorname{Mor}(\mathbf{L}_{Y|X}, M),$$

functorial in M. The representing object $\mathbf{L}_{Y|X}$ is called the cotangent complex for f, *and the morphism*

$$d : Y_{\mathbf{L}_{Y|X}} \longrightarrow Y$$

corresponding to the identity of $\mathbf{L}_{Y|X}$ is called the universal derivation for f.

1.8. — If R and S are (discrete) $\mathbf{Q}$-algebras, and if $R \longrightarrow S$ is a morphism thereof, then it can be shown that the cotangent complex $\mathbf{L}_{S|R}$ of Illusie coincides with $\mathbf{L}_{\operatorname{Spec} HS|\operatorname{Spec} HR}$. If, however, R and S are not $\mathbf{Q}$-algebras, this fails dramatically.

1.9. — Suppose $f : Y = \operatorname{Spec} B \longrightarrow \operatorname{Spec} A = X$ a morphism of affines, locally of finite presentation (so that B is a compact object in the category of E_∞ rings under A).
(1.9.1) One says that f is *smooth* if $\mathbf{L}_{Y|X}$ is compact in $\mathscr{M}od(X)$.
(1.9.2) One says that f is *étale* if $\mathbf{L}_{Y|X} \simeq 0$.

Theorem 1.10. — *The following are equivalent for a morphism $f : Y = \operatorname{Spec} B \longrightarrow \operatorname{Spec} A = X$ of affines.*
(1.10.1) The morphism f is étale.
(1.10.2) The induced morphism on topological Hochschild homology spectra

$$\operatorname{THH}(A) \longrightarrow \operatorname{THH}(B)$$

is an equivalence.
(1.10.3) The following pair of conditions is satisfied.
(1.10.3.1) The induced homomorphism $\operatorname{Spec} \pi_0 B \longrightarrow \operatorname{Spec} \pi_0 A$ is an étale morphism of ordinary schemes.
(1.10.3.2) For every integer $j \in \mathbf{Z}$, the homomorphism

$$\pi_j A \otimes_{\pi_0 A} \pi_0 B \longrightarrow \pi_j B$$

of $\pi_0 B$-modules is an isomorphism.

1.11. — Suppose X an affine. A family

$$\{V_i \longrightarrow U\}_{i \in I}$$

is an *étale covering* if each morphism $V_i \longrightarrow U$ is étale, and for some finite subset $I' \subset I$, the morphism

$$\coprod_{i \in I'} V_i \longrightarrow U$$

is faithfully flat.

These families generate the *étale topology* ét on the ∞-category $\mathscr{A}\!f\!f_{\geq 0,/X}$. The corresponding *étale ∞-topos* $\mathscr{S}^{\text{ét}}(\mathscr{A}\!f\!f_{\geq 0,/X})$ is not hypercomplete.

Affine ∞-gerbes. — A flat hypersheaf will be said to be a *affine ∞-gerbe* if it is locally (for the flat hypertopology) equivalent to a flat affine loopspace.

1.12. — For any affine X, a flat hypersheaf G over X is an *affine ∞-gerbe over X* if it satisfies the following pair of conditions.

(1.12.1) The hypersheaf G is locally nonempty for the flat hypertopology; that is, there exists a faithfully flat morphism $Y \longrightarrow X$ of affines such that $G(Y)$ is nonempty.

(1.12.2) For any morphism $U \longrightarrow X$ of affines and any pair of points $x, y \in G(U)$, the path space

$$\Omega_{x,y} G := U \times_{x,G,y} U$$

is representable and faithfully flat over X.

The ∞-category of affine ∞-gerbes over X will be denoted $\mathscr{G}\!erbe_{/X}$.

Theorem 1.13. — *Suppose $X = \operatorname{Spec} HR$ for a discrete ring R, and suppose G an affine ∞-gerbe over X. Then for any discrete R-algebra S, any point $x \in G(\operatorname{Spec} HS)$, and any positive integer $j > 0$, the sheaf $\pi_j(H^\star G, x)$ obtained by sheafifying the presheaf*

$$T \longmapsto \pi_j(G(HT), x)$$

on the large fpqc site of affine S-schemes is a proalgebraic group, which is unipotent if $j > 1$.

1.14. — The *classifying ∞-topos* of an affine ∞-gerbe G is the ∞-category

$$B^\flat(G) := \mathscr{C}\!art_{\mathscr{S}^\flat(\mathscr{A}\!f\!f_{\geq 0,/X})}(\Gamma^\flat(G), \operatorname{Fun}(\Delta^1, \mathscr{S}^\flat(\mathscr{A}\!f\!f_{\geq 0,/X})))$$

of cartesian morphisms $\Gamma^\flat(G) \longrightarrow \operatorname{Fun}(\Delta^1, \mathscr{S}^\flat(\mathscr{A}\!f\!f_{\geq 0,/X}))$ of cartesian fibrations over the flat ∞-topos $\mathscr{S}^\flat(\mathscr{A}\!f\!f_{\geq 0,/X})$. The objects of the classifying ∞-topos of G can be regarded as spaces with an *action* of G. There is, accordingly, a morphism of ∞-topoi

$$\tau_G : B^\flat(G) \longrightarrow \mathscr{S}^\flat(\mathscr{A}\!f\!f_{\geq 0,/X}),$$

where the pullback $\tau_G^\star F$ of a flat hypersheaf F can be regarded as endowing F with the *trivial* G-action. This functor also has a further left adjoint

$$\tau_{G,!} : B^\flat(G) \longrightarrow \mathscr{S}^\flat(\mathscr{A}\!f\!f_{\geq 0,/X}),$$

which can be interpreted in the following manner: for any object $K \in B^\flat(G)$, the flat hypersheaf $\tau_{G,!}K$ is the *quotient ∞-stack* $[K/G]$.

1.15. — The stable ∞-category of *perfect complexes* on an affine ∞-gerbe G over an affine X is the ∞-category

$$\mathscr{P}\!erf(G) := \mathscr{C}\!art_{\mathscr{S}^\flat(\mathscr{A}\!f\!f_{\geq 0,/X})}(\Gamma^\flat(G), \Gamma^\flat(\mathscr{P}\!erf))$$

of cartesian morphisms $\Gamma^\flat(G) \longrightarrow \Gamma^\flat(\mathscr{P}\!erf)$ of cartesian fibrations over the flat ∞-topos $\mathscr{S}^\flat(\mathscr{A}\!f\!f_{\geq 0,/X})$.

2. Mackey and Green functors for affine ∞-gerbes

Mackey and Green functors are now a standard tool in the representation theory of finite groups. Here I sketch how the theory may be adapted to the context of affine ∞-gerbes. Suppose here X an affine, and suppose G an affine ∞-gerbe over X.

The Burnside ∞-categories. — Classically, Mackey functors are additive functors indexed on a *Burnside category*, obtained by taking a group completion of a semi-additive category of spans. The ∞-categorical set-up is slightly more complicated than the classical description of the Burnside category; here I give a construction that is slightly circuitous, but it has the benefit of making it clear that the relevant categories exist and have the properties demanded of them. Unfortunately, the Burnside categories that I give here are quite large; smaller ones can be constructed, but this is beyond the scope of these notes.

2.1. — An object K of the classifying ∞-topos $B^\flat(G)$ will be said to be *finite* if it satisfies the following properties.

(2.1.1) The sheaf of sets $\pi_0\tau_{G,!}K$ obtained by sheafifying the assignment

$$U \longmapsto \pi_0((\tau_{G,!}K)(U))$$

is locally finite.

(2.1.2) The local isotropy of K is open. (This is most easily expressed using the language of bands of affine ∞-gerbes, but going into detail here might take us too far afield; one should have in mind the example of a proalgebraic group acting on a sheaf of spaces in such a way that all the local isotropy subgroups are *open* subgroups.)

Denote by $B^\flat(G)^{\mathrm{fin}}$ the full subcategory of $B^\flat(G)$ spanned by the finite objects. The objects of $B^\flat(G)^{\mathrm{fin}}$ will be called *finite G-spaces*.

Observe that this category is different from the category of compact objects of the ∞-topos $B^\flat(G)$.

2.2. — Define the *semiexcisive Burnside ∞-category* $\mathscr{B}^+_G$ in the following manner.

(2.2.1) The objects are finite G-spaces.

(2.2.2) A morphism $K \longrightarrow M$ of finite G-spaces is a diagram

$$K \longleftarrow L \longrightarrow M$$

in $B^\flat(G)$.

(2.2.3) Given two such diagrams

$$K \longleftarrow L \longrightarrow M \quad \text{and} \quad M \longleftarrow N \longrightarrow P,$$

their composition is defined (up to a contractible choice) as the top of the pullback

$$\begin{array}{ccccc} & & L \times_M N & & \\ & \swarrow & & \searrow & \\ & L & & N & \\ \swarrow & & \searrow \quad \swarrow & & \searrow \\ K & & M & & P. \end{array}$$

2.3. — Observe that the product $- \times -$ in $B^\flat(G)^{\mathrm{fin}}$ defines a symmetric monoidal structure on $\mathscr{B}^+_G$; note that the product of is *not* the cartesian product in $\mathscr{B}^+_G$.

Note also that there are two faithful, symmetric monoidal functors

$$\ell : B^\flat(G)^{\mathrm{fin,op}} \longrightarrow \mathscr{B}^+_G \quad \text{and} \quad r : B^\flat(G)^{\mathrm{fin}} \longrightarrow \mathscr{B}^+_G$$

that are each the identity on objects. Let's investigate the composites in $\mathscr{B}_G$ of morphisms induced by ℓ with morphisms induced by r.

(2.3.1) The composite of (the image under ℓ of) a morphism $L \longrightarrow K$ of finite G-spaces with (the image under r of) a morphism $L \longrightarrow M$ of finite G-spaces is the morphism

$$K \longleftarrow L \longrightarrow M.$$

(2.3.2) On the other hand, the composite of (the image under r of) a morphism $L \longrightarrow M$ of finite G-spaces with (the image under ℓ of) a morphism $N \longrightarrow M$ of finite G-spaces is given by the top of the pullback square

$$\begin{array}{ccc} & L \times_M N & \\ \swarrow & & \searrow \\ L & & N. \\ \searrow & & \swarrow \\ & M & \end{array}$$

2.4. — Call a functor $F : \mathscr{B}^+_G \longrightarrow D$ *admissible* if it satisfies the following properties.

(2.4.1) The functor F sends the zero object of $\mathscr{B}^+_G$ to an initial object.

(2.4.2) The functor $\ell^\star F : B^\flat(G)^{\mathrm{fin,op}} \longrightarrow D$ sends pushout squares of finite G-spaces to pushout squares in D.

(2.4.3) The functor $r^\star F : B^\flat(G)^{\mathrm{fin}} \longrightarrow D$ sends pushout squares of finite G-spaces to pushout squares in D.

Write $\mathrm{Adm}(\mathscr{B}^+_G, D)$ for the full subcategory of $\mathrm{Fun}(\mathscr{B}^+_G, D)$ spanned by the admissible functors.

2.5. — There exists a pointed ∞-category $\mathscr{B}^{\to}_G$ along with a functor $j : \mathscr{B}^+_G \longrightarrow \mathscr{B}^{\to}_G$ satisfying the following conditions.

(2.5.1) The ∞-category $\mathscr{B}^{\to}_G$ has all finite colimits.

(2.5.2) For any ∞-category D with all finite colimits, the functor j induces an equivalence of ∞-categories

$$\mathrm{Fun}_{\mathrm{rex}}(\mathscr{B}^{\to}_G, D) \longrightarrow \mathrm{Adm}(\mathscr{B}^+_G, D)$$

between the ∞-categories of right exact functors $\mathscr{B}^{\to}_G \longrightarrow D$ and the admissible functors $\mathscr{B}^+_G \longrightarrow D$.

The ∞-category $\mathscr{B}^{\to}_G$ will be called the *excisive Burnside ∞-category*. Since the product in $B^\flat(G)^{\mathrm{fin}}$ preserves finite colimits in each variable, the symmetric monoidal structure on $\mathscr{B}^+_G$ descends to $\mathscr{B}^{\to}_G$.

2.6. — Now define the *stable Burnside ∞-category* $\mathscr{B}_G$ as the colimit of the diagram

$$\mathscr{B}^{\to}_G \xrightarrow{\Sigma} \mathscr{B}^{\to}_G \xrightarrow{\Sigma} \dots$$

in the category of ∞-categories with right exact functors. This ∞-category has the property that $\mathrm{Ind}(\mathscr{B}_G)$ is the stabilization of $\mathrm{Ind}(\mathscr{B}^{\to}_G)$. It follows from a G-equivariant Blakers–Massey homotopy excision theorem that, in fact, $\mathscr{B}_G$ is stable. Moreover, if D is any stable ∞-category, the natural functor $\Sigma^\infty : \mathscr{B}^{\to}_G \longrightarrow \mathscr{B}_G$ induces an equivalence

$$\mathrm{Fun}_{\mathrm{ex}}(\mathscr{B}_G, D) \simeq \mathrm{Fun}_{\mathrm{rex}}(\mathscr{B}^{\to}_G, D).$$

The symmetric monoidal structure on $\mathscr{B}^{\to}_G$ descends to a symmetric monoidal structure $-\odot-$ on $\mathscr{B}_G$.

Mackey functors. — Mackey functors are certain functors indexed on the semi-excisive Burnside category; this idea can rephrased in a number of ways.

2.7. — In order to describe the notion of a Mackey functor, it is helpful to introduce some terminology. Suppose C an ∞-category containing all finite colimits, and suppose D an ∞-category containing all finite limits. Then a functor $C \longrightarrow D$ is said to be *excisive* if it sends an initial object of C to a terminal object of D, and if it sends a pushout square in C to a pullback square in D. The full subcategory of $\mathrm{Fun}(C, D)$ spanned by the excisive functors will be denoted $\mathrm{Exc}(C, D)$. Similarly, a functor $C^{\mathrm{op}} \longrightarrow D$ will be said to be *contra-excisive* if it sends initial object of C to a terminal object of D, and if it sends a pushout square in C to a pullback square in D. The full subcategory of $\mathrm{Fun}(C^{\mathrm{op}}, D)$ spanned by the excisive functors will be denoted $\mathrm{Exc}_{\mathrm{op}}(C^{\mathrm{op}}, D)$.

To illustrate, consider briefly the situation in which $C = \mathscr{S}^{\mathrm{fin}}$, the category of finite spaces, and $D = \mathscr{S}p^{\mathrm{fin}}$, the ∞-category of finite spectra. Then an excisive functor $F : \mathscr{S}^{\mathrm{fin}} \longrightarrow \mathscr{S}p^{\mathrm{fin}}$ amounts to a homology theory given by the formula

$$F_*(Y \subset X) := \pi_* \Omega^\infty F(X/Y).$$

A contra-excisive functor $G : \mathcal{S}^{\mathrm{fin,op}} \longrightarrow \mathcal{S}p^{\mathrm{fin}}$ amounts to a cohomology theory given by the formula

$$G^*(Y \subset X) := \pi_* \Omega^\infty G(X/Y).$$

There are a number of remarks to be made about excisive functors.

(2.7.1) If C is stable, then a functor $F : C \longrightarrow D$ is excisive if and only if it is left exact. If D is stable, then F is excisive if and only if it is right exact. A contra-excisive functor is always just a left exact functor $C^{\mathrm{op}} \longrightarrow D$.

(2.7.2) If D is a stable ∞-category, an excisive functor $F : C \longrightarrow D$ gives rise to an excisive functor $\widetilde{F} : C_\star \longrightarrow D$ by setting

$$\widetilde{F}(X) := F(X)/F(\star).$$

This defines an equivalence of ∞-categories

$$\mathrm{Exc}(C, D) \simeq \mathrm{Exc}(C_\star, D).$$

(2.7.3) Under Spanier–Whitehead duality, a contra-excisive functor $C^{\mathrm{op}} \longrightarrow \mathcal{S}p^{\mathrm{fin}}$ corresponds to an excisive functor

$$C \xrightarrow{G^{\mathrm{op}}} \mathcal{S}p^{\mathrm{fin,op}} \xrightarrow[\sim]{} \mathcal{S}p^{\mathrm{fin}}.$$

This defines an equivalence of ∞-categories

$$\mathrm{Exc}_{\mathrm{op}}(C^{\mathrm{op}}, \mathcal{S}p^{\mathrm{fin}}) \simeq \mathrm{Exc}(C, \mathcal{S}p^{\mathrm{fin}})$$

Provided that one is willing to work with pro-spectra, the finiteness restriction can be dropped:

$$\mathrm{Exc}_{\mathrm{op}}(C^{\mathrm{op}}, \mathcal{S}p) \simeq \mathrm{Exc}(C, \mathrm{pro}\,\mathcal{S}p).$$

(2.7.4) Stability of the target is actually unnecessary when considering excisive functors. The functor $\Omega^\infty : \mathcal{S}p \longrightarrow \mathcal{S}$ induces equivalences

$$\mathrm{Exc}(C, \mathcal{S}p^{\mathrm{fin}}) \simeq \mathrm{Exc}(C, \mathcal{S}^{\mathrm{fin}}) \qquad \text{and} \qquad \mathrm{Exc}_{\mathrm{op}}(C^{\mathrm{op}}, \mathcal{S}p^{\mathrm{fin}}) \simeq \mathrm{Exc}_{\mathrm{op}}(C^{\mathrm{op}}, \mathcal{S}^{\mathrm{fin}}).$$

2.8. — A *Mackey functor* for G is an admissible functor $\mathcal{B}_G^+ \longrightarrow \mathcal{S}p$. The ∞-category of Mackey functors for G will be denoted $\mathcal{M}ack_G$. This ∞-category can be described in a large number of ways:

$$\begin{aligned}
\mathcal{M}ack_G &:= \mathrm{Adm}(\mathcal{B}_G^+, \mathcal{S}p) \\
&\simeq \mathrm{Fun}_{\mathrm{rex}}(\mathcal{B}_G^\to, \mathcal{S}p) \simeq \mathrm{Fun}_{\mathrm{ex}}(\mathcal{B}_G, \mathcal{S}p) \\
&\simeq \mathrm{Exc}(\mathcal{B}_G^\to, \mathcal{S}p) \simeq \mathrm{Exc}(\mathcal{B}_G^\to, \mathcal{S}) \\
&\simeq \mathrm{Exc}(\mathcal{B}_G, \mathcal{S}p) \simeq \mathrm{Exc}(\mathcal{B}_G, \mathcal{S}) \\
&\simeq \mathrm{Fun}^{\mathrm{L}}(\mathrm{Ind}(\mathcal{B}_G), \mathcal{S}p) \simeq \mathrm{Fun}^{\mathrm{L}}(\mathrm{Ind}(\mathcal{B}_G^\to), \mathcal{S}p).
\end{aligned}$$

By construction, $\mathcal{M}ack_G$ is a presentable, stable ∞-category. The full subcategory $\mathcal{M}ack_{G,\geq 0}$ generated under extensions and colimits by the essential image of the functor

$$\Sigma^\infty : \mathrm{Fun}_{\mathrm{rex}}(\mathcal{B}_G^\to, \mathcal{S}) \longrightarrow \mathrm{Fun}_{\mathrm{rex}}(\mathcal{B}_G^\to, \mathcal{S}p) \simeq \mathcal{M}ack_G$$

defines an accessible t-structure on $\mathcal{M}ack_G$; this t-structure is both left and right complete. The heart $\mathcal{M}ack_G^\heartsuit$ of this t-structure is an abelian category of "classical" Mackey functors for the 1-truncation of G; there are corresponding functors $\pi_n : \mathcal{M}ack_G \longrightarrow \mathcal{M}ack_G^\heartsuit$.

2.9. — Given a Mackey functor A for G, one can define associated functors

$$A^\star := \ell^\star M : B^\flat(G)^{\mathrm{fin,op}} \longrightarrow \mathcal{S}p \quad \text{and} \quad A_\star := r^\star A : B^\flat(G)^{\mathrm{fin}} \longrightarrow \mathcal{S}p,$$

the first of which is contra-excisive, the second of which is excisive. This defines two "forgetful" functors

$$(-)^\star : \mathcal{M}ack_G \longrightarrow \mathrm{Exc}_{\mathrm{op}}(B^\flat(G)^{\mathrm{fin,op}}, \mathcal{S}p) \quad \text{and} \quad (-)_\star : \mathcal{M}ack_G \longrightarrow \mathrm{Exc}(B^\flat(G)^{\mathrm{fin}}, \mathcal{S}p).$$

Thus a Mackey functor for G splices together a homology theory for finite G-spaces together with a cohomology theory for finite G-spaces using a *base-change formula*; indeed, we see immediately that for any Mackey functor A for G and any pullback square

$$\begin{array}{ccccc} & & L\times_M N & & \\ & \overset{f}{\swarrow} & & \overset{g}{\searrow} & \\ L & & & & N. \\ & \underset{g}{\searrow} & & \underset{f}{\swarrow} & \\ & & M & & \end{array}$$

of $B^{\flat}(G)^{\mathrm{fin}}$, one must have a canonical homotopy

$$f^{\star}g_{\star}\simeq g_{\star}f^{\star}:A(L)\longrightarrow A(N).$$

The tensor product of Mackey functors. — The tensor product of Mackey functors is a special case of the Day convolution product, and it precisely codifies the interaction of the pullback and pushforward morphisms with the multiplicative structure that one sees in algebraic K-theory.

2.10. — Given two Mackey functors

$$A:\mathscr{B}_G\longrightarrow\mathscr{S}p\quad\text{and}\quad B:\mathscr{B}_G\longrightarrow\mathscr{S}p$$

for G, let us describe their tensor product. One can form their *external tensor product* $A\boxtimes B$:

$$\mathscr{B}_G\otimes\mathscr{B}_G\xrightarrow{(M,N)}\mathscr{S}p\otimes\mathscr{S}p\xrightarrow{-\wedge-}\mathscr{S}p.$$

Now one can form the spectrally-enriched left Kan extension of this composite along the symmetric monoidal structure

$$-\odot-:\mathscr{B}_G\otimes\mathscr{B}_G\longrightarrow\mathscr{B}_G.$$

This yields a Mackey functor $A\otimes B$:

$$\begin{array}{ccc} \mathscr{B}_G\otimes\mathscr{B}_G & \xrightarrow{(A,B)} & \mathscr{S}p\otimes\mathscr{S}p \\ {\scriptstyle -\odot-}\big\downarrow & & \big\downarrow{\scriptstyle -\wedge-} \\ \mathscr{B}_G & \xrightarrow[A\otimes B]{} & \mathscr{S}p. \end{array}$$

Here is a formula for the value of $A\otimes B$ on any object K of $B^{\flat}(G)^{\mathrm{fin}}$ using the spectrally-enriched coend:

$$(A\otimes B)(K):=\int^{L,M\in\mathscr{B}_G}\mathscr{B}_G(L\odot M,K)\wedge A(L)\wedge B(M).$$

Morally, given K, one forms the "colimit" over all morphisms $L\odot M\longrightarrow K$ in $\mathscr{B}_G$ of the smash product $A(L)\wedge B(N)$. The only reason that one must put "colimit" in scare quotes is that this colimit must be taken in a fashion that gives due and proper regard to the fact that the morphisms $L\odot M\longrightarrow K$ in $\mathscr{B}_G$ form a spectrum.

2.11. — Now the ∞-category $\mathscr{M}ack_G$ is symmetric monoidal with respect to this tensor product. In fact, the ∞-category $\mathscr{M}ack_G$ is *closed monoidal* with respect to this tensor product; that is, there is an internal Hom functor that is right adjoint to $-\otimes A$.

Green functors. — A *Green functor* is ordinarily defined as a monoid in the symmetric monoidal category of Mackey functors. But our Mackey functors are homotopical in nature; so instead we should ask for a *homotopy coherent monoid*.

2.12. — A *Green functor for G* is an A_∞ algebra in the symmetric monoidal category $\mathscr{M}ack_G$ of Mackey functors over S. A *commutative Green functor for G* is an E_∞ algebra in $\mathscr{M}ack_G$. More generally, for any operad $\mathscr{P}$, one may define a *$\mathscr{P}$-Green functor for G* simply as a $\mathscr{P}$-algebra in $\mathscr{M}ack_G$.

2.13. — Having a monoid structure on a Mackey functor A over S gives, for every finite G-space K, a morphism

$$(A\otimes A)(K) := \int^{L,M\in\mathscr{B}_G}(\mathscr{B}_G(L\odot M,K)\wedge A(L)\wedge A(M))\longrightarrow A(K).$$

In particular, if $K = L\times M$, this reduces to giving a product

$$A(L)\wedge A(M)\longrightarrow A(L\odot M),$$

and conversely, one can verify that it in order to define a multiplication $A\otimes A\longrightarrow A$, it is enough to define such maps in a suitably coherent manner. More precisely, the data of a Green functor is the data of a compatible system of 2-morphisms

$$\begin{array}{ccc} \mathscr{B}_G^{\otimes I} & \xrightarrow{(A,A,\dots,A)} & \mathscr{S}p^{\otimes I} \\ {\scriptstyle\odot_S}\downarrow & \Downarrow & \downarrow{\scriptstyle\wedge} \\ \mathscr{B}_G & \xrightarrow[A]{} & \mathscr{S}p, \end{array}$$

one for every totally ordered finite set I.

More expressively, the data of a Green functor for G is tantamount to the data of a Mackey functor A for G and a homotopy-coherently associative pairing

$$A(L)\wedge A(M)\longrightarrow A(L\odot M)$$

for every pair of finite G-spaces L and M, and a unit morphism

$$S^0\longrightarrow A(\star).$$

2.14. — If A is a Green functor over S, then it is in particular a Mackey functor for G. Thus there are two functors attached to A, namely,

$$A^\star : B^\flat(G)^{\mathrm{fin,op}}\longrightarrow\mathscr{S}p \quad\text{and}\quad A_\star : B^\flat(G)^{\mathrm{fin}}\longrightarrow\mathscr{S}p,$$

and the homotopy associative and unital pairing on A can be viewed as two morphisms of spectra

$$A^\star(L)\wedge A^\star(M)\longrightarrow A^\star(L\odot M) \quad\text{and}\quad A_\star(L)\wedge A_\star(M)\longrightarrow A_\star(L\odot M),$$

each of which is natural in L and M.

We internalize this external tensor product by pulling back along the diagonal functor; hence for any object $K\in B^\flat(G)^{\mathrm{fin}}$, the spectrum $A(K)$ is an A_∞ algebra. The pullback functors all preserve this structure, so

$$A^\star : B^\flat(G)^{\mathrm{fin,op}}\longrightarrow\mathscr{S}p$$

can be viewed as a presheaf of A_∞ ring spectra.

On the other hand, the pushforward maps all preserve the external product, but not necessarily its internalization. It therefore follows that for any morphism $f : L\longrightarrow M$, the morphism

$$f_\star : A_\star(L)\longrightarrow A_\star(M)$$

is a morphism of $A_\star(M)$-modules.

3. Equivariant derived algebraic geometry

Suppose throughout X an affine, and suppose G an affine ∞-gerbe over X.

3.1. — (3.1.1) Denote by $E_\infty(\mathscr{M}ack_G)$ the category of E_∞-algebras in $\mathscr{M}ack_G$, which will here be called *G-equivariant E_∞ rings*. Denote by $E_\infty(\mathscr{M}ack_G)_{\geq 0}$ the full subcategory of *connective G-equivariant E_∞ rings*, i.e., those G-equivariant E_∞ rings that are in the positive part of the t-structure described above.

(3.1.2) Denote by $\mathscr{A}ff_{G,\geq 0} := E_\infty(\mathscr{M}ack_G)^{\mathrm{op}}_{\geq 0}$. The objects of $\mathscr{A}ff_{G,\geq 0}$ will be called *equivariant (derived) affines*; for any connective G-equivariant E_∞ ring A, denote by $\operatorname{Spec} A$ the corresponding object of $\mathscr{A}ff_{G,\geq 0}$.

(3.1.3) Denote by $\mathscr{Aff}_{G,0}$ the opposite category to the category of commutative rings in $\mathscr{Mack}_G^{\heartsuit}$. (These would be the basic objects of "classical" G-equivariant algebraic geometry, had such a thing ever existed.) The Eilenberg-Mac Lane functor defines a functor

$$H : \mathscr{Aff}_{G,0} \longrightarrow \mathscr{Aff}_{G,\geq 0}.$$

(3.1.4) For any G-equivariant affine $S = \operatorname{Spec} A$, denote by $\mathscr{Perf}(S)$ the ∞-category of compact objects in the ∞-category $\mathscr{Mod}(S)$ of A-modules.

(3.1.5) For any G-equivariant affine S and any topology or hypertopology τ on $\mathscr{Aff}_{G,\geq 0,/S}$, we denote by $\mathscr{S}^{\tau}(\mathscr{Aff}_{G,\geq 0,/S})$ the corresponding ∞-topos. The objects of $\mathscr{S}^{\tau}(\mathscr{Aff}_{G,\geq 0,/S})$ are presheaves of spaces

$$F : \mathscr{Aff}^{\mathrm{op}}_{G,\geq 0,/S} \longrightarrow \mathscr{Kan}$$

that satisfy descent (or hyperdescent) with respect to τ. Pulling back τ along the Eilenberg-Mac Lane functor H defines a topology τ_0 on $\mathscr{Aff}_{G,0}$; there is an induced adjunction

$$H_! : \mathscr{S}^{\tau_0}(\mathscr{Aff}_{G,0}) \rightleftarrows \mathscr{S}^{\tau}(\mathscr{Aff}_{G,\geq 0}) : H^\star,$$

the left adjoint of which is fully faithful, and which is, again, *not* a morphism of ∞-topoi.

(3.1.6) Suppose X a G-equivariant affine, and suppose τ any topology or hypertopology on $\mathscr{Aff}_{\geq 0,/S}$. Attached to any sheaf F of ∞-categories on $\mathscr{Aff}_{G,\geq 0,/S}$ for τ is a cartesian fibration

$$\Gamma^{\tau}(F) \longrightarrow \mathscr{S}^{\tau}(\mathscr{Aff}_{G,\geq 0,/S}),$$

given by the ∞-categorical analogue of the Grothendieck construction. The fibers of this cartesian fibration over an affine S (more precisely, over the sheafification of the presheaf represented by S) is canonically equivalent to the ∞-category FS.

Flatness and the flat hypertopology. — Much of the theory of flat families from derived algebraic geometry translates directly to the G-equivariant context.

Theorem 3.2. *— The following are equivalent for a morphism $f : T = \operatorname{Spec} B \longrightarrow \operatorname{Spec} A = S$ of G-equivariant affines.*

(3.2.1) As an A-module, B can be written as a filtered colimit of finitely generated free A-modules.

(3.2.2) For any discrete A-module M, the B-module $f^\star M := M \otimes_A B$ is discrete.

(3.2.3) The functor

$$f^\star : \mathscr{Mod}(S) \longrightarrow \mathscr{Mod}(T)$$

is left t-exact, so that it carries $\mathscr{Mod}(S)_{\leq 0}$ into $\mathscr{Mod}(T)_{\leq 0}$.

(3.2.4) The following pair of conditions is satisfied.

(3.2.4.1) The induced homomorphism $\operatorname{Spec} \pi_0 B \longrightarrow \operatorname{Spec} \pi_0 A$ is a flat morphism of ordinary G-equivariant schemes.

(3.2.4.2) For every integer $j \in \mathbf{Z}$, the homomorphism

$$\pi_j A \otimes_{\pi_0 A} \pi_0 B \longrightarrow \pi_j B$$

of $\pi_0 B$-modules is an isomorphism.

In this case, the morphism f will be called flat.

Theorem 3.3. *— The following are equivalent for a flat morphism $f : T = \operatorname{Spec} B \longrightarrow \operatorname{Spec} A = S$ of G-equivariant affines.*

(3.3.1) The functor

$$f^\star : \mathscr{Mod}(S) \longrightarrow \mathscr{Mod}(T)$$

is conservative, so that for any nonzero A-module M, the B-module $f^\star M := M \otimes_A B$ is nonzero.

(3.3.2) The induced morphism $\operatorname{Spec} \pi_0 B \longrightarrow \operatorname{Spec} \pi_0 A$ of ordinary G-equivariant affines is faithfully flat.

In this case, the morphism f will be called faithfully flat.

3.4. — A simplicial object $V_\bullet$ of $\mathscr{A}\!\mathit{ff}_{G,\geq 0,/U}$ is a *flat hypercovering* of an affine U if for any integer $n \geq 0$, the morphism

$$(\mathrm{sk}_{n-1} V_\bullet)_n \longrightarrow U$$

is faithfully flat.

Along with Čech nerves of covering families

$$\{ U_i \longrightarrow \coprod_{j\in I} U_j \}_{i\in I},$$

the flat hypercoverings generate the *flat hypertopology* $\flat$ on the ∞-category $\mathscr{A}\!\mathit{ff}_{G,\geq 0,/S}$ for a G-equivariant affine S. The corresponding *flat ∞-topos* $\mathscr{S}^\flat(\mathscr{A}\!\mathit{ff}_{G,\geq 0,/S})$ is hypercomplete. A presheaf

$$F : \mathscr{A}\!\mathit{ff}^{\mathrm{op}}_{G,\geq 0,/S} \longrightarrow \mathscr{K}\!\mathit{an}$$

is a *flat hypersheaf over S* if it lies in $\mathscr{S}^\flat(\mathscr{A}\!\mathit{ff}_{G,\geq 0,/S})$, i.e., if the following two conditions are satisfied.

(3.4.1) For any object $U \in \mathscr{A}\!\mathit{ff}_{G,\geq 0,/S}$ and any flat hypercovering $V_\bullet$ of U, the induced morphism

$$FU \longrightarrow \lim FV_\bullet$$

is an equivalence.

(3.4.2) For any object $U = \coprod_{i\in I} U_i \in \mathscr{A}\!\mathit{ff}_{G,\geq 0,/S}$, the induced morphism

$$FU \longrightarrow \prod_{i\in I} FU_i$$

is an equivalence.

Theorem 3.5. — *Suppose S a G-equivariant affine. The flat hypertopology $\mathscr{A}\!\mathit{ff}_{G,\geq 0,/S}$ is subcanonical. Moreover, the assignments*

$$\mathscr{M}\!\mathit{od} : U \longmapsto \mathscr{M}\!\mathit{od}(U) \qquad \text{and} \qquad \mathscr{P}\!\mathit{erf} : U \longmapsto \mathscr{P}\!\mathit{erf}(U)$$

are flat hypersheaves of ∞-categories. In particular, the associated presheaves $\iota\mathscr{M}\!\mathit{od}$ and $\iota\mathscr{P}\!\mathit{erf}$ of spaces are flat hypersheaves.

The cotangent complex and the étale topology. — One of the easiest pieces of classical algebraic geometry to transfer to the derived setting is Illusie's *cotangent complex*.

3.6. — Suppose $f : T = \mathrm{Spec}\, B \longrightarrow \mathrm{Spec}\, A = S$ a morphism of G-equivariant affines. For any B-module M, one has an associated *square zero extension $B \oplus M$* of B. Write $T_M := \mathrm{Spec}(B \oplus M)$; there is an obvious morphism $T \longrightarrow T_M$. Now define the *space of derivations on T over S with coefficient in M* as the fiber $\mathrm{Der}_S(T;M)$ of the morphism of spaces

$$\mathrm{Mor}_S(T_M, T) \longrightarrow \mathrm{Mor}_S(T,T)$$

over the identity map. The result is a functor

$$\mathrm{Der}_S(T;-) : \mathscr{M}\!\mathit{od}(T) \longrightarrow \mathscr{S}.$$

Theorem 3.7. — *For any morphism $f : T = \mathrm{Spec}\, B \longrightarrow \mathrm{Spec}\, A = S$ of G-equivariant affines, the functor* $\mathrm{Der}_S(T;-)$ *is corepresentable; that is, there exists a T-module $\mathbf{L}_{T|S}$ and an equivalence*

$$\mathrm{Der}_S(T;M) \simeq \mathrm{Mor}(\mathbf{L}_{T|S}, M),$$

functorial in M. The representing object $\mathbf{L}_{T|S}$ is called the cotangent complex for f, *and the morphism*

$$d : T_{\mathbf{L}_{T|S}} \longrightarrow T$$

corresponding to the identity of $\mathbf{L}_{T|S}$ is called the universal derivation for f.

3.8. — Suppose $f : T = \operatorname{Spec} B \longrightarrow \operatorname{Spec} A = S$ a morphism of G-equivariant affines. By applying the functor $(-)^\star$, one obtains an induced morphism $f : T^\star = \operatorname{Spec} B^\star \longrightarrow \operatorname{Spec} A^\star = S^\star$ of diagrams of affines indexed on $B^\flat(G)^{\mathrm{fin}}$. One can therefore apply the cotangent complex "objectwise" to obtain $\mathbf{L}_{T^\star|S^\star}$. Since every derivation of B over A gives rise to a derivation of $B^\star$ over $A^\star$, there is an induced morphism of $T^\star$-modules

$$\mathbf{L}_{T^\star|S^\star} \longrightarrow \mathbf{L}_{T|S}^\star .$$

Adjoint to this morphism is a morphism

$$F_T \mathbf{L}_{T^\star|S^\star} \longrightarrow \mathbf{L}_{T|S} .$$

where $F_T \mathbf{L}_{T^\star|S^\star}$ denotes the free T-module generated by $\mathbf{L}_{T^\star|S^\star}$. Denote by $\mathbf{E}_{T|S}$ the fiber of this morphism; it corepresents the functor

$$M \longmapsto \operatorname{Der}_{S^\star}(T^\star; M^\star)/\operatorname{Der}_S(T; M).$$

This is often the beginning of a flitration

$$F_T \mathbf{L}_{T|S} = F^0 \mathbf{L}_{T|S} \subset F^1 \mathbf{L}_{T|S} \subset \cdots \subset \mathbf{L}_{T|S}$$

whose graded pieces $F^{j+1/j} \mathbf{L}_{T|S}$ involve only transfers "for a small collection of groups." In very good cases, one can actually analyze the corresponding spectral sequence.

3.9. — Suppose $f : T = \operatorname{Spec} B \longrightarrow \operatorname{Spec} A = S$ a morphism of G-equivariant affines, locally of finite presentation (so that B is a compact object in the category of G-equivariant E_∞ rings under A).

(3.9.1) One says that f is *smooth* if $\mathbf{L}_{T|S}$ is compact in $\mathcal{M}od(S)$.

(3.9.2) One says that f is *étale* if $\mathbf{L}_{T|S} \simeq 0$.

Theorem 3.10. — *The following are equivalent for a morphism $f : T = \operatorname{Spec} B \longrightarrow \operatorname{Spec} A = S$ of G-equivariant affines.*

(3.10.1) The morphism f is étale.

(3.10.2) The following pair of conditions is satisfied.

(3.10.2.1) The induced homomorphism $\operatorname{Spec} \pi_0 B \longrightarrow \operatorname{Spec} \pi_0 A$ *is an étale morphism of ordinary G-equivariant schemes.*

(3.10.2.2) For every integer $j \in \mathbf{Z}$, the homomorphism

$$\pi_j A \otimes_{\pi_0 A} \pi_0 B \longrightarrow \pi_j B$$

of $\pi_0 B$-modules is an isomorphism.

3.11. — Suppose S a G-equivariant affine. A family

$$\{ V_i \longrightarrow U \}_{i \in I}$$

is an *étale covering* if each morphism $V_i \longrightarrow U$ is étale, and for some finite subset $I' \subset I$, the morphism

$$\textstyle\coprod_{i \in I'} V_i \longrightarrow U$$

is faithfully flat.

These families generate the *étale topology* ét on the ∞-category $\mathcal{A}ff_{\geq 0,/S}$. The corresponding *étale ∞-topos* $\mathcal{S}^{\text{ét}}(\mathcal{A}ff_{G,\geq 0,/S})$ is not hypercomplete.

4. Example: The K-theory of tannakian ∞-categories

Tannakian ∞-categories over an affine X are symmetric monoidal ∞-categories that possess all the good properties of the symmetric monoidal ∞-category of perfect complexes on an affine ∞-gerbe over X. Suppose throughout this subsection that X is an affine.

Higher Tannaka duality. — There is a generalization of the classical Tannaka duality theorems of Saavedra-Rivano and Deligne, due to Toën, Lurie, and others. These involve Tannakian ∞-categories, which are well-behaved symmetric monoidal stable ∞-categories.

4.1. — A *tensor ∞-category* is a symmetric monoidal ∞-category A satisfying the following properties.

(4.1.1) The underlying ∞-category is stable.

(4.1.2) For any object M of A, the endofunctor $M \otimes -$ of A is exact.

Denote by $\mathscr{T}ens$ the ∞-category of tensor ∞-categories with symmetric monoidal exact functors.

Theorem 4.2. — *The functor*

$$\begin{aligned} \mathscr{M}od : E_\infty(\mathscr{S}p) &\longrightarrow \mathscr{T}ens \\ A &\longmapsto \mathscr{M}od(A) \end{aligned}$$

is fully faithful, with left adjoint given by the functor

$$\begin{aligned} \mathscr{T}ens &\longrightarrow E_\infty(\mathscr{S}p) \\ A &\longmapsto \mathrm{End}(\mathbf{1}_A). \end{aligned}$$

4.3. — A *rigid tensor ∞-category* is a tensor ∞-category A satisfying the following properties.

(4.3.1) As a symmetric monoidal ∞-category, A is *closed*, so that for any object M of A, the functor $M \otimes -$ admits a right adjoint $\underline{\mathrm{Mor}}_A$.

(4.3.2) For any object M of A the natural morphism

$$M^\vee \otimes M \longrightarrow \underline{\mathrm{Mor}}_A(M, M),$$

where we write $M^\vee := \underline{\mathrm{Mor}}_A(M, \mathbf{1}_A)$, is an equivalence.

Denote by $\mathscr{R}ig$ the ∞-category of rigid tensor ∞-categories with symmetric monoidal exact functors.

This can be relativized as well: a *rigid X-tensor ∞-category* is a symmetric monoidal functor $\mathscr{P}erf(X) \longrightarrow A$ of rigid tensor ∞-categories. The ∞-category of these is denoted $\mathscr{R}ig_{/X}$.

4.4. — A good supply of rigid tensor ∞-categories is provided by the functor

$$\begin{aligned} \mathscr{P}erf_{/X} : \mathscr{S}^\flat(\mathscr{A}ff_{\geq 0,/X}) &\longrightarrow \mathscr{R}ig_{/X} \\ F &\longmapsto \mathscr{P}erf(F). \end{aligned}$$

In the other direction, for any rigid X-tensor ∞-category A, denote by $\mathrm{Fib}_{/X}(A)$ the presheaf of spaces given by

$$\begin{aligned} \mathrm{Fib}_{/X}(A) : \mathscr{A}ff_{\geq 0,/X} &\longrightarrow \mathscr{K}an \\ U &\longmapsto \mathrm{Mor}_{\mathscr{R}ig_{/X}}(A, \mathscr{P}erf(U)). \end{aligned}$$

This presheaf is a flat hypersheaf.

Theorem 4.5. — *The resulting functor*

$$\begin{aligned} \mathrm{Fib}_{/X} : \mathscr{R}ig_{/X} &\longrightarrow \mathscr{S}^\flat(\mathscr{A}ff_{\geq 0,/X}) \\ A &\longmapsto \mathrm{Fib}_{/X}(A) \end{aligned}$$

is left adjoint to the functor $\mathscr{P}erf_{/X}$.

4.6. — A rigid X-tensor ∞-category A equipped with a t-structure is said to be *tannakian over X* or *X-tannakian* if it satisfies the following properties.

(4.6.1) The natural morphism $\operatorname{Spec}\mathrm{End}(\mathbf{1}_A) \longrightarrow X$ is an equivalence.

(4.6.2) There exists a faithfully flat affine Y over X along with a symmetric monoidal t-exact functor

$$\omega : A \longrightarrow \mathscr{P}erf(Y)$$

such that the induced functor $\overline{\omega} : \mathrm{Ind} A \longrightarrow \mathscr{M}od(Y)$ is conservative and t-exact.

The ∞-category of X-tannakian ∞-categories with exact, symmetric monoidal, t-exact functors is denoted $\mathscr{T}an_{/X}$.

4.7. — The adjunction $(\mathrm{Fib}_{/X}, \mathscr{P}\!\mathit{erf}_{/X})$ can be restricted to an adjunction between the ∞-category of X-tannakian categories and affine ∞-gerbes over X. Indeed, if G is an affine ∞-gerbe over X, then the rigid X-tensor ∞-category $\mathscr{P}\!\mathit{erf}(G)$ is tannakian over X. This defines a functor

$$\begin{aligned} \mathscr{P}\!\mathit{erf}^t_{/X} : \mathscr{G}\mathit{erbe}_{\geq 0,/X} &\longrightarrow \mathscr{T}\mathit{an}_{/X} \\ G &\longmapsto \mathscr{P}\!\mathit{erf}(G). \end{aligned}$$

In the other direction, for any X-tannakian category A, there is a presheaf of spaces

$$\begin{aligned} \mathrm{Fib}^t_{/X}(A) : \mathscr{A}\mathit{ff}_{\geq 0,/X} &\longrightarrow \mathscr{K}\mathit{an} \\ U &\longmapsto \mathrm{Mor}_{\mathscr{T}\mathit{an}_{/X}}(A, \mathscr{P}\!\mathit{erf}(U)). \end{aligned}$$

This is the *affine ∞-gerbe of fiber functors for A*. This defines a functor

$$\begin{aligned} \mathrm{Fib}^t_{/X} : \mathscr{T}\mathit{an}_{/X} &\longrightarrow \mathscr{G}\mathit{erbe}_{\geq 0,/X} \\ A &\longmapsto \mathrm{Fib}^t_{/X}(A), \end{aligned}$$

left adjoint to $\mathscr{P}\!\mathit{erf}^t_{/X}$.

***Theorem 4.8* (Higher Tannaka duality).** — *The adjunction $(\mathrm{Fib}^t_{/X}, \mathscr{P}\!\mathit{erf}^t_{/X})$ provides an equivalence of ∞-categories.*

K-theory of tannakian categories. — The algebraic K-theory of a tannakian category has a much richer structure than merely that of a spectrum. In fact, it can be regarded as an E_∞ Green functor for its affine ∞-gerbe of fiber functors. This provides an important example of equivariant affines.

4.9. — Suppose A an X-tannakian category, and let $G := \mathrm{Fib}_{/X}(A)$ be the corresponding affine ∞-gerbe of fiber functors. Define a functor

$$\begin{aligned} \mathbf{K}(A) : \mathscr{B}^+_G &\longrightarrow \mathscr{S}\!p \\ L &\longmapsto K\,\mathscr{P}\!\mathit{erf}(\tau_{G,!}L), \end{aligned}$$

assigning to any finite object L of the classifying ∞-topos of G the K-theory spectrum $K([L/G])$ of the category of perfect complexes on the quotient stack $[L/G]$. Observe that its value on the object $\star \in B^\flat(G)^{\mathrm{fin}}$ is simply the algebraic K-theory spectrum of A.

The finiteness assumptions guarantee that for any morphism $f : L \longrightarrow M$ of finite G-spaces, both pullback morphisms $f^\star$ and pushforward morphisms $f_\star$ are well-defined. It follows from a Zariski descent statement for algebraic K-theory that the functor $\mathbf{K}(A)$ is in fact admissible; hence $\mathbf{K}(A)$ is a Mackey functor.

For any morphism $f : L \longrightarrow M$, the functor $f^\star$ is compatible with the tensor products, in the sense that for any perfect complexes E and F on the stack $[M/G]$, there is a canonical isomorphism

$$f^\star(E \otimes F) \simeq f^\star E \otimes f^\star F,$$

so the induced morphism $f^\star$ on K-theory is a morphism of connective E_∞ ring spectra.

One may externalize this in the following manner: for any two finite G-spaces L and M, one may define an external tensor product:

$$\boxtimes : \mathscr{P}\!\mathit{erf}([L/G]) \times \mathscr{P}\!\mathit{erf}([M/G]) \longrightarrow \mathscr{P}\!\mathit{erf}([(L \times M)/G]).$$

This gives rise to an external pairing

$$K([L/G]) \wedge K([M/G]) \longrightarrow K([(L \times M)/G]).$$

Moreover, for any morphism $f : L \longrightarrow M$, one may regard $K([L/G])$ as a module over the E_∞ ring spectrum $K([M/G])$ via $f^\star$. For any perfect complexes E on $[L/G]$ and F on $[M/G]$, the canonical morphism

$$f_\star E \otimes F \longrightarrow f_\star(E \otimes f^\star F)$$

is an equivalence; this is the usual projection formula. At the level of K-theory, this translates to the observation that the diagram

$$\begin{array}{ccccc} & & K([L/G])\wedge K([L/G]) & \xrightarrow{\mu} & K([L/G]) \\ & {\scriptstyle (\mathrm{id}\wedge f^*)}\nearrow & & & \searrow{\scriptstyle f_\star} \\ K([L/G])\wedge K([M/G]) & \xrightarrow[(f_\star\wedge\mathrm{id})]{} & K([M/G])\wedge K([M/G]) & \xrightarrow[\mu]{} & K([M/G]) \end{array}$$

commutes up to coherent homotopy. In other words, the morphism

$$f_\star : K([L/G]) \longrightarrow K([M/G])$$

is a morphism of connective $K([M/G])$-modules.

Using these observations, one concludes that $\mathbf{K}(A)$ is in fact an E_∞ Green functor. One therefore defines a G-equivariant affine

$$\mathbf{S}(A) := \operatorname{Spec}\mathbf{K}(A)$$

over X. It's important to note that $\mathbf{S}(A)$ is totally intrinsic to the tannakian ∞-category over X; it depends on no additional structure. In fact, using an elaborated version of higher tannakian duality, one can construct $\mathbf{S}(A)$ entirely in terms of various categories related to A; one need not even mention the affine ∞-gerbe of fiber functors.

4.10. — Here is a relative version of the above construction. Suppose N a fixed G-space. Then one can define a functor

$$\begin{aligned} \mathbf{K}(A;N) : \mathscr{B}_G^+ &\longrightarrow \mathscr{S}p \\ L &\longmapsto K\,\mathscr{P}\!erf(\tau_{G,!}(L\times N)). \end{aligned}$$

The arguments above can be used to show that this too is an E_∞ Green functor. One therefore defines a G-equivariant affine

$$\mathbf{S}(A;N) := \operatorname{Spec}\mathbf{K}(A;N)$$

over X. This construction is functorial in N; it defines a functor

$$\begin{aligned} \mathbf{S}(A;-) : B^\flat(G) &\longrightarrow \mathscr{A}\!f\!f_{G,\geq 0} \\ N &\longmapsto \mathbf{S}(A;N). \end{aligned}$$

5. Application: K-theory of Galois representations

5.1. — Suppose k a field, and suppose $\overline{k}$ its algebraic closure. Write G_k for the absolute Galois group. Consider the k-tannakian ∞-category $A_k := \mathscr{P}\!erf(BG_k)$. The construction of the previous section permits us to form the G_k-equivariant affine $\mathbf{S}(A_k)$.

Suppose also that X is a geometrically connected variety over k. Write $\overline{X} := X \times_{\operatorname{Spec} k} \operatorname{Spec}\overline{k}$. One may form the G_k-equivariant affine $\mathbf{S}(A_k;\overline{X})$. One can also consider X itself with the trivial G_k action, yielding a G_k-equivariant affine $\mathbf{S}(A_k;X)$. There is a canonical morphism

$$\alpha : \mathbf{S}(A_k;\overline{X}) \longrightarrow \mathbf{S}(A_k;X).$$

We wish to study this morphism.

5.2. — In the G_k-equivariant setting, complete information about Green functors can be obtained by considering the values on finite G_k-sets, in particular on finite orbits (G_k/H). We have the following observations.

(5.2.1) The Green functor $\pi_*\mathbf{K}(A_k;X)$ assigns to any orbit (G_k/H) the K-theory of the category $\operatorname{Rep}_X[H]$ of variations of representations of H over X. In particular,

$$\pi_*^{\{1\}}\mathbf{K}(A_k;X) \cong K_*(X) \qquad \text{and} \qquad \pi_*^{G_k}\mathbf{K}(A_k;X) \cong K_*\operatorname{Rep}_X[G_k].$$

(Strictly speaking, of course, the subgroup $\{1\} \subset G_k$ isn't actually an option here; rather, I want to suggest that $\mathbf{K}(A_k;X)$ is an attempt at interpolation between $K(X)$ and $K\operatorname{Rep}_X[G_k]$.)

(5.2.2) The Green functor $\pi_*\mathbf{K}(A_k;\overline{X})$ assigns to any orbit (G_k/H) the K-theory of $X\times_{\operatorname{Spec}k}\operatorname{Spec}(\overline{k}^H)$. In particular,

$$\pi_*^{\{1\}}\mathbf{K}(A_k;\overline{X})\cong K_*(\overline{X}) \qquad \text{and} \qquad \pi_*^{G_k}\mathbf{K}(A_k;\overline{X})\cong K_*(X).$$

Observe that α is very from being an equivalence.

5.3. — By abuse, write $\mathbf{Z}$ for the *constant* (classical) Green functor for G_k at the integers. Recall that the constant Green functor has the following properties.

(5.3.1) For any subgroup $H\subset G_k$ of finite index, its value at (G_k/H) is $\mathbf{Z}$.
(5.3.2) For any subgroups $H\subset H'\subset G_k$ of finite index, the corresponding pullback morphism $\mathbf{Z}\longrightarrow\mathbf{Z}$ is the identity.
(5.3.3) For any subgroups $H\subset H'\subset G_k$ of finite index, the corresponding pushforward morphism $\mathbf{Z}\longrightarrow\mathbf{Z}$ is multiplication by the index of H in H'.

Now the *rank* homomorphism gives rise to a commutative diagram of G_k-equivariant affines:

$$\begin{array}{ccc} & \operatorname{Spec}H\mathbf{Z} & \\ \swarrow & & \searrow \\ \mathbf{S}(A_k;\overline{X}) & \xrightarrow[\alpha]{} & \mathbf{S}(A_k;X). \end{array}$$

Suppose now ℓ an odd prime with $1/\ell\in\mathscr{O}_X$; then the *mod ℓ rank* yields a triangle:

$$\begin{array}{ccc} & \operatorname{Spec}H(\mathbf{Z}/\ell) & \\ \swarrow & & \searrow \\ \mathbf{S}(A_k;\overline{X}) & \xrightarrow[\alpha]{} & \mathbf{S}(A_k;X). \end{array}$$

Thus $\operatorname{Spec}H(\mathbf{Z}/\ell)$ can be regarded as an equivariant derived subscheme of both $\mathbf{S}(A_k,\overline{X})$ and $\mathbf{S}(A_k,X)$, and the morphism α maps one subscheme isomorphically onto the other.

5.4. — Now denote by $\mathbf{S}(A_k;\overline{X})_\ell^{\langle n\rangle}$ and $\mathbf{S}(A_k;X)_\ell^{\langle n\rangle}$ the "2^{n-1}-th infinitesimal neighborhood" of $\operatorname{Spec}H(\mathbf{Z}/\ell)$ in $\mathbf{S}(A_k;\overline{X})$ and $\mathbf{S}(A_k;X)$, defined in the following manner. Set

$$\mathbf{S}(A_k;\overline{X})_\ell^{\langle 0\rangle}:=H(\mathbf{Z}/\ell)=:\mathbf{S}(A_k;X)_\ell^{\langle 0\rangle},$$

and for any $n>0$, let $\mathbf{S}(A_k;\overline{X})_\ell^{\langle n\rangle}$ and $\mathbf{S}(A_k;X)_\ell^{\langle n\rangle}$ be the Spec of the fibers of the universal derivations

$$\mathbf{K}(A_k;\overline{X})_\ell^{\langle n-1\rangle}\longrightarrow\mathbf{L}_{\mathbf{K}(A_k;\overline{X})|\mathbf{K}(A_k;\overline{X})_\ell^{\langle n-1\rangle}} \qquad \text{and} \qquad \mathbf{K}(A_k;X)_\ell^{\langle n-1\rangle}\longrightarrow\mathbf{L}_{\mathbf{K}(A_k;X)|\mathbf{K}(A_k;X)_\ell^{\langle n-1\rangle}}$$

(which have automatic E_∞ structures).

There is a morphism

$$\alpha_\ell^{\langle n\rangle}:\mathbf{S}(A_k;\overline{X})_\ell^{\langle n\rangle}\longrightarrow\mathbf{S}(A_k;X)_\ell^{\langle n\rangle}.$$

Note that this infinitesimal neighborhood is *not* obtained by taking forming the infinitesimal neighborhood objectwise, even on π_0.

Taking the colimit over n (in the category of flat hypersheaves), one obtains a commutative diagram

$$\begin{array}{ccc} \mathbf{S}(A_k;\overline{X})_\ell^\wedge & \longrightarrow & \mathbf{S}(A_k;X)_\ell^\wedge \\ \downarrow & & \downarrow \\ \mathbf{S}(A_k;\overline{X}) & \xrightarrow[\alpha]{} & \mathbf{S}(A_k;X). \end{array}$$

A computation shows that the top morphism is a π_0-isomorphism.

5.5. — We are studying the ℓ-adic completion of $\mathbf{S}(A_k;\overline{X})$ and $\mathbf{S}(A_k;\overline{X})$ relative to the rank. A warning about this: in general, the value of the ℓ-adic completion of a morphism of Green functors $A\longrightarrow H(\mathbf{Z}/\ell)$ on G is *not* simply the ℓ-adic completion of the value of A; one has

$$A_\ell^\wedge \simeq \lim_{\Delta}[\,\mathbf{n}\longmapsto H(\mathbf{Z}/\ell)^{\wedge_A n}\,]$$

in the ∞-category of Green functors for G_k. Since the smash product in Mackey functors for G is not the objectwise smash product, this is an intricate object. There is a Mackey functor-valued spectral sequence for computing the completion:

$$E^2_{p,q} = \pi_p((\pi_* A)^\wedge_\ell)_q \Longrightarrow \pi_{p+q}(A^\wedge_\ell),$$

but in general this is difficult to use, because the completion of $\pi_* A$ taken here is the completion of $H\pi_* A$.

5.6. — Let us assume from now on that k is perfect, and X is smooth. M. Walker, following ideas of Grayson, introduced a filtration on the K-theory of X:

$$\dots\longrightarrow W^2(X)\longrightarrow W^1(X)\longrightarrow W^0(X)=K(X),$$

whose successive quotients $W^{j/j+1}(X)$ are (at least rationally) pure of weight j. This filtration is a descending sequence of (E_∞) ideals in $K(X)$. Moreover, the filtration on $K_*(X)$ given by the spectral sequence

$$E_2^{p,q} = \pi_{p+q} W^{q/q+1}(X) \Longrightarrow K_{p+q}(X)$$

coincides rationally with the γ-filtration on $K_*(X)$.

In particular, the first quotient $W^{0/1}(X)$ is $H\mathbf{Z}$, and the second quotient $W^{1/2}(X)$ is a spectrum with homotopy groups

$$\pi_j W^{1/2}(X) \cong H^{2-j}_{\mathrm{mot}}(X;\mathbf{Z}(1)).$$

In general, the spectra $W^{j/j+1}(X)$ are $(j+1)$-truncated, and it follows from work of Suslin that

$$\pi_{2j-i}(W^{j/j+1}(X)) \cong H^i_{\mathrm{mot}}(X,\mathbf{Z}(j)).$$

For our purposes here, we shall regard this left hand homotopy group as the *definition* of motivic cohomology, despite the fact that there is another “official” definition.

The quotients $W^{0/t}(X)$ carry an E_∞ structure as well; in fact they can be shown to coincide with a certain K-theory spectrum.

5.7. — The W-filtration can be made to be fully equivariant for G_k, in the sense that one has filtrations

$$\dots\longrightarrow \mathbf{W}^2(A_k;X)\longrightarrow \mathbf{W}^1(A_k;X)\longrightarrow \mathbf{W}^0(A_k;X) = \mathbf{K}(A_k;X)$$

and

$$\dots\longrightarrow \mathbf{W}^2(A_k;\overline{X})\longrightarrow \mathbf{W}^1(A_k;\overline{X})\longrightarrow \mathbf{W}^0(A_k;\overline{X}) = \mathbf{K}(A_k;\overline{X}),$$

each of which is a descending sequence of E_∞ ideals.

The quotient maps $\mathbf{K}(A_k;X)\longrightarrow \mathbf{W}^{0/1}(A_k;X)$ and $\mathbf{K}(A_k;\overline{X})\longrightarrow \mathbf{W}^{0/1}(A_k;\overline{X})$ can be identified with the rank morphisms $\mathbf{K}(A_k;X)\longrightarrow H\mathbf{Z}$ and $\mathbf{K}(A_k;\overline{X})\longrightarrow H\mathbf{Z}$.

This leads us to contemplate two kinds of *G_k-equivariant motivic cohomology*; these are Mackey functors

$$\mathbf{H}^i_{\mathrm{mot}}(A_k,X;\mathbf{Z}(j)) := \pi_{2j-i}\mathbf{W}^{j/j+1}(A_k;X) \qquad \text{and} \qquad \mathbf{H}^i_{\mathrm{mot}}(A_k,\overline{X};\mathbf{Z}(j)) := \pi_{2j-i}\mathbf{W}^{j/j+1}(A_k;\overline{X}).$$

Let us define

$$\mathbf{T}^j(A_k;X) := \operatorname{Spec}\mathbf{W}^{0/j}(A_k;X) \qquad \text{and} \qquad \mathbf{T}^j(A_k;\overline{X}) := \operatorname{Spec}\mathbf{W}^{0/j}(A_k;\overline{X}).$$

One may contemplate the commtuative diagram

$$\begin{array}{ccccc} & & \mathbf{T}^0(A_k;\overline{X}) & \longrightarrow & \mathbf{T}^j(A_k;\overline{X}) \\ & \overset{\sim}{\nearrow} & & & \downarrow \\ \operatorname{Spec} H(\mathbf{Z}/\ell) \longrightarrow \operatorname{Spec} H\mathbf{Z} & & & & \\ & \underset{\sim}{\searrow} & & & \\ & & \mathbf{T}^0(A_k;X) & \longrightarrow & \mathbf{T}^j(A_k;X), \end{array}$$

and thus also the n-th infinitesimal formal neighborhoods of $\operatorname{Spec} H(\mathbf{Z}/\ell)$ in each: $\mathbf{T}^j(A_k;X)_\ell^{\langle n\rangle}$ and $\mathbf{T}^j(A_k;\overline{X})_\ell^{\langle n\rangle}$.

Lemma 5.8. — *Consider the following cube:*

$$\begin{array}{ccccccc} \mathbf{T}^j(A_k;\overline{X})_\ell^{\langle n\rangle} & & \longrightarrow & & \mathbf{T}^j(A_k;X)_\ell^{\langle n\rangle} & & \\ \downarrow & \searrow & & & \downarrow & \searrow & \\ & & \mathbf{S}(A_k;\overline{X})_\ell^{\langle n\rangle} & \longrightarrow & & & \mathbf{S}(A_k;X)_\ell^{\langle n\rangle} \\ \mathbf{T}^j(A_k;\overline{X}) & & \longrightarrow & & \mathbf{T}^j(A_k;X) & & \downarrow \\ & \searrow & \downarrow & & & \searrow & \\ & & \mathbf{S}(A_k;\overline{X}) & \longrightarrow & & & \mathbf{S}(A_k;X). \end{array}$$

The top and bottom squares are pullbacks.

Lemma 5.9. — *For any integer $j>0$, the connectivity of the cotangent complex*

$$\mathbf{L}_{(\mathbf{T}^j(A_k;\overline{X})_\ell^{\langle n\rangle} \mid \mathbf{T}^j(A_k;X)_\ell^{\langle n\rangle})}$$

increases without bound as $n\to\infty$.

Note. — The pullback diagrams permit one to compute this cotangent complex by pulling back the cotangent complex

$$\mathbf{L}_{(\mathbf{S}(A_k;\overline{X})_\ell^{\langle n\rangle} \mid \mathbf{S}(A_k;X))_\ell^{\langle n\rangle}}$$

to $\mathbf{T}^j(A_k;\overline{X})_\ell^{\langle n\rangle}$. Then the key point is that the spectra $W^{0/j}$ are j-truncated; hence the relevant spectral sequences are easier to manage. □

Theorem 5.10. — *The morphism on the completions*

$$\mathbf{S}(A_k;\overline{X})^\wedge_\ell \longrightarrow \mathbf{S}(A_k;X)^\wedge_\ell .$$

is an equivalence.

Theorem 5.11. — *The value of $\mathbf{K}(A_k;\overline{X})$ on (G_k/G_k) agrees with the ℓ-adic completion of $K(X)$.*

Note. — In effect, this follows from the fact that the action of G_k on $\mathbf{K}(A_k;\overline{X})$ is free in a suitable sense. □

5.12. — This theorem can be regarded as "Quillen–Lichtenbaum Lite." It confirms a related conjecture of Carlsson.

I expect that, in degrees above the ℓ-adic cohomological dimension of X, the value of $\mathbf{K}(A_k,X)^\wedge_\ell$ on (G_k/G_k) agrees with the ℓ-completion of the homotopy fixed point spectrum $K(\overline{X})^{hG_k}$. I have not been able to confirm this with current methods; however, I hope that a version of the slice spectral sequence developed by Mike Hill, Mike Hopkins, and Doug Ravenel may come in handy.

If one could verify this, the Quillen–Lichtenbaum conjecture would follow. This does not, however, seem to provide a filtration of $K(X)$ coming from the Beilinson–Lichtenbaum spectral sequence; I am unable to say anything intelligent about this.

6. Application: Crystalline realizations of K-theory

6.1. — Suppose V a complete discrete valuation ring of mixed characteristic $(0,p)$, with perfect residue field k and fraction field K; suppose $\pi \in V$ a uniformizer. Denote by $V_0 = W(k)$ the ring of Witt vectors, and denote by K_0 its fraction field.

Let us quickly recall Ogus's convergent site. Suppose X a k-scheme of finite type. An *affine enlargement* of X over V is a commutative diagram

$$\begin{array}{ccc} (T/\pi T)_{\mathrm{red}} & \longrightarrow & X \\ \downarrow & & \downarrow \\ T & \longrightarrow & \operatorname{Spf} V, \end{array}$$

wherein T is an affine, flat formal V-scheme. Endowed with the Zariski topology, the category of affine enlargements of X over V is a site $(X/V)_{\mathrm{conv}}$.

We define two sheaves on $(X/V)_{\mathrm{conv}}$:

$$\mathscr{O}_{X/V} : T \longmapsto \Gamma(T,\mathscr{O}_T) \qquad \text{and} \qquad \mathscr{K}_{X/V} : T \longmapsto \Gamma(T,\mathscr{O}_T) \otimes_V K.$$

The convergent ∞-topos is functorial in (X/V) in the usual manner for cocontinuous morphisms of large sites.

6.2. — The ∞-category of *(derived) crystals* on X over V is the defined as

$$\mathscr{C}ris(X/V) := \lim_{T \in (X/V)^{\mathrm{op}}_{\mathrm{conv}}} \mathscr{M}od(H\mathscr{O}_{X/V}(T)).$$

Put differently, a crystal $\mathscr{E}$ on X over V amounts to an assignment to any affine enlargement T a complex of $\mathscr{O}_T$-modules $\mathscr{E}_T$, and to any morphism $f : T' \longrightarrow T$ of affine enlargements a quasiisomorphism $f^{\star}\mathscr{E}_T \longrightarrow \mathscr{E}_{T'}$, to any composable sequence of morphisms $T'' \longrightarrow T' \longrightarrow T$ a homotopy between the two quasiisomorphisms $f^{\star}g^{\star}\mathscr{E}_T \longrightarrow \mathscr{E}_{T''}$, etc., etc. ...

The ∞-category of *(derived) isocrystals* on X over V is the homotopy limit

$$\mathscr{I}soc(X/V) := \lim_{T \in (X/V)^{\mathrm{op}}_{\mathrm{conv}}} \mathscr{M}od(H\mathscr{K}_{X/V}(T)).$$

Put differently, an isocrystal on X over V amounts to an assignment to any affine enlargement T a complex of $\mathscr{O}_T \otimes_V K$-modules $\mathscr{E}_T$, and to any morphism $f : T' \longrightarrow T$ of affine enlargements a quasiisomorphism $f^{\star}\mathscr{E}_T \longrightarrow \mathscr{E}_{T'}$, to any composable sequence of morphisms $T'' \longrightarrow T' \longrightarrow T$ a homotopy between the two quasiisomorphisms $f^{\star}g^{\star}\mathscr{E}_T \longrightarrow \mathscr{E}_{T''}$, etc., etc. ...

A rational point $x \in X(k)$ defines a fiber functor

$$\varpi_x : \mathscr{I}soc(X/V)^{\omega} \longrightarrow \mathscr{P}erf(\operatorname{Spec} K).$$

One verifies that $\mathscr{I}soc(X/V)^{\omega}$ is a K-tannakian ∞-category.

I emphasize that, despite the fact that the topology plays no role in these definitions, it follows from faithfully flat descent and the degeneration of the Tor spectral sequence that any crystal or isocrystal is additionally a sheaf for the fpqc topology on the category of affine enlargements of X over V.

6.3. — Suppose X a k-scheme of finite type. Denote by ϕ the absolute Frobenius endomorphism on X. Denote by σ a lift of the p-th power morphism on k to K. The pair (ϕ,σ) induces an endofunctor $\phi^{\star}$ of $\mathscr{I}soc(X/V)$. The ∞-category of *(derived) F-isocrystals* is the homotopy equalizer

$$\mathscr{F}\mathscr{I}soc(X/V) := \lim \left[\mathscr{I}soc(X/V) \underset{\mathrm{id}}{\overset{\phi^{\star}}{\rightrightarrows}} \mathscr{I}soc(X/V) \right].$$

Observe that $\mathscr{F}\mathscr{I}soc(X/V)$ is a rigid $\mathbf{Q}_p$-tensor ∞-category (*not* a rigid K-tensor ∞-category). A rational point $x \in X(k)$ defines a fiber functor

$$\varpi_x : \mathscr{F}\mathscr{I}soc(X/V)^{\omega} \longrightarrow \mathscr{P}erf(\operatorname{Spec} K).$$

One verifies that $\mathscr{F}\mathscr{I}soc(X/V)^{\omega}$ is a $\mathbf{Q}_p$-tannakian ∞-category.

6.4. — Denote by $F_k := \mathscr{F}\mathscr{I}soc(\operatorname{Spec} k/V)^{\omega}$. The objects here are thus perfect complexes over K equipped with a σ-linear autoequivalence. Let us write $G_{\mathscr{F}\mathscr{I}soc}$ for the Tannaka dual ∞-gerbe over $\mathbf{Q}_p$.

On the other hand, if X is a smooth and proper variety over k, denote by $X_{\mathscr{I}soc}$ the Tannaka dual ∞-gerbe for the K-tannakian ∞-category $\mathscr{I}soc(X/V)^{\omega,\mathrm{ss}} \subset \mathscr{I}soc(X/V)^{\omega}$ comprised of semistable objects. We call $X_{\mathscr{I}soc}$ the *crystalline homotopy type of X over V*.

The crystalline homotopy type $X_{\mathscr{I}soc}$ is naturally a $G_{\mathscr{F}\mathscr{I}soc}$-space; hence we may contemplate the K-theory Green functor $\mathbf{K}(F_k; X_{\mathscr{I}soc})$ for $G_{\mathscr{F}\mathscr{I}soc}$. Write

$$\mathbf{S}(F_k; X_{\mathscr{I}soc}) := \operatorname{Spec} \mathbf{K}(F_k; X_{\mathscr{I}soc})$$

for the corresponding F_k-equivariant affine.

6.5. — Suppose now that $\mathscr{X}$ is a proper, smooth model of X over V. One may regard $\mathscr{X}$ itself as a $G_{\mathscr{F}\mathscr{I}soc}$ space with a trivial action, and thus one has a K-theory Green functor $\mathbf{K}(F_k; \mathscr{X})$ for $G_{\mathscr{F}\mathscr{I}soc}$. Write

$$\mathbf{S}(F_k; \mathscr{X}) := \operatorname{Spec} \mathbf{K}(F_k; \mathscr{X})$$

for the corresponding F_k-equivariant affine.

One has a diagram of F_k-equivariant affines

$$\begin{array}{ccc} & H(\mathbf{Z}/p) & \\ \swarrow & & \searrow \\ \mathbf{S}(F_k; \mathscr{X}) & \longrightarrow & \mathbf{S}(F_k; X_{\mathscr{I}soc}) \end{array}$$

in which the vertical morphisms are given by the mod p rank.

Theorem 6.6. — *The morphism on the completions*

$$\mathbf{S}(F_k; \mathscr{X})^{\wedge}_{p} \longrightarrow \mathbf{S}(F_k; X_{\mathscr{I}soc})^{\wedge}_{p}$$

is an equivalence.

Note. — The proof is as in the Galois-equivariant case: one shows that the cotangent complex of

$$\mathbf{S}(F_k; \mathscr{X})^{\langle n \rangle}_{p} \longrightarrow \mathbf{S}(F_k; X_{\mathscr{I}soc})^{\langle n \rangle}_{p}$$

increases without bound as $n \to \infty$. For this, one need not use a filtration on the K-theory; instead, one is able to exploit the simple structure of $G_{\mathscr{F}\mathscr{I}soc}$. □

6.7. — This is a strange theorem. As far as I know, no one has predicted any result of this kind. However, it does suggest something bizarre: there is a shadow of the K-theory of a smooth proper model of X that does not actually depend on the model. Moreover, this suggests that some p-completion of motivic cohomology of a smooth proper model of X might not even require the existence of any smooth proper models: it can be defined directly using $X_{\mathscr{I}soc}$.

Remarks of J.–M. Fontaine suggest that there should be no corresponding “filtered” statement.

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