Syntomic regulators and special values of p-adic L-functions

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Plan

- Deligne cohomology
- Syntomic cohomology
- Regulators
- Beilinson conjecture
- p-adic Beilinson conjecture
- The Karoubi regulator

Motivation - complex theory

$X/\mathbb{C}$ - smooth proper irreducible of dimension d.
Cohomology of complex varieties
$M = H^i(X/\mathbb{Z})/\text{torsion}$ - a Hodge structure

- $M/\mathbb{Z}$ - free finite rank
- $M_{\mathbb{C}}$ - a finite dimensional complex vector space with a Hodge decomposition $M_{\mathbb{C}} = \oplus_{k+l=i} M^{k,l} \Rightarrow$ descending Hodge filtration $F^n = \oplus_{k \geq n} M^{k,l}$
- $\iota : M \otimes \mathbb{C} \to M_{\mathbb{C}}$ an isomorphism

Abel-Jacobi map

$\mathrm{CH}_0(X)$ - free abelian group on points of $X/$ rational equivalence
$\deg : \mathrm{CH}_0(X) \to \mathbb{Z} = H_0(X, \mathbb{Z})$
Abel-Jacobi map
$\alpha : \mathrm{Ker}(\deg) \to (\Omega^1(X))^*/\text{image of } H_1(X, \mathbb{Z})$

$$\alpha(\sum n_i x_i)(\omega) = \sum n_i \int_P^{x_i} \omega$$

$P \in X$ arbitrary. Integral taken on some path.

Cohomological interpretation

- $\mathrm{CH}_0(X) = \mathrm{CH}^d(X) =$ codimension d cycles.
- $H_0(X, \mathbb{Z}) = H^{2d}(X, \mathbb{Z})$
- $(\Omega^1(X))^* = H^{2d-1}(X, \mathbb{C})/F^d$ (Poincaré duality)
- $H_1(X, \mathbb{Z}) = H^{2d-1}(X, \mathbb{Z})$

$$\deg : \mathrm{CH}^d(X) \to H^{2d}(X, \mathbb{Z})$$

$$\alpha : \mathrm{Ker}(\deg) \to H^{2d-1}(X, \mathbb{C}) / \left[F^d + H^{2d-1}(X, \mathbb{Z}) \right]$$

Extensions interpretation

Define

- $M_0 = H^{2d}(X)$, $M_1 = H^{2d-1}(X)$
- $\mathbf{1} = \mathbb{Z}$ with $F^0\mathbf{1}_{\mathbb{C}} = \mathbf{1}_{\mathbb{C}}$, $F^1\mathbf{1}_{\mathbb{C}} = 0$
- Twists $M(n) = M$, $F^j M(n)_{\mathbb{C}} = F^{j+n}M_{\mathbb{C}}$, ι multiplied by $(2\pi i)^n$.

Then

- $H^{2d}(X,\mathbb{Z}) = M_0 = M_0 \cap F^d = M_0(d) \cap F^0 = \mathrm{Hom}(\mathbf{1}, M_0(d))$
- $H^{2d-1}(X,\mathbb{C})/\left[F^d + H^{2d-1}(X,\mathbb{Z})\right] = M_1(d)_{\mathbb{C}}/\left[F^0 + M_1(d)\right] = \mathrm{Ext}^1(\mathbf{1}, M_1(d))$

Higher dimensional analogues

$$c : \mathrm{CH}^k(X) \to \mathrm{Hom}(\mathbf{1}, H^{2k}(X)(k))$$
$$\alpha : \mathrm{Ker}(c) \to \mathrm{Ext}^1(\mathbf{1}, H^{2k-1}(X)(k))$$

Abel-Jacobi and extensions

Z - a cyle of codimension k on X, $c(Z) = 0$
$s(Z)$ - support of Z

$0 \to H^{2k-1}(X) \to H^{2k-1}(X - s(Z)) \to \mathrm{Ker}(H^{2k}_{s(Z)}(X) \to H^{2k}(X)) \to 0$

$c(Z) \in \mathrm{Hom}(\mathbf{1}, (\mathrm{Ker}(H^{2k}_{s(Z)}(X) \to H^{2k}(X)))(k))$
Pullback gives

$$0 \to H^{2k-1}(X)(k) \to H^{2k-1}(X - s(Z))(k) \to \mathbf{1} \to 0$$

whose extension class is $\alpha(Z)$

Deligne cohomology

Idea of Deligne: Both c and α should come from $\mathrm{CH}^k(X) \to H^{2k}_{\mathcal{D}}(X, \mathbb{Z}(k))$, with

$$0 \to \mathrm{Ext}^1(\mathbf{1}, H^{i-1}(X)(n)) \to H^i_{\mathcal{D}}(X, \mathbb{Z}(n)) \to \mathrm{Hom}(\mathbf{1}, H^i(X)(n)) \to 0 (*)$$

(for $i = 2k$, $k = n$).

How to define?

First step: find functorial complexes computing all cohomologies involved:

- $\mathbb{R}\Gamma_B(X, \mathbb{Z})$ computing integral cohomology.
- $\mathbb{R}\Gamma_{\mathrm{dR}}(X/\mathbb{C})$ - filtered complex computing de Rham cohomology with its filtrations (If X is non-proper take care to introduce log singularities).

Second step: Take $\mathbb{R}\,\mathrm{Hom}(\mathbf{1}, \bullet)$ on the resulting object.
Resulting spectral sequence gives $(*)$

Comments

- To get $\mathbb{R}\Gamma_{\mathrm{dR}}(X/\mathbb{C})$ take $\mathbb{R}\Gamma(\bar{X}, \Omega^{\bullet}_{\bar{X}/\mathbb{C}}(\log \bar{X} - X))$ for a compactification with normal crossings divisor as complement. To make it functorial take a limit over all possible compactifications.
- Set $A^{\bullet} \tilde{\times}_{C^{\bullet}} B^{\bullet} = MF(A^{\bullet} \oplus B^{\bullet} \to C^{\bullet})$. Then

$$\mathbb{R}\Gamma_{\mathcal{D}}(X, \mathbb{Z}(n)) = F^n \mathbb{R}\Gamma_{\mathrm{dR}}(X/\mathbb{C}) \tilde{\times}_{\mathbb{R}\Gamma_{\mathrm{dR}}(X/\mathbb{C})} \mathbb{R}\Gamma(X, \mathbb{Z})$$

p-adic theory

$K/\mathbb{Q}_p$ finite, $\mathcal{O}_K$ ring of integers, k-residue field
$X/\mathcal{O}_K$ smooth.
Similar to complex case: $V = H^i(X)$, an object of a category of "filtered Frobenius modules" consisting of

- V/K_0 finite dimensional vector space
- $\phi : V \to V$ a semi-linear Frobenius map
- V_K a K-vector space with desending filtration F^n.
- $\iota : V_K \to V \otimes K$ - comparison map

p-adic theory

Here

- $V = H^i_{\mathrm{rig}}(X_k/K_0)$ is the rigid cohomology of the special fiber.
- ϕ is the natural Frobenius
- $V_K = H^i_{\mathrm{dR}}(X_K/K)$ is the de Rham cohomology of the generic fiber
- ι is the "specialization map"

This category has

- $\mathbf{1}$ - object where $V = K_0$ with $\phi = \sigma$, $F^0 = V_K = K$, $F^1 = 0$
- Twists - filtrations shift as in Hodge, ϕ divided by p^n.

Syntomic cohomology

By analogy want

$$0 \to \mathrm{Ext}^1(\mathbf{1}, H^{i-1}(X)(n)) \to H^i_{\mathrm{syn}}(X,n) \to \mathrm{Hom}(\mathbf{1}, H^i(X)(n)) \to 0$$

How to define?

1. Define $\mathbb{R}\Gamma_{\mathrm{rig}}(X_k/K_0)$ with a Frobenius ϕ, $\mathbb{R}\Gamma_{\mathrm{rig}}(X_k/K)$ and a base change map
2. Define a specialization map on the level of complexes $\mathbb{R}\Gamma_{\mathrm{dR}}(X_K/K) \to \mathbb{R}\Gamma_{\mathrm{rig}}(X_k/K)$
3. Define

$$\mathbb{R}\Gamma_{\mathrm{syn}}(X,n) = MF(1-p^{-n}\phi)\tilde{\times}_{\mathbb{R}\Gamma_{\mathrm{rig}}(X_k/K)} F^n\mathbb{R}\Gamma_{\mathrm{dR}}(X_K/K)$$

$\mathrm{Hom}(\mathbf{1}, V(n)) = V^{\phi=p^n} \cap F^n V_K$
$\mathrm{Ext}^1(\mathbf{1}, V(n)) = V/(1-\phi/p^n)F^n$ (when $K = K_0$)

Other versions (often usefull)

Gros style - without log singularities
Niziol style
without log singularities and using convergent rather than rigid cohomology.
Same as above for proper X, hopeless otherwise.
Connection with étale cohomology (Niziol, Inventioness 127)
Y/K, K any field.
$H^i_{\text{ét}}(Y, \mathbb{Q}_p(n))$ - continuous étale cohomology (Jannsen).
Spectral sequence

$$E_2^{i,j} = H^i(K, H^j_{\text{ét}}(\bar{Y}, \mathbb{Q}_p(n))) \Rightarrow H^{i+j}_{\text{ét}}(Y, \mathbb{Q}_p(n))$$

Niziol's Theorem

For $X/\mathcal{O}_K$ proper smooth there is a functorial map

$$H^i_{\text{syn}}(X, n) \to H^i_{\text{ét}}(X_K, \mathbb{Q}_p(n))$$

compatible with Chern classes. It is compatible with the spectral sequences, e.g. with

$$\text{Ext}^1(\mathbf{1}, H^{i-1}(X)(n)) \to H^1(K, H^{i-1}_{\text{ét}}(X_{\bar{K}}, \mathbb{Q}_p(n)))$$

which turns out to be the Bloch-Kato exponential map.

Modified versions

Replace $\mathbb{R}\Gamma_{\mathrm{rig}}(X_k/K_0)$ with $\mathbb{R}\Gamma_{\mathrm{rig}}(X_k/K)$ and $1-\phi/p^n$ by $1-\phi^r/p^{rn}$ where ϕ^r is linear.
Benefit - Can use X to compute the rigid cohomology.
Obvious problem - It's a different cohomology (but see later).
Finite polynomial cohomology and Coleman integration
Replace $1-\phi^r/p^{rn}$ with more general polynomials $\Rightarrow$ (X proper smooth)

$$0 \to H_{\mathrm{dR}}^{i-1}(X_K/K)/F^n \to H_{\mathrm{fp}}^i(X,n) \to F^n H_{\mathrm{dR}}^i(X_K/K) \to 0$$

E.g. $i=n=1$

$$0 \to K \to H_{\mathrm{fp}}^1(X,1) \to \Omega^1(X_K/K) \to 0$$

$\omega \in \Omega^1(X_K/K) \Rightarrow \tilde{\omega} \in H_{\mathrm{fp}}^1(X,1)$ unique up to constant $=$ Coleman integral of ω.

Regulators (aka Chern classes)

$$c_n^s : K_s(X) \to H^{2n-s}_{\mathrm{syn}}(X, n)$$

Main interest: explicitly compute these

For X proper and $s > 0$ the target is very simple

1. $H^{2n-s}_{\mathrm{rig}}(X_k/K_0)^{\phi=p^n} = 0$ for weight reasons
2. $1 - \phi/p^n$ is invertible on $H^{2n-s-1}_{\mathrm{rig}}(X_k/K_0) \Rightarrow$ $\mathrm{Ext}^1(\mathbf{1}, H^{2n-s-1}(X)(n)) \cong H^{2n-s-1}_{\mathrm{rig}}(X_k/K)/F^n$

(Modified versions are the same) So usually

$$c_n^s : K_s(X) \to H^{2n-s-1}_{\mathrm{rig}}(X_k/K)/F^n$$

Construction when $X = \mathrm{Spec}(A)$

First compute $H^{2n}_{\mathrm{syn}}(BGL_i/\mathcal{O}_K, n)$, $i >> 0$

- BGL_i has cohomology only in even degrees
- The standard classes $x_n \in H^{2n}(BGL_i)$ exist in de Rham and rigid cohomology and are in $\mathrm{Hom}(\mathbf{1}, H^{2n}(BGL_i)(n)) = H^{2n}_{\mathrm{syn}}(BGL_i/\mathcal{O}_K, n)$
- These classes are compatible so give classes in $H^{2n}_{\mathrm{syn}}(BGL/\mathcal{O}_K, n)$

Chern classes now follow from standard machinary:

1. Hurewich $K_s(X) \to H_s(BGL(A)^+, \mathbb{Z}) \cong H_s(BGL(A), \mathbb{Z})$
2. $H_s(BGL(A), \mathbb{Z})$ is computed with a complex having in degree s maps from $X = \mathrm{Spec}(A)$ to the degree s part of $BGL(A)$.
3. using pullbacks get

$$H_s(BGL(A), \mathbb{Z}) \times H^k(BGL, n) \to H^{k-s}(X, n)$$

4. Compose the Hurevich map with the pairing applied to x_n.

p-adic Abel Jacobi

For K_0 and X proper Chern classes induce a p-adic Abel-Jacobi map

$$\alpha_p : \mathrm{Ker}(\deg) \to \Omega^1(X_K/K)^*$$

This has the same formula as in the classical case, only with ordinary integration replaced by Coleman integration.

Beilinson's conjecture: Motivic cohomology

$$H^i_{\mathcal{M}}(X, \mathbb{Q}(n)) = (K_{2n-i}(X) \otimes \mathbb{Q})^{(n)}$$

(n) refers to the part where the Adams operation ψ^k acts by k^n.

- An integral version can be defined using Bloch's higher Chow groups.
- The indexing is such that the Chern character defines $H^i_{\mathcal{M}}(X, \mathbb{Q}(n)) \to H^i_{\mathcal{D}}(X, \mathbb{Q}(n))$ and $H^i_{\mathcal{M}}(X, \mathbb{Q}(n)) \to H^i_{\text{syn}}(X, n)$ in the appropriate situations.
- For Beilinson conjectures need to introduce subspace $H^i_{\mathcal{M}}(X, \mathbb{Q}(n))_{\mathbb{Z}}$ (Scholl).

Deligne cohomology over $\mathbb{R}$

$X/\mathbb{R} \Rightarrow X(\mathbb{C})$ has an action of complex conjugation c.
Definition: $H^i_{\mathcal{D}}(X/\mathbb{R}, \mathbb{R}(n)) = H^i_{\mathcal{D}}(X/\mathbb{C}, \mathbb{R}(n))^{c=1}$ where c acts both on space and on coefficients.
For $s > 0$

$$H^{2n-s}_{\mathcal{D}}(X/\mathbb{C}, \mathbb{R}(n)) = H^{2n-s-1}_{\mathrm{dR}}(X/\mathbb{C}) / \left(F^n + H^{2n-s-1}(X(\mathbb{C}), (2\pi i)^n \mathbb{R})\right)$$

Taking c invariants

$$\begin{aligned} 0 \to H^{2n-s-1}(X(\mathbb{C}), (2\pi i)^n \mathbb{R})^{c=1} &\to H^{2n-s-1}_{\mathrm{dR}}(X/\mathbb{R})/F^n \\ &\to H^{2n-s}_{\mathcal{D}}(X/\mathbb{R}, \mathbb{R}(n)) \to 0 \end{aligned}$$

Suppose $X/\mathbb{Q}$ smooth and projective. $H^{2n-s-1}(X(\mathbb{C}), (2\pi i)^n \mathbb{R})^{c=1}$ and $H^{2n-s-1}_{\mathrm{dR}}(X/\mathbb{R})/F^n$ have obvious $\mathbb{Q}$ structures on them, hence $H^{2n-s}_{\mathcal{D}}(X/\mathbb{R}, \mathbb{R}(n))$ has one, denoted $\mathcal{D}$.

Beilinson's conjecture

- $H_{\mathcal{M}}^{2n-s}(X, \mathbb{Q}(n))_{\mathbb{Z}} \otimes \mathbb{R} \xrightarrow{\sim} H_{\mathcal{D}}^{2n-s}(X/\mathbb{R}, \mathbb{R}(n))$ hence provides a $\mathbb{Q}$ structure $\mathcal{M}$ on $H_{\mathcal{D}}^{2n-s}(X/\mathbb{R}, \mathbb{R}(n))$.
- $\det \mathcal{M} = L(H^{2n-s-1}(X), n) \det \mathcal{D}$.
- More concretely, if bases $\{\mathcal{M}_i\}$ and $\{\mathcal{D}_i\}$ are chosen

$$\alpha \cdot \bigwedge \mathcal{M}_i = L(H^{2n-s-1}(X), n) \cdot \bigwedge \mathcal{D}_i$$

with $\alpha \in \mathbb{Q}^\times$

Example: number fields

$X = \mathrm{Spec}(L)$, L a number field with r_1 real embeddings and $2r_2$ complex embeddings.
What is $H^1_{\mathcal{D}}(\mathrm{Spec}\, L/\mathbb{R}, \mathbb{R}(n))$?

- $X(\mathbb{C})$ = set of embeddings, permuted by c
- c acts on $(2\pi i)^n$ as $(-1)^n$.
- $\dim H^0(X(\mathbb{C}), (2\pi i)^n\mathbb{R})^{c=1} = r_2$ $(+r_1$ if $2|n)$
- $\dim H^1_{\mathcal{D}}(X/\mathbb{R}, \mathbb{R}(n)) = r_2$ $(+r_1$ if $2|n+1)$

In this case, Beilinson's conjecture is Borel's Theorem.
Important observation: K-theory knows information at infinity.

p-adic Beilinson conjecture

$X/\mathbb{Q}$ smooth proper, $s > 0$
X_p - an integral model for $X \otimes \mathbb{Q}_p$, still assumed smooth and proper.

$$H_{\mathcal{M}}^{2n-s}(X, \mathbb{Q}(n)) \to H_{\text{syn}}^{2n-s}(X_p, n) = (H_{\text{dR}}^{2n-s-1}(X/\mathbb{Q})/F^n) \otimes \mathbb{Q}_p$$

Problem: dimensions don't match !!!
Missing the $H^{2n-s-1}(X(\mathbb{C}), (2\pi i)^n \mathbb{R})^{c=1}$ piece.
A possible solution: import this part using comparison maps.
Perrin-Riou's solution: Make everything depend on the choice of a basis for a complimentary subspace of dimension
$d_{\pm} = \dim H^{2n-s-1}(X(\mathbb{C}), (2\pi i)^n \mathbb{R})^{c=1}$!!!

The conjecture

- Fixing the bases as for Beilinson's conjecture.
- Adding the complelementary basis to the image of the regulator.
- Replacing Beilinson's regulator by the syntomic regulator.
- Replacing the L-function by the *p*-adic L-function (for the complementary basis).

It is the same conjecture!!! (with the same rational factor up to trivial terms).

What are p-adic L-functions?

3 answers

- The precise - An element of an Iwasawa algebra coming via a logrithm map from an Euler system...
- The easy - It is defined by the above conjecture: Rephrase to say: There is a p-adic continuous (or even analytic) function that interpolates the numbers arising in the conjecture above (with L-factor at p removed)
- The practical - For some (so called critical) n no motivic cohomology is needed. The p-adic L-function interpolates these values.

Better yet: It is a measure on $\mathbb{Z}_p$ such that the integral of $x^n\chi$, for a Dirichlet character χ, gives α corresponding to $L(H^{2n-s-1}(X)\otimes\chi, n)$ for critical n, χ.

Evidence for p-adic Beilinson

- X an elliptic curve with CM over $\mathbb{Q}$, $s = n = 2$ (Coleman-de Shalit, B.)
- Certain numerical evidence for totally real fields and totally real Artin motives (B. Buckingham, de Jeu, Roblot)
- Modular curves (Gealy)

Relation with the Karoubi regulator

Problem: The syntomic regulator depends on an integral model
Should expect dependence on the generic fiber only: It's étale cohomology determines the filtered Frobenius module via Fontaine's functor.
Karoubi's regulator may provide the solution

Karoubi's regulator

A - non archimedean Banach ring
create a (rigid) homotopy invariant K-theory
Make a simplicial ring $A_\bullet$

$$A_n = A\{x_0, \ldots, x_n\}/(\sum x_i - 1)$$

Definition: $K_{\text{top}}(A) = BGL(A_\bullet)$.
Map $K(A) \to K_{\text{top}}(A)$
$K_{\text{rel}}(A) = \text{MF}(K(A) \to K_{\text{top}}(A))$
de Rham Chern classes
$K(A) \to F^n\Omega^{\bullet+2n}(A)$
By integrating on simplices
$K_{\text{top}}(A) \to \Omega^{\bullet+2n}(A)$
$K_{\text{rel}}(A) \to \text{MF}(F^n \to \Omega^\bullet(A))$
$K_s^{\text{rel}}(A) \to H_{\text{dR}}^{2n-s-1}(A)/F^n$
just like the syntomic regulator

Relations with the syntomic regulator

$K_s^{\text{top}}(A)$ is torsion in some cases. so K_{rel} is like K.

- Karoubi conjectures a relation with p-adic polylogarithms for p-adic fields
- B. conjectured a relation with the syntomic regulator
- Hamida - formulas for Karoubi's regulator for p-adic fields
- Tamme - relation with the syntomic regulator for fields (maybe in general)

Expectation: $MF(K_{\text{top}} \oplus F^n \to \Omega^{\bullet+2n})$ should be a replacement for syntomic cohomology.