

p -adic Hodge theory and comparison theorems.

A flavor of the syntomic technics

ring = commutative ring. A -algebra = ring B with a morphism $A \rightarrow B$.

General construction : A_0 a ring, A = an A_0 -algebra, $p \in A_0 \Rightarrow B_{dR}^+(A/A_0)$.

We define a decreasing sequence of A_0 -algebras

$$A = A^{(0)} \supset A^{(1)} \supset \dots \supset A^{(n-1)} \supset A^{(n)} \supset \dots$$

by

$$0 \rightarrow A^{(n)} \rightarrow A^{(n-1)} \rightarrow A \otimes_{A^{(n-1)}} \Omega_{A^{(n-1)}/A_0}^1$$

$$\text{Set } \widehat{A}^{(n)} = \varprojlim_{r \in \mathbb{N}} A^{(n)} / p^r A^{(n)}, \quad B_m = \widehat{A}^{(m-1)}[1/p] \text{ and } B_{dR}^+ = \varprojlim_{m \in \mathbb{N}} B_m$$

Application :

K is a field of characteristic 0 complete with respect to a discrete valuation.

\mathcal{O}_K = the valuation ring, \mathfrak{m}_K = the maximal ideal

$k = \mathcal{O}_K / \mathfrak{m}_K$ = the residue field (assumed to be perfect of characteristic $p > 0$).

\overline{K} = a chosen algebraic closure of K , $\mathcal{O}_{\overline{K}}$ = integral closure of \mathcal{O}_K in \overline{K} ,

$G_K = \text{Gal}(\overline{K}/K)$.

$$B_{dR}^+ = B_{dR}^+(\mathcal{O}_{\overline{K}}/\mathcal{O}_K) .$$

For $n > 0$, get a short exact sequence

$$0 \rightarrow \mathcal{O}_{\overline{K}}^{(n)} \rightarrow \mathcal{O}_{\overline{K}}^{(n-1)} \rightarrow \Omega^{(n)} \rightarrow 0 \text{ with } \Omega^{(n)} = \mathcal{O}_{\overline{K}} \otimes_{\mathcal{O}_{\overline{K}}^{(n-1)}} \Omega_{\mathcal{O}_{\overline{K}}^{(n-1)}/\mathcal{O}_K}^1 .$$

Hence

$$0 \rightarrow (\Omega^{(n)})_{p^r} \rightarrow \mathcal{O}_{\overline{K}}^{(n)} / p^r \mathcal{O}_{\overline{K}}^{(n)} \rightarrow \mathcal{O}_{\overline{K}}^{(n-1)} / p^r \mathcal{O}_{\overline{K}}^{(n-1)} \rightarrow 0$$

$$0 \rightarrow T_p(\Omega^{(n)}) \rightarrow \widehat{\mathcal{O}_{\overline{K}}^{(n)}} \rightarrow \widehat{\mathcal{O}_{\overline{K}}^{(n-1)}} \rightarrow 0 .$$

Get $B_1 = C$ and, for any n , $V_p(\Omega^{(n)}) = \mathbb{Q}_p \otimes_{\mathbb{Z}_p} T_p(\Omega^{(n)}) = C(n)$ and

B_{dR}^+ is a complete discrete valuation ring whose residue field is C .

$$B_{dR} = \text{Frac } B_{dR}^+ .$$

Theorem. — *Let X a proper and smooth variety over K , then, for any $m \in \mathbb{N}$,*

$$B_{dR} \otimes_K H_{dR}^m(X) = B_{dR} \otimes_{\mathbb{Q}_p} H_{\acute{e}t}^m(X_{\overline{K}}, \mathbb{Q}_p) .$$

This is the p -adic analogue of the classical comparison theorem

$$\mathbb{C} \otimes_K H_{dR}^m(X) = \mathbb{C} \otimes_{\mathbb{Q}} H^m(X_{\sigma}(\mathbb{C}), \mathbb{Q})$$

($\sigma : K \rightarrow \mathbb{C}$ is an embedding and $H_{dR}^m(X) = \mathbb{H}_{Zar}^m(X, \Omega_{X/K})$).

In the above theorem,

– $X_{\overline{K}}$ is the variety over \overline{K} obtained from X by base change,

– $H_{\acute{e}t}^m(X_{\overline{K}}, \mathbb{Q}_p) = \mathbb{Q}_p \otimes_{\mathbb{Z}_p} \varprojlim_{n \in \mathbb{N}} H^m((X_{\overline{K}})_{\acute{e}t}, \mathbb{Z}/p^n \mathbb{Z})$,

– "=" means that this is a canonical isomorphism which is functorial and compatible with everything you want.

In particular,

1 – The isomorphism is compatible with the action of G_K

(on the LHS, $g(b \otimes x) = g(b) \otimes x$, on the RHS, $g(b \otimes y) = g(b) \otimes g(y)$).

2 – The isomorphism is compatible with the filtration :

(on the LHS, $F^i(B_{dR} \otimes H^m(X)) = \sum_{i_1+i_2=i} F^{i_1} B_{dR} \otimes F^{i_2} H_{dR}^m(X)$,

on the RHS, $F^i(B_{dR} \otimes H_{\acute{e}t}^m(X_{\overline{K}}, \mathbb{Q}_p)) = F^i B_{dR} \otimes H_{\acute{e}t}^m(X_{\overline{K}}, \mathbb{Q}_p)$.)

($F^i B_{dR} = i$ -th power of the maximal ideal of B_{dR}^+)

Moreover $(B_{dR})^{G_K} = K$ and

$$H_{dR}^m(X_K) = D_{dR}(H_{\acute{e}t}^m(X_{\overline{K}}, \mathbb{Q}_p))$$

(for any p -adic representation V of G_K , we set $D_{dR}(V) =$ the filtered K -vector space $(B_{dR} \otimes_{\mathbb{Q}_p} V)^{G_K}$. Get $\dim_K D_{dR}(V) \leq \dim_{\mathbb{Q}_p} V$ with equality iff V is de Rham).

But the theorem is more precise : there are always more algebraic structures which can be defined on $H_{dR}^m(X)$.

$$H_{\acute{e}t}^m(X_{\overline{K}}, \mathbb{Q}_p) = V(H_{dR}^m(X) \text{ with additional structures }) \text{ (ad hoc functor) } .$$

$W = W(k)$, $K_0 = W[1/p] = \text{Frac } W$, equipped with the absolute Frobenius σ .

φ -isocrystal over $k =$ finite dimensional K_0 -vector space D equipped with a σ -semi-linear bijective map $\varphi : D \rightarrow D$.

1 – The easiest case : If X has *good reduction*, i.e. if there exists \mathcal{X} proper and smooth over \mathcal{O}_K such that $\text{Spec } K \times \mathcal{X} = X$, get the *crystalline cohomology of the special fiber \mathcal{X}_k of \mathcal{X}*

$$H_{cris}^m(X) = K_0 \otimes_W \varprojlim_{n \in \mathbb{N}} H^m((\mathcal{X}_k/W_n)_{cris}, \text{struct.sheaf})$$

Depends only on \mathcal{X}_k , this is a φ -isocrystal and

$$K \otimes_{K_0} H_{cris}^m(X) = H_{dR}^m(X) \quad (\text{Berthelot-Ogus})$$

Get a *filtered φ -module over K* and

$$V_{cris}(H_{dR}^m(X)) = H_{\acute{e}t}^m(X_{\overline{K}}, \mathbb{Q}_p) \text{ and } D_{cris}(H_{\acute{e}t}^m(X_{\overline{K}}, \mathbb{Q}_p)) = H_{dR}^m(X) .$$

Remark : The $H^m((\mathcal{X}_k/W_n)_{cris}, \text{struct.sheaf})$ are also the cohomology groups of the de Rham-Witt complex $W_n \Omega_{\mathcal{X}_k/k}^*$ (see Illusie's talk).

2 – The case where X has *semi-stable reduction*, i.e. there exists a proper regular scheme \mathcal{X} over \mathcal{O}_K whose general fiber \mathcal{X}_K is X and whose special fiber $\mathcal{X}_k \hookrightarrow \mathcal{X}$ is a divisor with normal crossings.

Need in this case to use log-geometry and to replace the crystalline cohomology with the *log-crystalline cohomology* (or *crystalline-cohomology with log poles* or *Hyodo-Kato cohomology*) (see the talk of Ogus).

To the log-special fiber of \mathcal{X} correspond the $H_{st}^m(X) =$ a φ -isocrystal equipped with a K_0 -linear map $N : D \rightarrow D$ such that $N\varphi = p\varphi N$.

We have also

$$K \otimes_{K_0} H_{st}^m(X) = H_{dR}^m(X) \quad (\text{Hyodo-Kato})$$

Get a *filtered (φ, N) -module over K* and

$$V_{st}(H_{dR}^m(X)) = H_{\acute{e}t}^m(X_{\overline{K}}, \mathbb{Q}_p) \text{ and } D_{st}(H_{\acute{e}t}^m(X_{\overline{K}}, \mathbb{Q}_p)) = H_{dR}^m(X) .$$

3 – General case : Use de Jong's alterations to reduce the problem to the semi-stable case. We have slightly more involved algebraic structures (filtered (φ, N, G_K) -modules over K).

Remark : The *p-adic monodromy theorem* (Berger + André, Mebkhout, Kedlaya) says that any de Rham representation is potentially semi-stable.

We get an equivalence of categories

Admissible filtered (φ, N, G_K) – modules over $K \iff$ de Rham repr's of G_K .

Three distinct methods have given the complete results.

1 – *Syntomic method* : F-Messing, Kato,....., Breuil, Tsuji.
to be discussed

2 – *The almost étale approach* due to Gerd Faltings.

3 – *The K-theoretic approach* due to Wieslawa Niziol.
(see her talk)

W. Niziol also proved that the three comparison theorems gives the same maps (by proving that there are the solution of a universal problem).

A new approach consists of using rigid cohomology. It is very promising and should be useful for comparisons theorems with non constant coefficients. Recently

- *p*-adic Hodge theory became closer to rigid analytic geometry,
- very important progress have been made in rigid cohomology (see Kedlaya's talks).

**What are *p*-adic Hodge theory and *p*-adic comparison theorems good for?
!!!!!!**

In today's arithmetic geometry, both *p*-adic Hodge theory and automorphic forms play a crucial role.

The syntomic topos

A morphism $\alpha : X \rightarrow S$ of schemes is *syntomic* (Mazur) if

- a) α is flat,
- b) α is a locally complete intersection, i.e., Zarisky locally, it may be written

$$\alpha : \text{Spec } B \rightarrow \text{Spec } A \text{ with } B = A[X_1, X_2, \dots, X_m]/(f_1, f_2, \dots, f_n)$$

and f_1, f_2, \dots, f_n a regular sequence.

If S is a scheme, the *big* and the *small syntomic sites* are

S_{SYN} : the underlying category is the category of S -schemes locally of finite type.

S_{syn} : the underlying category is the full sub-category of the previous one whose objects are S -schemes such that the structural morphism is syntomic.

For both sites, covering are surjective families of syntomic morphisms.

The sheaf \mathcal{O}_n^{cris}

$k =$ perfect field of characteristic $p > 0$ $W_n = W_n(k) =$ ring of Witt vector of length n .

For any topos and any W_n -algebra A over this topos, let $\mathcal{E}^{dp}(A/W_n)$ the category of W_n -divided power thickenings of A :

– An object is a triple $(\mathcal{A}, \rho, \gamma)$ where \mathcal{A} is a (sheaf of) W_n -algebra(s), $\rho : \mathcal{A} \rightarrow A$ is an epimorphism of (sheaves of) W_n -algebras and γ is a divided power structure on the kernel of ρ such that $\gamma_m(px) = (p^m/m!)x^m$, for all $x \in \mathcal{A}$.

– A morphism is a momorphism of the underlying (sheaves of) W_n -algebras which is compatible with the ρ 's and the γ 's.

Theorem. — *Let \mathcal{O} be the structural sheaf over $(\text{Spec } k)_{SYN}$.
The category $\mathcal{E}^{dp}(\mathcal{O}/W_n)$ has an initial object.*

We call it \mathcal{O}_n^{cris} . If A is any k -algebra, we have

$$\mathcal{O}_n^{cris}(\text{Spec } A) = \mathcal{O}_n^{cris}(A) = \varprojlim_{\mathcal{A} \in \mathcal{C}_A} \mathcal{A}$$

(this is a direct inverse system).

Extends uniquely to a sheaf of W_n -algebras over $(\text{Spec } k)_{SYN}$ (plus an epimorphism $\rho : \mathcal{O}_n^{cris} \rightarrow \mathcal{O}$ and a divided power structure on the ideal $\text{Ker } \rho$).

Moreover, by functoriality, the Frobenius $a \mapsto a^p$ on \mathcal{O} induces an endomorphism $\varphi : \mathcal{O}_n^{cris} \rightarrow \mathcal{O}_n^{cris}$

Warning : for a given k -algebra A , the ring $\mathcal{O}_n^{cris}(A)$ itself is not, in general, an object of $\mathcal{E}^{dp}(A/W_n)$ (the map $\rho_A : \mathcal{O}_n^{cris}(A) \rightarrow A$ may not be surjective!)

Moreover

- For $m, n \in \mathbb{N}$, $\mathcal{O}_{m+n}^{cris} \rightarrow \mathcal{O}_n^{cris}$ is an epimorphism,
- Over $(\text{Spec } k)_{syn}$, we have a short exact sequence

$$0 \rightarrow \mathcal{O}_m^{cris} \rightarrow \mathcal{O}_{m+n}^{cris} \rightarrow \mathcal{O}_n^{cris} \rightarrow 0 .$$

$(\mathcal{O}_n^{cris})_{n \in \mathbb{N}}$ is a p -divisible sheaf.

Alternative descriptions :

1 – If $A = B/I$ with B a smooth W_n -algebra and if B^{dp} is the divided power envelope of B with respect to I (compatible with canonical divided powers on the ideal generated by p),

$$0 \rightarrow \mathcal{O}_n^{cris}(A) \rightarrow B^{dp} \rightarrow B^{dp} \otimes_B \Omega_{B/W_n}^1$$

2 – For any scheme X over k , we have

$$\mathcal{O}_n^{cris}(X) = H^0(X/W_n)_{cris, \text{struct.sheaf}} .$$

3 – For any k -algebra A , get $\rho_A : W_n(A) \rightarrow A$ via $(a_0, a_1, \dots, a_{n-1}) \mapsto a_0^{p^n}$

$W_n^{DP}(A) =$ divided power envelope of $W_n(A)$ with respect to the kernel of ρ_A , compatible with canonical divided powers on $VW_{n-1}(A)$.

Extends uniquely to a sheaf for the Zariski topology. Set \tilde{W}_n^{DP} the sheafification of W_n^{DP} for the syntomic topology.

$$\tilde{W}_n^{DP} \rightarrow \mathcal{O}_n^{cris} \text{ is an isomorphism.}$$

Moreover, if $A = B/(f_1^{p^n}, f_2^{p^n}, \dots, f_s^{p^n})$, with B smooth over k and f_1, f_2, \dots, f_s a regular sequence, then

$$\tilde{W}_n^{DP}(A) = W_n^{DP}(A).$$

Projection on the Zariski site

Let $X \rightarrow k$ syntomic, Consider $u : X_{syn} \rightarrow X_{Zar}$.

Let $Y \rightarrow X$ a syntomic covering such that any local section of \mathcal{O}_X has a p^n -th root in \mathcal{O}_Y . Then, *the complex*

$$\mathcal{O}_n^{cris}(Y) \rightarrow \mathcal{O}_n^{cris}(Y \times_X Y) \rightarrow \mathcal{O}_n^{cris}(Y \times_X Y \times_X Y) \rightarrow \dots$$

(where $(\mathcal{O}_n^{cris}(Y \times_X \dots \times_X Y))$ means the projection onto X_{Zar} of the restriction of \mathcal{O}_n^{cris} to $Y \times_X \dots \times_X Y$)

represents $Ru_ \mathcal{O}_n^{cris}$.*

This complex computes also the crystalline cohomology, i.e. if $\pi : (X/W_n)_{cris} \rightarrow X_{Zar}$ is the natural projection, the above complex represents also $R\pi_*(\text{struct.sheaf})$.

The smooth case

If $X \rightarrow k$ is smooth there is a canonical choice for Y

$$Y = X^{(n)} \rightarrow X \text{ (the map is Frob}^n \text{).}$$

- This complex is also quasi-isomorphic to the de Rham-Witt complex $W_n \Omega_{X/k}^*$ (see Illusie's talk). For $n = 1$, $W_1 \Omega_{X/k}^* = \Omega_{X/k}^*$, the usual de Rham complex.

The crystalline comparison theorem

Let $\mathcal{O}_{\overline{K}}$ be the integral closure of \mathcal{O}_K in \overline{K} . Then $H^m((\mathcal{O}_{\overline{K}}/p\mathcal{O}_{\overline{K}})_{syn}, \mathcal{O}_n^{cris}) = 0$ for $m > 0$. Set

$$A_{cris} = \varprojlim_{n \in \mathbb{N}} \mathcal{O}_n^{cris}(\mathcal{O}_{\overline{K}}/p\mathcal{O}_{\overline{K}}) \text{ and } B_{cris}^+ = A_{cris}[1/p].$$

Then $A_{cris}/p^n A_{cris} = \mathcal{O}_n^{cris}(\mathcal{O}_{\overline{K}}/p\mathcal{O}_{\overline{K}})$. Get also a surjective homomorphism of $W(k)$ -algebras :

$$\theta : A_{cris} \rightarrow \mathcal{O}_C$$

and the kernel of θ is a divided power ideal.

The p-adic $2\pi i$:

Let $\varepsilon = (\varepsilon^{(n)})_{n \in \mathbb{N}}$ a generator of $\mathbb{Z}_p(1)$ viewed multiplicatively, i.e. a sequence of elements of $\mathcal{O}_{\overline{K}}$ such that $\varepsilon^{(0)} = 1$, $\varepsilon^{(1)} \neq 1$ and $(\varepsilon^{(n+1)})^p = \varepsilon^{(n)}$ for $n > 0$. For $n \in \mathbb{N}$ let $\varepsilon_n = x_n^{p^n}$ where x_n is any lifting in $\mathcal{O}_n^{cris}(\mathcal{O}_{\overline{K}}/p\mathcal{O}_{\overline{K}})$ of the image of $\varepsilon^{(n)}$ in $\mathcal{O}_{\overline{K}}/p\mathcal{O}_{\overline{K}}$. Then $[\varepsilon] = (\varepsilon_n)_{n \in \mathbb{N}} \in A_{cris}$ and

$$\text{"}2\pi i\text{"} = t = \log([\varepsilon]) \in A_{cris}$$

Set $B_{cris} = B_{cris}^+[1/t]$ (action of φ extends ($\varphi t = pt$, $\varphi(1/t) = 1/pt$)).

The map θ extends to a map

$$\theta : B_{cris}^+ \rightarrow C.$$

$$B_{dR}^+ = \varprojlim_{n \in \mathbb{N}} B_{cris}^+ / (\text{Ker } \theta)^n, \text{ } t \text{ is a generator of the maximal ideal of } B_{dR}^+$$

and $K \otimes_{K_0} B_{cris} \rightarrow B_{dR}$ is injective

(Analogue for $\mathcal{O}_{\overline{K}}$ of Berthelot-Ogus theorem).

Theorem. — *Let X be a proper and smooth variety over K with good reduction. For all $m \in \mathbb{N}$,*

$$B_{cris} \otimes_{K_0} H_{cris}^m(X) = B_{cris} \otimes_{\mathbb{Q}_p} H_{\acute{e}t}^m(X_{\overline{K}}, \mathbb{Q}_p) .$$

(We choose a proper and smooth model \mathcal{X} of X over \mathcal{O}_K and we set

$$H_{cris}^m(X) = H_{cris}^m(\mathcal{X}_k) \text{ with the Hodge filtration on } K \otimes_{K_0} H_{cris}^m(X) = H_{dR}^m(X) .)$$

$$\begin{aligned} D_{cris}(H_{\acute{e}t}^m(X_{\overline{K}})) &= H_{cris}^m(X) \\ V_{cris}(H_{cris}^m(X)) &= H_{\acute{e}t}^m(X_{\overline{K}}, \mathbb{Q}_p) \end{aligned}$$

where

$$\begin{aligned} D_{cris}(V) &= (B_{cris} \otimes_{\mathbb{Q}_p} V)^{G_K} \\ V_{cris}(D) &= (B_{cris} \otimes_{K_0} D)_{\varphi=1} \cap F^0(B_{dR} \otimes_K D_K) \end{aligned}$$

Sketch of the syntomic proof (in the case $K = K_0 = W[1/p]$)

For any $m \in \mathbb{N}$, we may consider the small syntomic site $(\text{Spec } W_m)_{syn}$ over $W_m = W_m(k)$.

For $n \leq m$, \mathcal{O}_n is the reduction mod p^n of the structural sheaf. Get a short exact sequence

$$0 \rightarrow J_n \rightarrow \mathcal{O}_n^{cris} \rightarrow \mathcal{O}_n \rightarrow 0$$

and J_n is a (sheaf of) divided power ideal(s). For $r \in \mathbb{N}$, we call $J_n^{[r]}$ the r -th divided power of J_n .

Assume $r \leq p-1$ and $n+r \leq m$. We have two different maps from $J_n^{[r]}$ to \mathcal{O}_n^{cris} . The first one, ι is the natural inclusion, the second one is $\varphi_r = "p^{-r}\varphi"$

(More precisely, if x is a section in $J_{n+r}^{[r]}$, $\varphi(x)$ is divisible by p^r in \mathcal{O}_{n+r}^{cris} , hence come from a well defined section $y \in \mathcal{O}_n^{cris}$. The so defined map $J_{n+r}^{[r]} \rightarrow \mathcal{O}_n^{cris}$ factors through $J_n^{[r]}$ and φ_r is the induced map).

We call S_n^r the kernel of the map $\varphi_r - \iota : J_n^{[r]} \rightarrow \mathcal{O}_n^{cris}$. For $r < p-1$, we have a short exact sequence

$$0 \rightarrow S_n^r \rightarrow J_n^{[r]} \rightarrow \mathcal{O}_n^{cris} \rightarrow 0 .$$

Proposition. — *Let $m \in \mathbb{N}$, Y_m a proper and smoth scheme over $\text{Spec } W_m$ and $\overline{Y}_m = \text{Spec } \mathcal{O}_{\overline{K}} \times_{\text{Spec } W} Y_m$. Let $i, r, n \in \mathbb{N}$ such that $i \leq r < p-1$ and $r+n \leq m$. Then $H^i((\overline{X})_{syn}, S_n^r)(-r)$ is a finite representation of G_K independent of the choice of r . Moreover, the sequence*

$$0 \rightarrow H^i((\overline{X})_{syn}, S_n^r) \rightarrow H^i((\overline{X})_{syn}, J_n^{[r]}) \rightarrow H^i((\overline{X})_{syn}, \mathcal{O}_n^{cris}) \rightarrow 0$$

is exact.

Consider the sites

$$\mathrm{Spf} W_{syn} \quad \text{and} \quad \mathrm{Spf} W_{sét} :$$

For both sites, the underlying category is the full sub-category of formal schemes $U = (U_m)_{m \in \mathbb{N}}$ over W which are syntomic (i.e. $U_m \rightarrow \mathrm{Spec} W_m$ is syntomic for all m).

Covering are surjective families of quasi-finite syntomic morphisms (resp. quasi-finite syntomic morphisms with étale (rigid) generic fiber).

$S_{\mathbb{Q}_p}^r$ = the \mathbb{Q}_p -sheaf over one of these sites defined using the direct images of the S_n^r .

Exercise : Adapt the previous construction to define the \mathbb{Q}_p -sheaves $S_{\mathbb{Q}_p}^r$ for $r > p - 1$.

Theorem. — *Let X be a proper and smooth scheme over W . Let $\bar{Y} = (\bar{Y}_m)_{m \in \mathbb{N}}$ with $\bar{Y}_m = \mathrm{Spec} \mathcal{O}_{\bar{K}}/p^m \times_{\mathrm{Spec} W} X$. Then, for all $i \in \mathbb{N}$, $D_{cris}(H^i(X_K))$ is a finite dimensional \mathbb{Q}_p -vector space and we get*

$$D_{cris}(H_{cris}^i(X_K)) = H^i(\bar{Y}_{syn}, S_{\mathbb{Q}_p}^r)(-r) = H^i(\bar{Y}_{sét}, S_{\mathbb{Q}_p}^r)(-r) \text{ for all } r \geq i .$$

For $i < p - 1$, the proof is easy (dévissages, linear algebra, F-Laffaille theory). The general case requires more work!

To complete the proof in the case $K = K_0 = W[1/p]$, it is enough to prove that

Under the assumptions of the previous theorem, if $X_{\bar{K}} = \mathrm{Spec} \bar{K} \times_{\mathrm{Spec} W} X$, for $i, r \in \mathbb{N}$, there is a canonical homomorphism

$$H^i(\bar{Y}_{sét}, S_{\mathbb{Q}_p}^r) \simeq H^i((X_{\bar{K}})_{ét}, \mathbb{Q}_p(r))$$

which is an isomorphism if $r \geq i$.

Consider the following diagram of sites

$$(\mathrm{Spf} W)_{\mathrm{sét}} \xrightarrow{i} (\mathrm{Spec} W)_{\mathrm{sét}} \xleftarrow{j} (\mathrm{Spec} K_0)_{\mathrm{ET}}$$

For any sheaf \mathcal{G} over $(\mathrm{Spec} W)_{\mathrm{sét}}$, the square

$$\begin{array}{ccc} \mathcal{G} & \rightarrow & j_*j^*\mathcal{G} \\ \downarrow & & \downarrow \\ i_*i^*\mathcal{G} & \rightarrow & i_*i^*j_*j^*\mathcal{G} \end{array}$$

is cartesian.

This means that *the functor* $\mathcal{G} \rightarrow (i^*\mathcal{G}, j^*\mathcal{G}, \alpha)$ (here α is i^* of the morphism $\mathcal{G} \rightarrow j_*j^*\mathcal{G}$ defined by adjunction) *gives an equivalence* between the category of sheaves over $(\mathrm{Spec} W)_{\mathrm{sét}}$ and the category of triples $(\mathcal{F}, \mathcal{H}, \alpha)$ consisting of a sheaf \mathcal{F} over $(\mathrm{Spf} W)_{\mathrm{sét}}$, a sheaf \mathcal{H} over $(\mathrm{Spec} K)_{\mathrm{ET}}$ and a morphism $\alpha : \mathcal{F} \rightarrow i^*j_*\mathcal{H}$.

We then define a sheaf \mathcal{S}_n^r over $(\mathrm{Spec} W)_{\mathrm{sét}}$ by gluing S_n^r on $(\mathrm{Spec} W)_{\mathrm{sét}}$ and $(\mathbb{Z}/p^n\mathbb{Z})(r)$ over $(\mathrm{Spec} K_0)_{\mathrm{ET}}$. We have $\mathcal{S}_n^0 = \mathbb{Z}/p^n\mathbb{Z}$ and $\mathcal{S}_n^1 = \mu_{p^n}$ (if $p \neq 2$).

One first proves

Proposition. — *Let* $\overline{X} = \mathrm{Spec} \mathcal{O}_{\overline{K}} \times_{\mathrm{Spec} W} X$.

i) For $i, r, n \in \mathbb{N}$ *with* $i, r < p - 1$, *the natural map*

$$H^i((\overline{X})_{\mathrm{sét}}, \mathcal{S}_n^r) \rightarrow H^i((\overline{Y})_{\mathrm{sét}}, \mathcal{S}_n^r)$$

is an isomorphism.

ii) For $i, r \in \mathbb{N}$, *the natural map*

$$H^i((\overline{X})_{\mathrm{sét}}, \mathcal{S}_{\mathbb{Q}_p}^r) \rightarrow H^i((\overline{Y})_{\mathrm{sét}}, \mathcal{S}_{\mathbb{Q}_p}^r)$$

is an isomorphism.

The proof is as follow : We have a short exact sequence

$$0 \rightarrow j!(\mathbb{Z}/p^n\mathbb{Z}) \rightarrow \mathcal{S}_n^r(-r) \rightarrow i_*S_n^r(-r) \rightarrow 0$$

and we are reduced to show that $H^m((\overline{X})_{\mathrm{sét}}, j\mathbb{Z}/p^n\mathbb{Z}) = 0$.

We have also the exact sequence

$$0 \rightarrow j!(\mathbb{Z}/p^n\mathbb{Z}) \rightarrow \mathbb{Z}/p^n\mathbb{Z} \rightarrow i_*(\mathbb{Z}/p^n\mathbb{Z}) \rightarrow 0$$

and we are reduced to prove that $H^*((\overline{X})_{\mathrm{sét}}, \mathbb{Z}/p^n\mathbb{Z}) \rightarrow H^*((\overline{Y})_{\mathrm{sét}}, \mathbb{Z}/p^n\mathbb{Z})$ is an isomorphism. One can then checks that the proof of proper base change theorem for étale cohomology extends word to word to syntomic étale cohomology.

We concludes the proof with

Proposition. — *Let $\bar{X} = \text{Spec } \mathcal{O}_{\bar{K}} \times_{\text{Spec } W} X$.*

i) For $i, r, n \in \mathbb{N}$ with $i \leq r < p - 1$, the natural map

$$H^i((\bar{X})_{\text{set}}, \mathcal{S}_n^r) \rightarrow H^i((X_{\bar{K}})_{\text{ét}}, \mathbb{Z}/p^n\mathbb{Z})(r)$$

is an isomorphism.

ii) For $i, r \in \mathbb{N}$ with $r \leq \min\{i, \dim X\}$, the natural map

$$H^i((\bar{X})_{\text{set}}, \mathcal{S}_{\mathbb{Q}_p}^r) \rightarrow H^i((X_{\bar{K}})_{\text{ét}}, \mathbb{Q}_p)(r)$$

is an isomorphism.

The proof of (ii) does not require too much work : The two \mathbb{Q}_p -vector spaces have the same finite dimension. Hence, it's enough to prove that the map is injective.

The map is compatible with product structures and Poincaré duality. Hence, if X is of dimension d , it is enough to check that the map $H^{2d}((\bar{X})_{\text{set}}, \mathcal{S}_{\mathbb{Q}_p}^r) \rightarrow H^{2d}((X_{\bar{K}})_{\text{ét}}, \mathbb{Q}_p)(r)$ is an isomorphism. It suffices to check it for $r = d$, in which case it results, by standard arguments, of compatibility with Chern classes.

The proof of (i) relies on Kazuya Kato's computation of vanishing (or nearby) cycles in terms of Milnor K-theory. This computation implies (Kurihara)

Proposition. — *Let X be a smooth scheme over W and r an integer satisfying $0 \leq r < p - 1$. Consider the following diagram of sites*

$$(\bar{X}_{n+r})_{\text{syn}} \xrightarrow{\varepsilon} (\bar{X}_{n+r})_{\text{ét}} = (\bar{Y})_{\text{ét}} \xrightarrow{i} \bar{X}_{\text{ét}} \xleftarrow{j} (X_{\bar{K}})_{\text{ét}} .$$

There exists a canonical isomorphism

$$R\mathcal{E}_* \mathcal{S}_n^r \rightarrow \tau_{\leq r} i^* Rj_*(\mathbb{Z}/p^n\mathbb{Z})(r)$$

The results follow easily from this statement.