p-adic Hodge theory and comparison theorems.

A flavor of the syntomic technics

ring = commutative ring. A-algebra = ring B with a morphim $A \to B$.

General construction : A_0 a ring, A = an A_0-algebra, $p \in A_0 \Rightarrow\ B^+_{dR}(A/A_0)$.

We define a decreasing sequence of A_0-algebras

$$A = A^{(0)} \supset A^{(1)} \supset \ldots \supset A^{(n-1)} \supset A^{(n)} \supset \ldots$$

by

$$0 \to A^{(n)} \to A^{(n-1)} \to A \otimes_{A^{(n-1)}} \Omega^1_{A^{(n-1)}/A_0}$$

$$\text{Set } \widehat{A}^{(n)} = \varprojlim_{r \in \mathbb{N}} A^{(n)}/p^r A^{(n)}\ ,\ B_m = \widehat{A}^{(m-1)}[1/p] \text{ and } B^+_{dR} = \varprojlim_{m \in \mathbb{N}} B_m$$

Application :
K is a field of characteristic 0 complete with respect to a discrete valuation.
$\mathcal{O}_K$ = the valuation ring, $\mathfrak{m}_K$ = the maximal ideal
$k = \mathcal{O}_K/\mathfrak{m}_K$ = the residue field (assumed to be perfect of characteristic $p > 0$).
$\overline{K}$ = a choosen algebraic closure of K, $\mathcal{O}_{\overline{K}}$ = integral closure of $\mathcal{O}_K$ in $\overline{K}$,
$G_K = \mathrm{Gal}(\overline{K}/K)$.

$$B^+_{dR} = B^+_{dR}(\mathcal{O}_{\overline{K}}/\mathcal{O}_K)\ .$$

For $n > 0$, get a short exact sequence

$$0 \to \mathcal{O}^{(n)}_{\overline{K}} \to \mathcal{O}^{(n-1)}_{\overline{K}} \to \Omega^{(n)} \to 0 \text{ with } \Omega^{(n)} = \mathcal{O}_{\overline{K}} \otimes_{\mathcal{O}^{(n-1)}_{\overline{K}}} \Omega^1_{\mathcal{O}^{(n-1)}_{\overline{K}}/\mathcal{O}_K}\ .$$

Hence

$$0 \to (\Omega^{(n)})_{p^r} \to \mathcal{O}^{(n)}_{\overline{K}}/p^r\mathcal{O}^{(n)}_{\overline{K}} \to \mathcal{O}^{(n-1)}_{\overline{K}}/p^r\mathcal{O}^{(n-1)}_{\overline{K}} \to 0$$

$$0 \to T_p(\Omega^{(n)}) \to \widehat{\mathcal{O}_{\overline{K}}}^{(n)} \to \widehat{\mathcal{O}_{\overline{K}}}^{(n-1)} \to 0\ .$$

Get $B_1 = C$ and, for any $n, V_p(\Omega^{(n)}) = \mathbb{Q}_p \otimes_{\mathbb{Z}_p} T_p(\Omega^{(n)}) = C(n)$ and

$$B^+_{dR} \text{ is a complete discrete valuation ring whose residue field is } C\ .$$

$$B_{dR} = \mathrm{Frac}\ B^+_{dR}\ .$$

Theorem. — *Let X a proper and smooth variety over K, then, for any $m \in \mathbb{N}$,*

$$B_{dR} \otimes_K H^m_{dR}(X) = B_{dR} \otimes_{\mathbb{Q}_p} H^m_{ét}(X_{\overline{K}}, \mathbb{Q}_p) \ .$$

This is the p-adic analogue of the classical comparison theorem

$$\mathbb{C} \otimes_K H^m_{dR}(X) = \mathbb{C} \otimes_{\mathbb{Q}} H^m(X_\sigma(\mathbb{C}), \mathbb{Q})$$

($\sigma : K \to \mathbb{C}$ is an embedding and $H^m_{dR}(X) = \mathbb{H}^m_{Zar}(X, \Omega^{\cdot}_{X/K})$).

In the above theorem,

– $X_{\overline{K}}$ is the variety over $\overline{K}$ obtained from X by base change,

– $H^m_{ét}(X_{\overline{K}}, \mathbb{Q}_p) = \mathbb{Q}_p \otimes_{\mathbb{Z}_p} \varprojlim_{n \in \mathbb{N}} H^m((X_{\overline{K}})_{ét}, \mathbb{Z}/p^n\mathbb{Z})$,

– "=" means that this is a canonical isomorphim which is functorial and compatible with everything you want.

In particular,

1 – The isomorphism is compatible with the action of G_K

(on the LHS, $g(b \otimes x) = g(b) \otimes x$, on the RHS, $g(b \otimes y) = g(b) \otimes g(y)$).

2 – The isomorphism is compatible with the filtration :

(on the LHS, $F^i(B_{dR} \otimes H^m(X)) = \sum_{i_1+i_2=i} F^{i_1} B_{dR} \otimes F^{i_2} H^m_{dR}(X)$,

on the RHS, $F^i(B_{dR} \otimes H^m_{ét}(X_{\overline{K}}, \mathbb{Q}_p)) = F^i B_{dR} \otimes H^m_{ét}(X_{\overline{K}}, \mathbb{Q}_p)$.)

($F^i B_{dR} = i$-th power of the maximal ideal of B^+_{dR})

Moreover $(B_{dR})^{G_K} = K$ and

$$H^m_{dR}(X_K) = D_{dR}(H^m_{ét}(X_{\overline{K}}, \mathbb{Q}_p))$$

(for any p-adic representation V of G_K, we set $D_{dR}(V)$ = the filtered K-vector space $(B_{dR} \otimes_{\mathbb{Q}_p} V)^{G_K}$. Get $\dim_K D_{dR}(V) \leq \dim_{\mathbb{Q}_p} V$ with equality iff V is de Rham).

But the theorem is more precise : there are always more algebraic structures which can be defined on $H^m_{dR}(X)$.

$$H^m_{ét}(X_{\overline{K}}, \mathbb{Q}_p) = V(H^m_{dR}(X) \text{ with additional structures }) \text{ (ad hoc functor) .}$$

$W = W(k)$, $K_0 = W[1/p] = \mathrm{Frac}\, W$, equipped with the absolute Frobenius σ.

φ-isocrystal over k = finite dimensional K_0-vector space D equipped with a σ-semi-linear bijective map $\varphi : D \to D$.

1 – The easiest case : If X has *good reduction*, i.e. if there exists $\mathcal{X}$ proper and smooth over $\mathcal{O}_K$ such that Spec $K \times \mathcal{X} = X$, get the *crystalline cohomology of the special fiber $\mathcal{X}_k$ of $\mathcal{X}$*

$$H^m_{cris}(X) = K_0 \otimes_W \varprojlim_{n\in\mathbb{N}} H^m((\mathcal{X}_k/W_n)_{cris}, \text{struct.sheaf})$$

Depends only on $\mathcal{X}_k$, this is a φ-isocrystal and

$$K \otimes_{K_0} H^m_{cris}(X) = H^m_{dR}(X) \quad \text{(Berthelot-Ogus)}$$

Get a *filtered φ-module over K* and

$$V_{cris}(H^m_{dR}(X)) = H^m_{\acute{e}t}(X_{\overline{K}}, \mathbb{Q}_p) \text{ and } D_{cris}(H^m_{\acute{e}t}(X_{\overline{K}}, \mathbb{Q}_p)) = H^m_{dR}(X)\ .$$

Remark : The $H^m(\mathcal{X}_k/W_n)_{cris}$, struct.sheaf) are also the cohomology groups of the de Rham-Witt complex $W_n\Omega^*_{\mathcal{X}_k/k}$ (see Illusie's talk).

2 – The case where X has *semi-stable reduction*, i.e. there exists a proper regular scheme $\mathcal{X}$ over $\mathcal{O}_K$ whose general fiber $\mathcal{X}_K$ is X and whose special fiber $\mathcal{X}_k \hookrightarrow \mathcal{X}$ is a divisor with normal crossings.

Need in this case to use log-geometry and to replace the crystalline cohomology with the *log-crystalline cohomology* (or *crystalline-cohomology with log poles* or *Hyodo-Kato cohomology*) (see the talk of Ogus).

To the log-special fiber of $\mathcal{X}$ correspond the $H^m_{st}(X)$ = a φ-isocrystal equipped with a K_0-linear map $N : D \to D$ such that $N\varphi = p\varphi N$.

We have also

$$K \otimes_{K_0} H^m_{st}(X) = H^m_{dR}(X) \quad \text{(Hyodo-Kato)}$$

Get a *filtered (φ, N)-module over K* and

$$V_{st}(H^m_{dR}(X)) = H^m_{\acute{e}t}(X_{\overline{K}}, \mathbb{Q}_p) \text{ and } D_{st}(H^m_{\acute{e}t}(X_{\overline{K}}, \mathbb{Q}_p)) = H^m_{dR}(X)\ .$$

3 – General case : Use de Jong's alterations to reduce the problem to the semi-stable case. We have slightly more involved algebraic structures (filtered (φ, N, G_K)-modules over K).

Remark : The *p-adic monodromy theorem* (Berger + André, Mebkhout, Kedlaya) says that any de Rham representation is potentially semi-stable.

We get an equivalence of categories

$$\text{Admissible filtered } (\varphi, N, G_K) - \text{modules over } K \iff \text{de Rham repr's of } G_K \ .$$

Three distinct methods have given the complete results.

1 – *Syntomic method* : F-Messing, Kato,...., Breuil, Tsuji.
to be discussed

2 – *The almost étale approach* due to Gerd Faltings.

3 – *The K-theoretic approach* due to Wieslawa Niziol.
(see her talk)

W. Niziol also prouved that the three comparison theorems gives the same maps (by proving that there are the solution of a universal problem).

A new approach consists of using rigid cohomology. It is very promising and should be useful for comparisons theorems with non constant coefficients. Recently
– p-adic Hodge theory became closer to rigid analytic geometry,
– very important progress have been made in rigid cohomology (see Kedlaya's talks).

What are p-adic Hodge theory and p-adic comparison theorems good for ?
!!!!!
In today's arithmetic geometry, both p-adic Hodge theory and automorphic forms play a crucial role.

The syntomic topos

A morphism $\alpha : X \to S$ of schemes is *syntomic* (Mazur) if
a) α is flat,
b) α is a locally complete intersection, i.e., Zarisky locally, it may be written

$$\alpha : \text{Spec } B \to \text{Spec} A \text{ with } B = A[X_1, X_2, \ldots, X_m]/(f_1, f_2, \ldots, f_n)$$

and $f_1, f_2, \ldots, f_n$ a regular sequence.

If S is a scheme, the *big* and the *small syntomic sites* are

S_{SYN} : the underlying category is the category of S-schemes locally of finite type.

S_{syn} : the underlying category is the full sub-category of the previous one whose objects are S-schemes such that the structural morphism is syntomic.

For both sites, covering are surjective families of syntomic morphisms.

The sheaf $\mathcal{O}_n^{cris}$

k = perfect field of characteristic $p > 0$ $W_n = W_n(k)$ = ring of Witt vector of length n.

For any topos and any W_n-algebra A over this topos, let $\mathcal{E}^{dp}(A/W_n)$ the category of *W_n-divided power thickenings of A* :

– An object is a triple $(\mathcal{A}, \rho, \gamma)$ where $\mathcal{A}$ is a (sheaf of) W_n-algebra(s), $\rho : \mathcal{A} \to A$ is an epimorphism of (sheaves of) W_n-algebras and γ is a divided power structure on the kernel of ρ such that $\gamma_m(px) = (p^m/m!)x^m$, for all $x \in \mathcal{A}$.

– A morphism is a momorphism of the underlying (sheaves of) W_n-algebras which is compatible with the ρ's and the γ's.

Theorem. — *Let $\mathcal{O}$ be the structural sheaf over* (Spec $k)_{SYN}$.
The category $\mathcal{E}^{dp}(\mathcal{O}/W_n)$ has an initial object.

We call it $\mathcal{O}_n^{cris}$. If A is any k-algebra, we have

$$\mathcal{O}_n^{cris}(\text{Spec } A) = \mathcal{O}_n^{cris}(A) = \varprojlim_{\mathcal{A} \in \mathcal{C}_A} \mathcal{A}$$

(this is a direct inverse system).

Extends uniquely to a sheaf of W_n-algebras over (Spec $k)_{SYN}$ (plus an epimorphism $\rho : \mathcal{O}_n^{cris} \to \mathcal{O}$ and a divided power structure on the ideal Ker ρ).

Moreover, by functoriality, the Frobenius $a \mapsto a^p$ on $\mathcal{O}$ induces an endomorphism $\varphi : \mathcal{O}_n^{cris} \to \mathcal{O}_n^{cris}$

Warning : for a given k-algebra A, the ring $\mathcal{O}_n^{cris}(A)$ itself is not, in general, an object of $\mathcal{E}^{dp}(A/W_n)$ (the map $\rho_A : \mathcal{O}_n^{cris}(A) \to A$ may not be surjective!)

Moreover
– For $m, n \in \mathbb{N}$, $\mathcal{O}_{m+n}^{cris} \to \mathcal{O}_n^{cris}$ is an epimorphim,
– Over (Spec $k)_{syn}$, we have a short exact sequence

$$0 \to \mathcal{O}_m^{cris} \to \mathcal{O}_{m+n}^{cris} \to \mathcal{O}_n^{cris} \to 0 \ .$$

$(\mathcal{O}_n^{cris})_{n \in \mathbb{N}}$ is a p-divisible sheaf.

Alternative descriptions :

1 – If $A = B/I$ with B a smooth W_n-algebra and if B^{dp} is the divided power enveloppe of B with respect to I (compatible with canonical divided powers on the ideal generated by p),

$$0 \to \mathcal{O}_n^{cris}(A) \to B^{dp} \to B^{dp} \otimes_B \Omega^1_{B/W_n}$$

2 – For any scheme X over k, we have

$$\mathcal{O}_n^{cris}(X) = H^0(X/W_n)_{cris}, \text{struct.sheaf}) \ .$$

3 – For any k-algebra A, get $\rho_A : W_n(A) \to A$ via $(a_0, a_1, \ldots, a_{n-1}) \mapsto a_0^{p^n}$

$W_n^{DP}(A)$ = divided power enveloppe of $W_n(A)$ with respect to the kernel of ρ_A, compatible with canonical divided powers on $VW_{n-1}(A)$.

Extends uniquely to a sheaf for the Zariski topology. Set $\tilde{W}_n^{DP}$ the sheafification of W_n^{DP} for the syntomic topology.

$$\tilde{W}_n^{DP} \to \mathcal{O}_n^{cris} \text{ is an isomorphism.}$$

Moreover, if $A = B/(f_1^{p^n}, f_2^{p^n}, \ldots, f_s^{p^n})$, with B smooth over k and $f_1, f_2, \ldots, f_s$ a regular sequence, then

$$\tilde{W}_n^{DP}(A) = W_n^{DP}(A).$$

Projection on the Zariski site

Let $X \to k$ syntomic, Consider $u : X_{syn} \to X_{Zar}$.

Let $Y \to X$ a syntomic covering such that any local section of $\mathcal{O}_X$ has a p^n-th root in $\mathcal{O}_Y$. Then, *the complex*

$$\mathcal{O}_n^{cris}(Y) \to \mathcal{O}_n^{cris}(Y \times_X Y) \to \mathcal{O}_n^{cris}(Y \times_X Y \times_X Y) \to \ldots$$

(where $(\mathcal{O}_n^{cris}(Y \times_X \ldots \times_X Y)$ means the projection onto X_{Zar} of the restriction of $\mathcal{O}_n^{cris}$ to $Y \times_X \ldots Y \times_X Y)$

represents $Ru_*\mathcal{O}_n^{cris}$.

This complex computes also the crystalline cohomology, i.e. if $\pi : (X/W_n)_{cris} \to X_{Zar}$ is the natural projection, the above complex represents also $R\pi_*(\text{struct.sheaf})$.

The smooth case

If $X \to k$ is smooth there is a canonical choice for Y

$$Y = X^{(n)} \to X \text{ (the map is Frob}^n) \ .$$

- This complex is also quasi-isomorphic to the de Rham-Witt complex $W_n\Omega^*_{X/k}$ (see Illusie's talk). For $n = 1$, $W_1\Omega^*_{X/k} = \Omega^*_{X/k}$, the usual de Rham complex.

The crystalline comparison theorem

Let $\mathcal{O}_{\overline{K}}$ be the integral closure of $\mathcal{O}_K$ in $\overline{K}$. Then $H^m((\mathcal{O}_{\overline{K}}/p\mathcal{O}_{\overline{K}})_{syn}, \mathcal{O}_n^{cris}) = 0$ for $m > 0$. Set

$$A_{cris} = \varprojlim_{n\in\mathbb{N}} \mathcal{O}_n^{cris}(\mathcal{O}_{\overline{K}}/p\mathcal{O}_{\overline{K}}) \text{ and } B^+_{cris} = A_{cris}[1/p] \ .$$

Then $A_{cris}/p^n A_{cris} = \mathcal{O}_n^{cris}(\mathcal{O}_{\overline{K}}/p\mathcal{O}_{\overline{K}})$. Get also a surjective homomorphism of $W(k)$-algebras :

$$\theta : A_{cris} \to \mathcal{O}_C$$

and the kernel of θ is a divided power ideal.

The p-adic $2\pi i$:

Let $\varepsilon = (\varepsilon^{(n)})_{n\in\mathbb{N}}$ a generator of $\mathbb{Z}_p(1)$ viewed multiplicatively, i.e. a sequence of elements of $\mathcal{O}_{\overline{K}}$ such that $\varepsilon^{(0)} = 1$, $\varepsilon^{(1)} \neq 1$ and $(\varepsilon^{(n+1)})^p = \varepsilon^{(n)}$ for $n > 0$. For $n \in \mathbb{N}$ let $\varepsilon_n = x_n^{p^n}$ where x_n is any lifting in $\mathcal{O}_n^{cris}(\mathcal{O}_{\overline{K}}/p\mathcal{O}_{\overline{K}})$ of the image of $\varepsilon^{(n)}$ in $\mathcal{O}_{\overline{K}}/p\mathcal{O}_{\overline{K}}$. Then $[\varepsilon] = (\varepsilon_n)_{n\in\mathbb{N}} \in A_{cris}$ and

$$\text{``}2\pi i\text{''} = t = log([\varepsilon]) \in A_{cris}$$

Set $B_{cris} = B^+_{cris}[1/t]$ (action of φ extends ($\varphi t = pt$, $\varphi(1/t) = 1/pt$).

The map θ extends to a map

$$\theta : B^+_{cris} \to C \ .$$

$$B^+_{dR} = \varprojlim_{n\in\mathbb{N}} B^+_{cris}/(\text{Ker } \theta)^n \quad , t \text{ is a generator of the maximal ideal of } B^+_{dR}$$

$$\text{and } K \otimes_{K_0} B_{cris} \to B_{dR} \text{ is injective}$$

(Analogue for $\mathcal{O}_{\overline{K}}$ of Berthelot-Ogus theorem).

Theorem. — *Let X be a proper and smooth variety over K with good reduction. For all $m \in \mathbb{N}$,*

$$B_{cris} \otimes_{K_0} H^m_{cris}(X) = B_{cris} \otimes_{\mathbb{Q}_p} H^m_{ét}(X_{\overline{K}}, \mathbb{Q}_p)\ .$$

(We choose a proper and smooth model $\mathcal{X}$ of X over $\mathcal{O}_K$ and we set

$H^m_{cris}(X) = H^m_{cris}(\mathcal{X}_k)$ with the Hodge filtration on $K \otimes K_0 H^m_{cris}(X) = H^m_{dR}(X)$.)

$$D_{cris}(H^m_{ét}(X_{\overline{K}}) = H^m_{cris}(X)$$
$$V_{cris}(H^m_{cris}(X)) = H^m_{ét}(X_{\overline{K}}, \mathbb{Q}_p)$$

where

$$D_{cris}(V) = (B_{cris} \otimes_{\mathbb{Q}_p} V)^{G_K}$$
$$V_{cris}(D) = (B_{cris} \otimes_{K_0} D)_{\varphi=1} \cap F^0(B_{dR} \otimes_K D_K)$$

Sketch of the syntomic proof (in the case $K = K_0 = W[1/p]$)

For any $m \in \mathbb{N}$, we may consider the small syntomic site $(\mathrm{Spec}\ W_m)_{syn}$ over $W_m = W_m(k)$.

For $n \leq m$, $\mathcal{O}_n$ is the reduction mod p^n of the structural sheaf. Get a short exact sequence

$$0 \to J_n \to \mathcal{O}_n^{cris} \to \mathcal{O}_n \to 0$$

and J_n is a (sheaf of) divided power ideal(s). For $r \in \mathbb{N}$, we call $J_n^{[r]}$ the r-th divided power of J_n.

Assume $r \leq p-1$ and $n + r \leq m$. We have two different maps from $J_n^{[r]}$ to $\mathcal{O}_n^{cris}$. The first one, ι is the natural inclusion, the second one is $\varphi_r =$"$p^{-r}\varphi$"

(More precisely, if x is a section in $J_{n+r}^{[r]}$, $\varphi(x)$ is divisible by p^r in $\mathcal{O}_{n+r}^{cris}$, hence come from a well defined section $y \in \mathcal{O}_n^{cris}$. The so defined map $J_{n+r}^{[r]} \to \mathcal{O}_n^{cris}$ factors through $J_n^{[r]}$ and φ_r is the induced map).

We call S_n^r the kernel of the map $\varphi_r - \iota : J_n^{[r]} \to \mathcal{O}_n^{cris}$. For $r < p-1$, we have a short exact sequence

$$0 \to S_n^r \to J_n^{[r]} \to \mathcal{O}_n^{cris} \to 0\ .$$

Proposition. — *Let $m \in \mathbb{N}$, Y_m a proper and smoth scheme over* Spec W_m *and* $\overline{Y}_m =$ Spec $\mathcal{O}_{\overline{K}} \times_{\mathrm{Spec}\ W} Y_m$. *Let $i, r, n \in \mathbb{N}$ such that $i \leq r < p-1$ and $r + n \leq m$ Then $H^i((\overline{X})_{syn}, S_n^r)(-r)$ is a finite representation of G_K independent of the choice of r. Moreover, the sequence*

$$0 \to H^i((\overline{X})_{syn}, S_n^r) \to H^i((\overline{X})_{syn}, J_n^{[r]}) \to H^i((\overline{X})_{syn}, \mathcal{O}_n^{cris}) \to 0$$

is exact.

Consider the sites

$$\text{Spf } W_{syn} \quad \text{and} \quad \text{Spf } W_{s\acute{e}t} \ :$$

For both sites, the underlying category is the full sub-category of formal schemes $U = (U_m)_{m\in\mathbb{N}}$ over W which are syntomic (i.e. $U_m \to \text{Spec } W_m$ is syntomic for all m).

Covering are surjective families of quasi-finite syntomic morphisms (resp. quasi-finite syntomic morphisms with étale (rigid) generic fiber).

$S^r_{\mathbb{Q}_p}$ = the $\mathbb{Q}_p$-sheaf over one of these sites defined using the direct images of the S^r_n.

Exercise : Adapt the previous contruction to define the $\mathbb{Q}_p$-sheaves $S^r_{\mathbb{Q}_p}$ for $r > p-1$.

Theorem. — *Let X be a proper and smooth scheme over W. Let $\overline{Y} = (\overline{Y}_m)_{m\in\mathbb{N}}$ with $\overline{Y}_m = \text{Spec } \mathcal{O}_{\overline{K}}/p^m \times_{\text{Spec } W} X$. Then, for all $i \in \mathbb{N}$, $D_{cris}(H^i(X_K))$ is a finite dimensional $\mathbb{Q}_p$-vector space and we get*

$$D_{cris}(H^i_{cris}(X_K)) = H^i(\overline{Y}_{syn}, S^r_{\mathbb{Q}_p})(-r) = H^i(\overline{Y}_{s\acute{e}t}, S^r_{\mathbb{Q}_p})(-r) \textit{ for all } r \geq i \ .$$

For $i < p-1$, the proof is easy (dévissages, linear algebra, F-Laffaille theory). The general case requires more work !

To complete the proof in the case $K = K_0 = W[1/p]$, it is enough to prouve that

Under the asumptions of the previous theorem, if $X_{\overline{K}} = \text{Spec } \overline{K} \times_{\text{Spec } W} X$, for $i, r \in \mathbb{N}$, there is a canonical homomorphism

$$H^i(\overline{Y}_{s\acute{e}t}, S^r_{\mathbb{Q}_p}) \simeq H^i((X_{\overline{K}})_{\acute{e}t}, \mathbb{Q}_p(r))$$

which is an isomorphism if $r \geq i$.

Consider the following diagram of sites

$$(\mathrm{Spf}\ W)_{s\acute{e}t} \xrightarrow{i} (\mathrm{Spec}\ W)_{s\acute{e}t} \xleftarrow{j} (\mathrm{Spec}\ K_0)_{ET}$$

For any sheaf $\mathcal{G}$ over $(\mathrm{Spec}\ W)_{set}$, the square

$$\begin{array}{ccc} \mathcal{G} & \rightarrow & j_*j^*\mathcal{G} \\ \downarrow & & \downarrow \\ i_*i^*\mathcal{G} & \rightarrow & i_*i^*j_*j^*\mathcal{G} \end{array}$$

is cartesian.

This means that *the functor* $\mathcal{G} \rightarrow (i^*\mathcal{G}, j^*\mathcal{G}, \alpha)$ (here α is i^* of the morphism $\mathcal{G} \rightarrow j_*j^*\mathcal{G}$ defined by adjunction) *gives an equivalence* between the category of sheaves over $(\mathrm{Spec}\ W)_{s\acute{e}t}$ and the category of triples $(\mathcal{F}, \mathcal{H}, \alpha)$ consisting of a sheaf $\mathcal{F}$ over $(\mathrm{Spf}\ W)_{s\acute{e}t}$, a sheaf $\mathcal{H}$ over $(\mathrm{Spec}\ K)_{ET}$ and a morphism $\alpha : \mathcal{F} \rightarrow i^*j_*\mathcal{H}$.

We then define a sheaf $\mathcal{S}_n^r$ over $(\mathrm{Spec}\ W)_{set}$ by gluing S_n^r on $(\mathrm{Spec}\ W)_{s\acute{e}t}$ and $(\mathbb{Z}/p^n\mathbb{Z})(r)$ over $(\mathrm{Spec}\ K_0)_{ET}$. We have $\mathcal{S}_n^0 = \mathbb{Z}/p^n\mathbb{Z}$ and $\mathcal{S}_n^1 = \mu_{p^n}$ (if $p \neq 2$).

One first proves

Proposition. — *Let* $\overline{X} = \mathrm{Spec}\ \mathcal{O}_{\overline{K}} \times_{\mathrm{Spec}\ W} X$.

i) For $i, r, n \in \mathbb{N}$ *with* $i, r < p - 1$, *the natural map*

$$H^i((\overline{X})_{s\acute{e}t}, \mathcal{S}_n^r) \rightarrow H^i((\overline{Y})_{s\acute{e}t}, S_n^r)$$

is an isomorphism.

ii) For $i, r \in \mathbb{N}$, *the natural map*

$$H^i((\overline{X})_{s\acute{e}t}, \mathcal{S}_{\mathbb{Q}_p}^r) \rightarrow H^i((\overline{Y})_{s\acute{e}t}, S_{\mathbb{Q}_p}^r)$$

is an isomorphism.

The proof is as follow : We have a short exact sequence

$$0 \rightarrow j_!(\mathbb{Z}/p^n\mathbb{Z}) \rightarrow \mathcal{S}_n^r(-r) \rightarrow i_*S_n^r(-r) \rightarrow 0$$

and we are reduced to show that $H^m((\overline{X})_{s\acute{e}t}, j\mathbb{Z}/p^n\mathbb{Z}) = 0$.

We have also the exact sequence

$$0 \rightarrow j_!(\mathbb{Z}/p^n\mathbb{Z}) \rightarrow \mathbb{Z}/p^n\mathbb{Z} \rightarrow i_*(\mathbb{Z}/p^n\mathbb{Z}) \rightarrow 0$$

and we are reduced to prove that $H^*((\overline{X})_{s\acute{e}t}, \mathbb{Z}/p^n\mathbb{Z}) \rightarrow H^*((\overline{Y})_{s\acute{e}t}, \mathbb{Z}/p^n\mathbb{Z})$ is an isomorphism. One can then checks that the proof of proper base change theorem for étale cohomology extends word to word to syntomic étale cohomology.

We concludes the proof with

Proposition. — *Let* $\overline{X} = \mathrm{Spec}\ \mathcal{O}_{\overline{K}} \times_{\mathrm{Spec}\ W} X$.
i) For $i, r, n \in \mathbb{N}$ *with* $i \leq r < p-1$, *the natural map*

$$H^i((\overline{X})_{s\acute{e}t}, \mathcal{S}_n^r) \to H^i((X_{\overline{K}})_{\acute{e}t}, \mathbb{Z}/p^n\mathbb{Z})(r)$$

is an isomorphism.
ii) For $i, r \in \mathbb{N}$ *with* $r \leq \min\{i, \dim X\}$, *the natural map*

$$H^i((\overline{X})_{set}, \mathcal{S}_{\mathbb{Q}_p}^r) \to H^i((X_{\overline{K}})_{\acute{e}t}, \mathbb{Q}_p)(r)$$

is an isomorphism.

The proof of (ii) does not require too much work : The two $\mathbb{Q}_p$-vector spaces have the same finite dimension. Hence, it's enough to prove that the map is injective.

The map is compatible with product structures and Poincaré duality. Hence, if X is of dimension d, it is enough to check that the map $H^{2d}((\overline{X})_{s\acute{e}t}, \mathcal{S}_{\mathbb{Q}_p}^r) \to H^{2d}((X_{\overline{K}})_{\acute{e}t}, \mathbb{Q}_p)(r)$ is an isomorphism. It suffises to check it for $r = d$, in which case it results, by standard arguments, of compatibility with Chern classes.

The proof of (i) relies on Kazuya Kato's computation of vanishing (or nearby) cycles in terms of Milnor K-theory. This computation implies (Kurihara)

Proposition. — *Let* X *be a smooth scheme over* W *and* r *an integer satisfying* $0 \leq r < p-1$. *Consider the following diagram of sites*

$$(\overline{X}_{n+r})_{syn} \xrightarrow{\varepsilon} (\overline{X}_{n+r})_{\acute{e}t} = (\overline{Y})_{\acute{e}t} \xrightarrow{i} \overline{X}_{\acute{e}t} \xleftarrow{j} (X_{\overline{K}})_{\acute{e}t} \ .$$

There exists a canonical isomorphism

$$R\varepsilon_* S_n^r \to \tau_{\leq r} i^* Rj_*(\mathbb{Z}/p^n\mathbb{Z})(r)$$

The results follow easily from this statement.