On the slice filtration and the Kervaire invariant one problem

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Main Result

Theorem (H.-Hopkins-Ravenel)

There are smooth Kervaire invariant one manifolds only in dimensions 2, 6, 14, 30, 62, *and maybe* 126.

Exemplars:

- 2: $S^1 \times S^1$
- 6: $SU(2) \times SU(2)$
- 14: $S(\mathbb{O}) \times S(\mathbb{O})$
- 30: (Bökstedt) Related to $E_6/\big(U(1) \times Spin(10)\big)$
- 62: Possibly a similar construction.

Geometry and History

1930s Pontryagin proves
$\{\text{framed } n-\text{manifolds}\}/\text{cobordism} \cong \pi_n^S$.

Tries to use surgery to reduce to spheres & misses an obstruction.

1950s Kervaire-Milnor show can always reduce to case of spheres...

Except possibly in dimension $4k + 2$, where there is an obstruction: Kervaire Invariant.

Adams Spectral Sequence

$$[X, Y] \rightsquigarrow \mathrm{Hom}_{\mathcal{A}}\big(H^*(Y), H^*(X)\big)$$

Have a SS with

$$E_2 = \mathrm{Ext}_{\mathcal{A}}\big(H^*(Y), H^*(X)\big)$$

and converging to $[X, Y]$.

- (Adem) $\mathrm{Ext}^1(\mathbb{F}_2, \mathbb{F}_2)$ is generated by classes h_i, $i \geq 0$.
- h_j survives the Adams SS if $\mathbb{R}^{2^j}$ admits a division algebra structure.

Browder's Reformulation

Theorem (Browder 1969)

1. *There are no smooth Kervaire invariant one manifolds in dimensions not of the form $2^{j+1}-2$.*
2. *There is such a manifold in dimension $2^{j+1}-2$ iff h_j^2 survives the Adams spectral sequence.*

Adams showed that h_j itself survives only if $j < 4$

$$d_2(h_{j+1}) = h_0 h_j^2.$$

Previous Progress

h_1^2, h_2^2, and h_3^2 classically exist.

Theorem (Mahowald-Tangora)

The class h_4^2 survives the Adams SS.

Theorem (Barratt-Jones-Mahowald)

The class h_5^2 survives the Adams SS.

Theorem (H.-Hopkins-Ravenel)

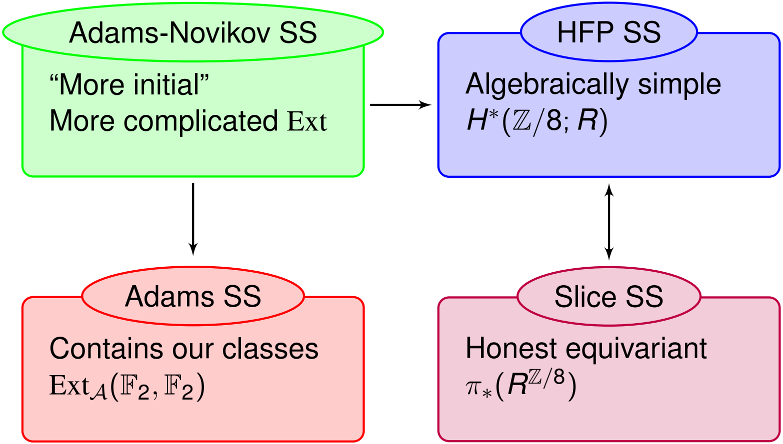
For $j \geq 7$, h_j^2 does not survive the Adams SS.

General Outline

There are four main steps

1. Reduce to a simpler homotopy computation which faithfully sees the Kervaire classes
2. Rigidify the problem to get more structure and less wiggle-room
3. Show homotopy is automatically zero in dimension -2
4. Show homotopy is periodic with period 2^8

Reduction to Simpler Cases



Benefits of Reduction

Reduction is purely algebraic!

1. Lifting from Adams to Adams-Novikov is well understood.
2. Reduction from Adams-Novikov to homotopy fixed points is formal deformation theory.

So good choice of R gives us something that is

- easily computable
- strong enough to detect the classes.

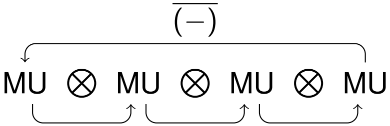
Why Go Equivariant?

- Simplify computation by adding extra structure: equivariance.
- Here have fixed points, rather than homotopy fixed points.
- And there are spheres for every real representation.

Example

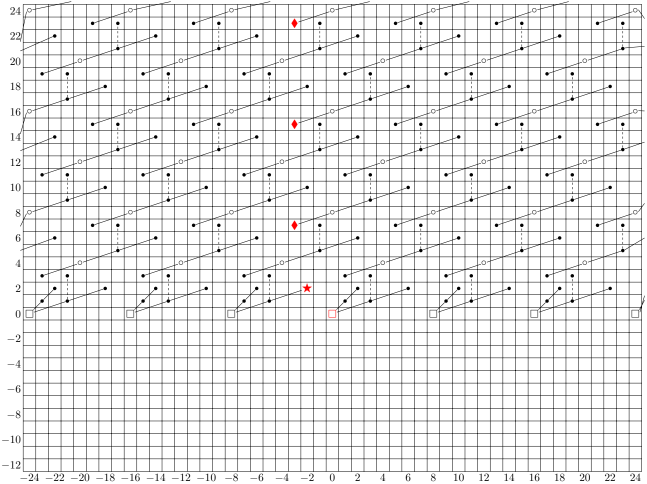
If $G = \mathbb{Z}/2$, then have $S^{\rho_2} = \mathbb{C}^+$ and S^2.

1. Begin with MU with $\mathbb{Z}/2$ given by complex conjugation.
2. "induce" up to a $\mathbb{Z}/8$ spectrum:

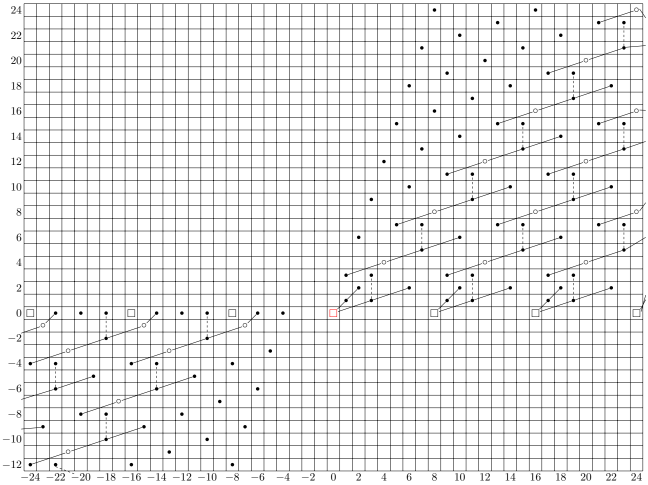


3. The geometric fixed points spectrum for the $\mathbb{Z}/8$-action is geometric.
4. This is a piece of the simplicial THH construction Teena gave for MU.

Advantages of the Slice SS



Advantages of the Slice SS



Basic Idea of Slices

Want to decompose an equivariant X into computable pieces.
Similar to Postnikov tower.
Key difference: don't use all spheres!

Acceptable Ones

1. $S^{k\rho_8}$, $S^{k\rho_8-1}$
2. $\mathbb{Z}/8 \otimes_{\mathbb{Z}/4} S^{k\rho_4}$
3. $\mathbb{Z}/8 \otimes_{\mathbb{Z}/2} S^{k\rho_2}$
4. $\mathbb{Z}/8 \otimes S^{k}$

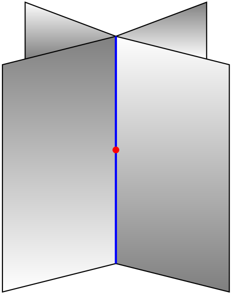
Unacceptable Ones

1. $S^{k\rho_8-2}$
2. $\mathbb{Z}/8 \otimes_{\mathbb{Z}/4} S^{\sigma}$
3. $\mathbb{Z}/8 \otimes_{\mathbb{Z}/2} S^{\sigma-1}$
4. S^{k}

Key Fact

For spectra like MU, slices can be computed from equivariant simple chain complexes.

These algebraically describe the fixed points of the acceptable spheres.



Cellular Chains for $S^{\rho_4 - 1}$

Gives the chain complex

$$\mathbb{Z}^4 \to \mathbb{Z}^4 \to \mathbb{Z}^2 \to \mathbb{Z} = C_\bullet.$$

Maps determined by $H_*(C_\bullet) = \tilde{H}_*(S^3)$.

Gaps

Theorem

For any non-trivial subgroup H of $\mathbb{Z}/8$ and for any slice sphere $\mathbb{Z}/8 \otimes_H S^{\rho_H}$,

$$H_{-2}(C_*^{\mathbb{Z}/8}) = 0$$

The proof is an easy direct computation:

1. If $k \geq 0$, then we are looking at something connected.
2. If $k \leq 0$, then we look at the associated *co*chain algebra.
3. In the relevant degrees, the complex is $\mathbb{Z} \to \mathbb{Z}^2$ by $1 \mapsto (1, 1)$.

Gap Theorem

Theorem

$\pi_{-2}(R) = 0$.

Proof.

- Slices of $MU \otimes MU \otimes MU \otimes MU$ are all of the form

$$H\mathbb{Z} \otimes (\mathbb{Z}/8 \otimes_H S^{k\rho_H}).$$

- Class we are inverting is carried by an $S^{k\rho_8}$.
- Inversion is a colimit and first steps show $\pi_{-2} = 0$. □

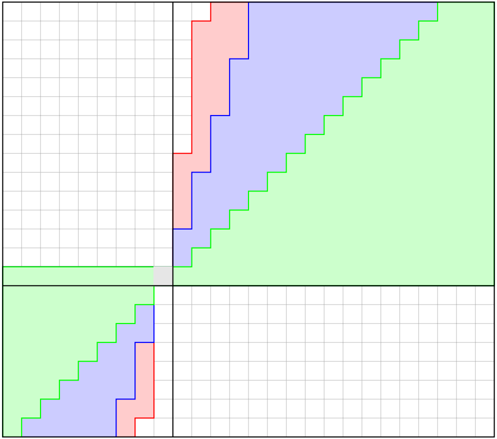
Connections to K-Theory

Recall from Teena's Talk:
$TC(A)$ can be computed from $TR^{\cdot}_{*}(A)$

$$TR^n_q(A,p) = \pi_q\left(THH(A)^{C_{p^{n-1}}}\right).$$

These are homotopy groups of honest fixed points and the maps essentially come from equivariant structure
All of this is computable with slice machinery

General Benefits of Slices



Computing *TR*

Slice machine can much more general. Can build slice spectral sequences for other collections of cells

1. Should have nice connectivity properties
2. Should be “closed under restriction” $=$ a Mackey functor in sets or subcategories
3. Should make cellular objects being considered simpler.

Possible Collection: $L = \mathbb{C}$, $k \cdot L$, $L \oplus L^2 \oplus \cdots \oplus L^d$.