Rigid cohomology and its coefficients

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These slides can be found online:

`http://math.mit.edu/~kedlaya/papers/talks.shtml`

Overview

Rigid cohomology is "the" Weil cohomology for varieties X over a perfect (for ease of exposition) field k of characteristic $p > 0$ taking values in $K = \operatorname{Frac} W(k)$. (In this talk, we mostly ignore integral and torsion coefficients. We also ignore ramified base fields, even though everything carries over nicely.)

In this talk, we'll survey several different constructions of rigid cohomology. We'll also describe categories of smooth and constructible coefficients.

In general, definitions can be more unwieldy than for étale cohomology: ensuring functoriality and various compatibilities can be awkward. But the resulting objects are more directly related to geometry. For instance, they are excellently suited for machine computation of zeta functions. Also, apparently links to K-theory can be made quite explicit.

Why "rigid"?

Basic idea of Tate's *rigid analytic geometry* over K (or any complete nonarchimedean field): the ring T_n of formal power series in $x_1, \dots, x_n$ convergent on the *closed* unit disc is excellent ($\Rightarrow$ noetherian). Define analytic spaces by pasting together closed subspaces of $\operatorname{Maxspec} T_n$.

For a good theory of coherent sheaves, one must restrict *admissible coverings* to a suitable Grothendieck topology (hence "rigid"). E.g.,

$$\{x_1 \in \operatorname{Maxspec} T_1 : |x_1| \leq |p|\} \cup \{x_1 \in \operatorname{Maxspec} T_1 : |x_1| \geq |p|\}$$

is admissible, but not

$$\{x_1 \in \operatorname{Maxspec} T_1 : |x_1| \leq |p|\} \cup \{x_1 \in \operatorname{Maxspec} T_1 : |x_1| > |p|\}.$$

Berkovich's alternate approach: replace $\operatorname{Maxspec} T_n$ by a richer set with a "realer" topology, in which opens are defined by open inequalities.

Contents

Contents

What we'll find

For X of finite type over k and Z a closed subscheme, we produce finite dimensional K-vector spaces $H^i_{Z,\mathrm{rig}}(X)$ (cohomology with supports in Z) and $H^i_{c,\mathrm{rig}}(X)$ (cohomology with compact supports).

These enjoy appropriate functoriality, excision, Künneth formula, Poincaré duality, cycle classes, Gysin maps, Grothendieck-Riemann-Roch, Lefschetz-Verdier trace formula, cohomological descent for proper hypercoverings, etc.

Also available in logarithmic form.

Crystalline cohomology

Modeled on Grothendieck's infinitesimal site (computing algebraic de Rham cohomology), Berthelot constructed a *crystalline site* consisting of nilpotent thickenings of Zariski opens, equipped with divided power structures. This gives a cohomology $H^i_{\text{crys}}(X)$ with coefficients in W. If X is smooth proper, $H^i_{\text{rig}}(X) = H^i_{\text{crys}}(X)_K$.

Deligne and Illusie gave an alternate construction using the de Rham-Witt complex $W\Omega^{\cdot}_{X/k}$, a certain quotient of $\Omega^{\cdot}_{W(X)/W}$ carrying actions of F and V. See previous lecture.

If $X \to \overline{X}$ is a good compactification with boundary Z (i.e., $(\overline{X}, Z)$ is log-smooth over k), one can define $H^i_{\text{rig}}(X) = H^i_{\text{crys}}(\overline{X}, Z)_K$. One can extend this definition using de Jong's alterations (since we ignore torsion) and lots of descent data.

Monsky-Washnitzer cohomology

Suppose X is smooth affine. Let $\mathfrak{X}/W$ be a smooth affine lift of X. The p-adic completion $\widehat{\mathfrak{X}}$ along the special fibre computes crystalline cohomology. Monsky-Washnitzer instead form the *weak completion* $\mathfrak{X}^\dagger$ and show that $H^i_{\mathrm{dR}}(\mathfrak{X}^\dagger_K)$ is functorial in X.

For instance, if $X = \operatorname{Spec} k[x_1, \dots, x_n]$ and $\mathfrak{X} = \operatorname{Spec} W[x_1, \dots, x_n]$, $\widehat{\mathfrak{X}} = \operatorname{Spf} W\langle x_1, \dots, x_n \rangle$ for

$$W\langle x_1, \dots, x_n \rangle = \left\{ \sum_{i_1, \dots, i_n = 0}^{\infty} c_I x^I : c_I \to 0 \right\},$$

while the global sections of $\mathfrak{X}^\dagger$ are

$$W\langle x_1, \dots, x_n \rangle^\dagger = \bigcup_{a,b} \left\{ \sum_{i_1, \dots, i_n = 0}^{\infty} c_I x^I : v_p(c_I) \geq a(i_1 + \cdots + i_n) - b \right\}.$$

Monsky-Washnitzer cohomology: theorems

Theorem (Monsky-Washnitzer, van der Put)

Let $f : A \to B$ be a morphism of smooth k-algebras relative to some endomorphism of k (e.g., Frobenius).

(a) *There exist flat W-algebras $\mathfrak{A}^\dagger, \mathfrak{B}^\dagger$ which are quotients of $W\langle x_1, \dots, x_n\rangle^\dagger$ for some n, and identifications*

$$\mathfrak{A}^\dagger \otimes_W k \cong A, \qquad \mathfrak{B}^\dagger \otimes_W k \cong B.$$

(b) *Given any such $\mathfrak{A}^\dagger, \mathfrak{B}^\dagger$, there exists a continuous lift $\tilde{f}^\dagger : \mathfrak{A}^\dagger \to \mathfrak{B}^\dagger$ of f.*

(c) *For any two continuous lifts $\tilde{f}_1^\dagger, \tilde{f}_2^\dagger : \mathfrak{A}^\dagger \to \mathfrak{B}^\dagger$ of f, the maps*

$$\tilde{f}_1^\dagger, \tilde{f}_2^\dagger : \Omega^{\cdot,\mathrm{cts}}_{\mathfrak{A}^\dagger/W} \to \Omega^{\cdot,\mathrm{cts}}_{\mathfrak{B}^\dagger/W}$$

are chain-homotopic, so induce the same maps $H^i_{\mathrm{dR}}(\mathfrak{A}^\dagger_K) \to H^i_{\mathrm{dR}}(\mathfrak{B}^\dagger_K)$.

Note: $\mathfrak{A}^\dagger, \mathfrak{B}^\dagger$ are unique up to **noncanonical** isomorphism.

Extension to nonaffine schemes using global lifts

If X is smooth over k and admits a smooth lift $\mathfrak{X}$ over W, one can glue overconvergent lifts to define $H^i_{\mathrm{rig}}(X)$. One obtains functoriality between such lifts using the previous theorem plus a Čech spectral sequence.

For example, if X and $\mathfrak{X}$ are proper, then $H^i_{\mathrm{rig}}(X)$ is the de Rham cohomology of the rigid analytic space $\mathfrak{X}_K^{\mathrm{an}}$. This is $H^i_{\mathrm{dR}}(\mathfrak{X}_K)$ by rigid GAGA, but now it also has an action of Frobenius on X.

If a smooth lift is not available, one can still define $H^i_{\mathrm{rig}}(X)$ as limit Čech cohomology for covering families of local lifts (Berthelot, Arabia-Mebkhout), or by forming a suitable site (le Stum; see later). One can also define a more canonical lifting construction...

Overconvergent de Rham-Witt (Davis-Langer-Zink)

For X smooth, there is a canonical subcomplex $W^\dagger\Omega^\cdot_{X/k}$ (actually a Zariski and étale subsheaf) of the usual de Rham-Witt complex $W\Omega^\cdot_{X/k}$, such that $H^i(W^\dagger\Omega^\cdot_{X/k}[1/p]) = H^i_{\mathrm{rig}}(X)$.

E.g., $W^\dagger\Omega^0_{X/k} = W^\dagger(k[x_1,\dots,x_n])$ consists of $\sum_{i=0}^\infty p^i[y_i^{1/p^i}]$ with $y_i \in k[x_1,\dots,x_n]$, such that for some a,b,

$$p^{-i}\deg(y_i) \leq ai - b.$$

Problem

Does this construction have any K-theoretic significance?

Extension to nonsmooth schemes: motivation

Recall: to make algebraic de Rham cohomology for a scheme X of finite type over a field of characteristic zero, locally immerse $X \hookrightarrow Y$ with Y smooth, then locally define $H^i_{\mathrm{dR}}(X)$ as the de Rham cohomology of the formal completion of Y along X. (Use an infinitesimal site to globalize.) Similarly define cohomology $H^i_{Z,\mathrm{dR}}(X)$ for $Z \hookrightarrow X$ a closed immersion.

Extension to nonsmooth schemes: construction

Now say $X \to Y$ is an immersion of k-schemes, with Y smooth proper and admitting a smooth proper lift $\mathfrak{Y}$. Let $\overline{X}$ be the Zariski closure of X in Y.

Let $\mathrm{sp} : \mathfrak{Y}_K^{\mathrm{an}} \to Y$ be the specialization map. E.g., for $Y = \mathbb{P}^n_k, \mathfrak{Y} = \mathbb{P}^n_W$, given $[x_0 : \cdots : x_n] \in \mathfrak{Y}(\overline{K})$, normalize so that $\max\{|x_0|, \ldots, |x_n|\} = 1$; then

$$\mathrm{sp}([x_0 : \cdots : x_n]) = [\overline{x_0} : \cdots : \overline{x_n}].$$

Then $H^i_{\mathrm{rig}}(X)$ is the direct limit of de Rham cohomology of *strict neighborhoods* (a/k/a Berkovich neighborhoods, defined by strict inequalities) of $]X[= \mathrm{sp}^{-1}(X)$ (the *tube of* X) in $]\overline{X}[$. Similarly define $H^i_{Z,\mathrm{rig}}(X)$.

Cohomology with compact supports

For Poincaré duality, we also need cohomology with compact supports. We construct $H^i_{c,\mathrm{rig}}(X)$ as the hypercohomology of the complex

$$0 \to \Omega_{]\overline{X}[/K} \to i_* i^* \Omega_{]\overline{X}[/K} \to 0$$

for $i :]\overline{X} - X[\hookrightarrow \mathfrak{Y}_K^{\mathrm{an}}$ the inclusion. (This agrees with $H^i_{\mathrm{rig}}(X)$ if X is proper, as then $X = \overline{X}$.)

Problem

Give an overconvergent de Rham-Witt construction of the general $H^i_{Z,\mathrm{rig}}(X)$ and $H^i_{c,\mathrm{rig}}(X)$.

The overconvergent site (le Stum)

Consider triples $X \subset P \leftarrow V$, where X is immersed into a formal W-scheme P and $V \to P_K$ is a morphism of analytic spaces. Put $]X[_V = \lambda^{-1}(\mathrm{sp}^{-1}(X))$. A morphism

$$(X \subset P \leftarrow V) \to (X' \subseteq P' \leftarrow V')$$

consists of morphisms $X \to X', V \to V'$ which are compatible on $]X[_V$. We *do not* require a compatible map $P \to P'$!

The *overconvergent site* consists of these, after inverting *strict neighborhoods*:

$$(X \subset P \leftarrow W) \to (X \subset P \leftarrow V)$$

where W is a Berkovich neighborhood of $]X[_V$ in V. Coverings should be set-theoretic coverings of a Berkovich neighborhood of X in V. Cohomology of the structure sheaf computes $H^i_{\mathrm{rig}}(X)$.

Problem

Extend to $H^i_{Z,\mathrm{rig}}(X), H^i_{c,\mathrm{rig}}(X)$.

Contents

What we'll find

We construct "locally constant" coefficients admitting excision, Gysin maps, Künneth formula, Poincaré duality, pullbacks, finite étale pushforwards, generic higher direct images, Lefschetz trace formula for Frobenius, cohomological descent.

Problem (Berthelot)

Construct direct images under smooth proper morphisms (for overconvergent isocrystals; the convergent case is treated by Ogus).

Problem

Check the Lefschetz-Verdier trace formula (probably by reducing to log-crystalline cohomology using "semistable reduction").

Problem

Give a good logarithmic theory (for overconvergent isocrystals; the convergent case is treated by Shiho).

Convergent (F-)isocrystals

Suppose $X \to Y$ is an immersion, with Y smooth proper and admitting a smooth proper lift $\mathfrak{Y}$. Put $\mathscr{Y} = \mathfrak{Y}_K^{\mathrm{an}}$. A *convergent isocrystal* on X is a vector bundle on $]X[_{\mathscr{Y}}$ equipped with an integrable connection ∇, such that the formal Taylor isomorphism converges on all of $]X[_{\mathscr{Y}\times\mathscr{Y}} \subseteq]X[_{\mathscr{Y}} \times]X[_{\mathscr{Y}}$. This definition is functorial in X.

E.g., for $X = \mathbb{A}_k^n$, this is a finite free module M over $W\langle x_1, \dots, x_n\rangle_K$ such that

$$\sum_{i_1,\dots,i_n=0}^{\infty} \frac{\lambda_1^{i_1}\cdots\lambda_n^{i_n}}{i_1!\cdots i_n!} \left(\frac{\partial}{\partial x_1}\right)^{i_1} \cdots \left(\frac{\partial}{\partial x_n}\right)^{i_n} (m)$$

converges for any $m \in M$ and $\lambda_i \in \overline{K}$ with $|\lambda_i| < 1$.

Theorem (Berthelot, Ogus)

The category of convergent isocrystals is equivalent to the isogeny category of crystals of finite W-modules.

These have good cohomological properties only if X is smooth proper.

Overconvergent (F-)-isocrystals

An *overconvergent isocrystal* on X is a vector bundle on a strict neighborhood of $]X[_{\mathscr{Y}}$ in $]\overline{X}[_{\mathscr{Y}}$ equipped with an integrable connection ∇, such that the formal Taylor isomorphism converges on a strict neighborhood of $]X[_{\mathscr{Y}\times\mathscr{Y}}$ in $]\overline{X}[_{\mathscr{Y}} \times]\overline{X}[_{\mathscr{Y}}$. These can also be described using le Stum's overconvergent site (and probably also using overconvergent de Rham-Witt).

E.g., for $X = \mathbb{A}^n_k$, this is a finite free module M over $W\langle x_1, \dots, x_n\rangle^\dagger_K$ such that

$$\sum_{i_1,\dots,i_n=0}^{\infty} \frac{\lambda_1^{i_1}\cdots\lambda_n^{i_n}}{i_1!\cdots i_n!} \left(\frac{\partial}{\partial x_1}\right)^{i_1} \cdots \left(\frac{\partial}{\partial x_n}\right)^{i_n} (m)$$

converges *for the direct limit topology* for all $m \in M$ and $\lambda_i \in \overline{K}$ with $|\lambda_i| < 1$.

These have good cohomological properties if we assume also the existence of an isomorphism $F : \phi^*\mathscr{E} \to \mathscr{E}$, where ϕ is a power of absolute Frobenius. (This also forces the convergence of the Taylor isomorphism.)

Newton polygons

Over Spec(k), an F-isocrystal is a K-vector space equipped with an invertible semilinear F-action. For k algebraically closed, the *Newton polygon* is a complete isomorphism invariant (Dieudonné-Manin). That is, each F-isocrystal has the form $\oplus_{r/s} M_{r/s}^{e(r/s)}$, where

$$M_{r/s} = Ke_1 \oplus \cdots \oplus Ke_s, \qquad F(e_1) = e_2, \dots, F(e_{s-1}) = e_s, F(e_s) = p^r e_1;$$

the polygon has slope r/s on width $s \cdot e(r/s)$. For de Rham cohomology of a smooth proper W-scheme, this lies above the Hodge polygon (Mazur).

Theorem (Grothendieck, Katz, de Jong, Oort)

The Newton polygon is upper semicontinuous, i.e., it jumps up on a closed subscheme of X. Moreover, this closed subscheme is of pure codimension 1.

Problem (easy)

If the Newton polygon of X is constant, then X admits a filtration whose successive quotients have constant straight Newton polygons.

Unit-root objects and p-adic étale sheaves

On this slide, assume F is an action of *absolute* Frobenius. An F-isocrystal is *unit-root* if its Newton polygon at each point has slope 0.

Theorem (Katz, Crew)

The category of convergent unit-root F-isocrystals on X is equivalent to the category of lisse étale $\mathbb{Q}_p$-sheaves.

Theorem (Crew, Tsuzuki, KSK)

*The category of overconvergent unit-root F-isocrystals on X is equivalent to the category of **potentially unramified** lisse étale $\mathbb{Q}_p$-sheaves.*

There is no such simple description for non-unit-root objects, which makes life complicated. There is an analogue of the potentially unramified property ("semistable reduction").

Finiteness properties

Theorem (Crew, KSK)

For $\mathscr{E}$ an overconvergent F-isocrystal on X and $Z \hookrightarrow X$ a closed immersion, $H^i_{Z,\mathrm{rig}}(X,\mathscr{E}), H^i_{c,\mathrm{rig}}(X,\mathscr{E})$ are finite dimensional over K. Also, Poincaré duality and the Künneth formula hold. (These can be used for a p-adic proof of the Weil conjectures, using Fourier transforms after Deligne-Laumon.)

Theorem ("Semistable reduction theorem", KSK)

Let $\mathscr{E}$ be an overconvergent F-isocrystal on X. Then there exists an alteration (proper, dominant, generically finite) $f : Y \to X$ and a closed immersion $j : Y \to \overline{Y}$ such that $\overline{Y}_{\log} = (\overline{Y}, \overline{Y} - Y)$ is proper and log-smooth over k, and $f^*\mathscr{E}$ extends to a convergent F-log-isocrystal on $\overline{Y}_{\log}$.

The latter allows for reductions to log-crystalline cohomology. (Sibling theorem: de Rham representations are potentially semistable.)

Contents

What we'll find

A category containing overconvergent F-isocrystals, whose associated derived category admits direct/inverse image, exceptional direct/inverse image, tensor product, duality (the **six operations**). Also: local cohomology, dévissage in overconvergent F-isocrystals.

Arithmetic $\mathscr{D}$-modules

Over a field of characteristic zero, vector bundles carrying integrable connection can be viewed as coherent modules over a coherent (noncommutative) ring $\mathscr{D}$ of differential operators. This gives a good theory of constructible coefficients in algebraic de Rham cohomology, once one adds the condition of *holonomicity* to enforce finiteness.

Berthelot observed that one can imitate this theory working modulo p^n, up to a point. One does get a category of *arithmetic $\mathscr{D}$-modules* whose derived category carries cohomological operations (including duality *if* one adds Frobenius structure).

One sticking point is that the analogue of Bernstein's inequality fails. It can be salvaged using *Frobenius descent*, leading to a notion of *holonomic* arithmetic $\mathscr{D}$-modules, but it was unknown until recently whether the resulting derived category is stable under cohomological operations.

Overholonomicity (Caro)

Caro introduced the category of *overholonomic* (complexes of) F-$\mathscr{D}$-modules; roughly by design, these have bounded $\mathscr{D}$-coherent cohomology, and keep it under extraordinary inverse image and duality.

Theorem (Caro)

These are also stable under direct image, extraordinary direct image, inverse image. Moreover, any such complex has holonomic cohomology.

Theorem (Caro)

These admit dévissage in overconvergent F-isocrystals. (Such dévissable complexes are stable under tensor product.)

Theorem (Caro, Tsuzuki; uses "semistable reduction")

Conversely, any complex admitting dévissage in overconvergent F-isocrystals is overholonomic. Corollary: overholonomic complexes are stable under the six operations.

Overholonomicity and holonomicity (Caro)

Using these results, Caro has established stability of Berthelot's original notion of holonomicity over smooth projective varieties.

Theorem (Caro)

The category of complexes of arithmetic $\mathscr{D}$-modules over quasiprojective k-varieties with bounded, F-holonomic cohomology is stable under the six operations.

Contents

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More references available upon request!