A theory of base motives

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This is a sequel to and continuation of a talk at last summer's conference in Bonn honoring Haynes Miller [15]. I owe many mathematicians thanks for helpful conversations and encouragement, but want to single out John Rognes particularly, and thank him as well for organizing this wonderful conference.

§1 Prologue

The homotopy groups of the map

$$\mathrm{O}_n \to \mathrm{O} \to \lim_{n\to\infty} \Omega^{n-1} S^{n-1} := Q(S^0)$$

define a homomorphism

$$J : \pi_{k-1}\mathrm{O} = KO_k(*) \to \lim_{n\to\infty} \pi_{n+k-1}(S^n) = \pi_k^S(*)$$

on homotopy groups, which factors through

$$KO_{4k} = \mathbb{Z} \to \zeta(1-2k)\cdot \mathbb{Z}/\mathbb{Z} \subset \pi_{4k-1}^S(*)$$

(at least, away from two).

Alternately, a real vector bundle over S^{4k} defines a stable cofiber sequence

$$S^{4k-1} \xrightarrow{\alpha} S^0 \longrightarrow \mathrm{cof}\ \alpha \longrightarrow S^{4k} \cdots$$

and hence an extension

$$[0 \to KO(S^{4k}) \to \cdots \to KO(S^0) \to 0]$$

in

$$\mathrm{Ext}_{\mathrm{Adams}}(KO(S^0), KO(S^{4k}) \cong H^1_c(\hat{\mathbb{Z}}^\times, \hat{K}O(S^{4k}))\ ,$$

with $\psi^\alpha,\ \alpha \in \hat{\mathbb{Z}}^\times$ acting on $\hat{K}O(S^{2k})$ by $\psi^\alpha(\beta^k) = \alpha^k\beta^k$. This (essentially Galois) cohomology can be evaluated, via von Staudt's theorem, in terms of Bernoulli numbers.

Deligne and Goncharov [cf [17] for a more homotopy-theoretic account] have constructed an abelian tensor $\mathbb{Q}$-linear category MTM of **mixed Tate motives** over $\mathbb{Z}$, generated by objects $\mathbb{Q}(n)$ satisfying a (small, ie trivial when $* > 1$) Adams-style spectral sequence

$$\mathrm{Ext}^*_{\mathsf{MTM}}(\mathbb{Q}(0), \mathbb{Q}(n)) \Rightarrow K_{2n-*}(\mathbb{Z}) \otimes \mathbb{Q}$$

The groups on the right have rank one in degree $4k+1$, with generators corresponding (via Borel regulators) to $\zeta(1+2k)$.

These same zeta-values appear in differential topology [13], in the classification of smooth ('Euclidean') cell bundles over the $4k+2$-sphere: there these even and odd zeta-values can be seen as having a common origin, summarized by a diagram

$$\begin{tikzcd}
 & X \arrow[dl, dotted] \arrow[d] \arrow[dr, dashed] & \\
BO_n \arrow[r] \arrow[d] & B\mathrm{Diff}(E^n) \arrow[r] & B\mathrm{Diff}_c(\mathbb{R}^n) \arrow[d] \\
BQ(S^0) & & \Omega\mathrm{Wh}(*) \, .
\end{tikzcd}$$

The space $\mathrm{Wh}(*)$ on the bottom right is Waldhausen's smooth pseudoisotopy space, which appears in

$$K(\mathbb{S}) = A(*) = \mathbb{S} \vee \mathrm{Wh}(*) \, .$$

The two-dimensional difference in the cell vs. vector bundle story is accounted for by the factor B on the lower left, and Ω on the lower right. Odd zeta-values appear in both contexts because the natural map

$$K(\mathbb{Z}) \to K(\mathbb{S})$$

is a rational equivalence.

This suggests reformulating some of the ideas of differential topology in terms of a category of 'motives over $\mathbb{S}$' analogous to the arithmetic geometers' motives over $\mathbb{Z}$, replacing the algebraic K-spectrum of the integers by Waldhausen's A-theory, in the hope that these zeta- values will provide a trail of breadcrumbs leading us toward some general insights. §2 below recalls some machinery from homological algebra, regarding

$$K(\mathbb{S}) = A(*) = \mathbb{S} \vee \mathrm{Wh}(*) \to \mathbb{S}$$

and

$$\mathrm{TC}(\mathbb{S}) \sim \mathbb{S} \vee \Sigma \mathbb{C}P^{\infty}_{-1} \to \mathbb{S}$$

(mod completions) as analogs of local rings over $\mathbb{S}$, with the appropriate trace maps interpreted as quotients by maximal ideals. Note that the algebraic K-theory spectrum of $\mathbb{Z}$ lacks such an augmentation.

Tannakian formalism identifies MTM as a category of representations of a certain pro-affine $\mathbb{Q}$-groupscheme with free graded Lie algebra with conjectural relations to other areas of mathematics (eg algebras of multiple zeta-values and renormalization theory). In the context proposed here, a similar group

$$\mathrm{Spec}\ \mathbb{S} \wedge_A \mathbb{S}$$

appears as derived automorphisms of A. §3 proposes to define a kind of cycle map from arithmetic motives to their A-theoretic analogs, conjecturally identifying their motivic groups.

§2 Brave new local rings

2.1 I'll start with work on commutative local rings, eg $A \to k$ with maximal ideal I, with roots in the very beginnings [18] of homological algebra. Eventually A will be graded, or a DGA.

The functors

$$H_*(A, -) := \mathrm{Tor}^A_*(k, -)$$

and

$$H^*(A, -) := \mathrm{Ext}^*_A(k, -)$$

appear in Cartan-Eilenberg; the first is covariant, and the second is contravariant, in A. I'll be concerned mostly with

$$H_*(A, k) = \mathrm{Tor}^A_*(k, k) \text{ and}$$

$$H^*(A, k) = \mathrm{Ext}^*_A(k, k)\ .$$

Under reasonable finiteness conditions, these are dual k-vector spaces [the associativity sseqs [7 XVI §4] degenerate]: in fact they are dual Hopf algebras, with $H^*(A, k)$ being the universal enveloping algebra of a graded Lie algebra [2].

2.1.1 ex: If $A = \mathbb{Z}_p \to \mathbb{F}_p$ then

$$H_*(A,k) = E(Q_0)$$

is an exterior algebra on a degree one (Bockstein) element. If $A = k[\epsilon]/(\epsilon^2)$ then $H_*(A,k) = k[x]$ ($|x| = 2$) is the Hopf algebra of the additive group. This is just the first manifestation of some kind of generalized Koszul duality.

2.1.2 remarks: For local rings this homology is closely related to Hochschild theory [7 X §2], so it may also be related to recent work [3] on Hopf algebra structures on THH).

2.1.3 Proposition: The homological functor

$$M \mapsto H_*(M \otimes^L_{\mathbb{Z}_p} \mathbb{F}_p) := \overline{M} : D(\mathbb{Z}_p - \mathrm{Mod}) \to \mathbb{F}_p - \mathrm{Mod}$$

lifts to the category of $E(Q_0)$-comodules. There is a Bockstein spectral sequence

$$\mathrm{Ext}^*_{E(Q_0)-\mathrm{Comod}}(\overline{M}, \overline{N}) \Rightarrow \mathrm{Hom}_{D(\mathbb{Z}_p - \mathrm{Mod})}(M, N) \ .$$

2.1.4 Definition: $\mathbb{G}(A) := \mathrm{Spec}\ H_*(A,k)$ is an affine (super) k-groupscheme; its grading is encoded by an action of the multiplicative group

$$\mathbb{G}_m = \mathrm{Spec}\ k[\beta^{\pm 1}] \ ,$$

and $\tilde{\mathbb{G}}A) := \mathbb{G}(A) \rtimes \mathbb{G}_m$.

2.2.1 The Bockstein spectral sequence generalizes: if $M \in D(A - \mathrm{Mod})$, let

$$\overline{M} = H_*(M \otimes^L_A k) = H_*(M \otimes_A \mathbf{A}) \in (k - \mathrm{Mod}) \ ,$$

where $A \to \mathbf{A} \to k$ is a factorization of the quotient map through a cofibration and a weak equivalence (ie $\mathbf{A}$ is a resolution of k, eg

$$0 \longrightarrow \mathbb{Z}_p \xrightarrow{p} \mathbb{Z}_p \longrightarrow 0 \ ) \ .$$

Proposition: The functor $M \to \overline{M}$ lifts to a homological functor

$$D(A - \mathrm{Mod}) \to (\tilde{G}(A) - \mathrm{reps}) \ ,$$

and there is an 'ascent' sseq

$$\mathrm{Ext}^*_{\tilde{G}(A)-\mathrm{reps}}(\overline{M}, \overline{N}) \Rightarrow \mathrm{Hom}_{D(A-\mathrm{Mod})}(M, N)$$

of Adams (alt: Bockstein) type ...

The **Proof** is as in Adams' Chicago notes [1], replacing the map $\mathbb{S} \to \mathrm{MU}$ with $A \to k$: thus $MU_*(X)$ becomes an MU_*MU-comodule by taking homotopy groups of the composition

$$X \wedge \mathrm{MU} = X \wedge \mathbb{S} \wedge \mathrm{MU} \to X \wedge \mathrm{MU} \wedge \mathrm{MU}$$

$$= (X \wedge \mathrm{MU}) \wedge_{\mathrm{MU}} (\mathrm{MU} \wedge \mathrm{MU}) \;,$$

yielding

$$MU_*(X) \to MU_*(X) \otimes_{MU_*} MU_*MU \;.$$

In the present context the comodule structure map comes from taking the homology of the composition

$$M \otimes_A \mathbf{A} = M \otimes_A A \otimes_A \mathbf{A} \to M \otimes_A \mathbf{A} \otimes_A \mathbf{A}$$

$$= (M \otimes_A \mathbf{A}) \otimes_{\mathbf{A}} (\mathbf{A} \otimes_A \mathbf{A}) \;,$$

resulting in $\overline{M} \to \overline{M} \otimes_k H_*(A, k)$. $\square$

2.2.2 ex: The bar construction provides a cofibrant replacement for k, with underlying algebra

$$\oplus_{n \geq 0} \otimes^n I[1]$$

and a suitable differential. When $A = k \oplus I$ is a singular ($I^2 = 0$) extension, the differential is trivial, and $\mathrm{Ext}^*_A(k, k)$ is the universal enveloping algebra of the free Lie algebra on $I^*[1]$.

2.3.1 Examples closer to homotopy theory appear in recent work of Dwyer, Greenlees, and Iyengar. Suppose for example that X is a connected pointed space (eg with finitely many cells in each dimension), and let $X_+ = X \vee S^0$ be X with a disjoint basepoint appended. Its Spanier-Whitehead dual

$$X^D := \mathrm{Maps}_{\mathbb{S}}(\Sigma^\infty X_+, \mathbb{S})$$

is an E_∞ ring-spectrum, with augmentation $X^D \to \mathbb{S}$ given by the basepoint.

The Rothenberg-Steenrod construction [12 §4.22] then yields an equivalence

$$\mathrm{Hom}_{X^D-\mathrm{Mod}}(\mathbb{S}, \mathbb{S}) \sim \mathbb{S}[\Omega X]$$

of (A_∞, $\mathrm{co} - E_\infty$) Hopf algebra objects in the category of spectra.

If X is simply connected, there is a dual result with coefficients in the Eilenberg-MacLane spectrum $k = Hk$ of a field: then the 'double commutator'

$$\mathrm{Hom}_{k[\Omega X]-\mathrm{Mod}}(k,k) \sim C^*(X,k)$$

is homotopy equivalent to the (commutative) cochain algebra of X. [This puts the homotopy groups

$$\pi_* C^*(X,k) \cong H^{-*}(X,k)$$

in negative dimension.] This sharpens a classical [4] analogy between the homology of loopspaces and local rings.

The functor

$$M \mapsto \mathrm{Hom}_{X^D}(M,\mathbb{S}) : (X^D - \mathrm{Mod}) \to (\mathbb{S}[\Omega X] - \mathrm{Mod}) \ .$$

seems worth further investigation ...

2.3.2 ex: Suspensions are formal, so if $k = \mathbb{Q}$ and $X = \Sigma Y$ then

$$X^D \otimes \mathbb{Q} \sim H^{-*}(\Sigma Y, \mathbb{Q})$$

is a singular extension of $\mathbb{Q}$, so $\mathbb{Q}[\Omega\Sigma Y]$ is the universal enveloping algebra of the free Lie algebra on the graded dual of $\tilde{H}^{-*-1}(Y,\mathbb{Q})$[1].

Recent work of Baker and Richter [5] identifies the integral homology $H_*(\Omega\Sigma\mathbb{C}P^\infty)$ with the Hopf algebra of noncommutative symmetric functions: the universal enveloping algebra of a free graded Lie algebra. The dual Hopf algebra $H^*(\Omega\Sigma\mathbb{C}P^\infty)$ is the (commutative) algebra of quasi-symmetric functions.

2.3.3 The topological cyclic homology $\mathrm{TC}(\mathbb{S};p)$ of the sphere spectrum (at p) is an E_∞ ringspectrum, equivalent to the p-completion of $\mathbb{S} \vee \Sigma\mathbb{C}P^\infty_{-1}$ [14]; the subscript signifies a twisted desuspension of projective space by the Hopf line bundle.

From now on I'll be working over the rationals, eg with the graded algebra

$$\mathrm{TC}_{2n-1}(\mathbb{S};\mathbb{Q}_p) \cong \mathbb{Q}_p \oplus \mathbb{Q}_p\langle e_{2n-1}\rangle \ ,$$

$n \geq 0$ (with trivial multiplication).

2.4.1 The multiplication on a ring-spectrum A defines a composition

$$[X, A \wedge Y] \wedge [Y, A \wedge Z] \to [X, A \wedge Z]$$

(on morphism objects in spectra) by

$$X \to A \wedge Y \to A \wedge A \wedge Z \to A \wedge Z \ .$$

The map $X \to A \wedge X$ defines a functor from the category with $\mathbb{S}$-modules (eg X, Y) as objects, and

$$\mathrm{Corr}_A(X, Y) := [X, A \wedge Y]$$

as morphisms, to the category of A-modules, because

$$\mathrm{Corr}_A(X, Y) = [X, A \wedge Y] \to [X, [A, A \wedge Y]_A]$$
$$\cong [A \wedge X, A \wedge Y]_A \ .$$

Let $(A - \mathrm{Corr})$ be the triangulated subcategory of $(A - \mathrm{Mod})$ generated by the image of this construction. The augmentation of A defines a functor from $(A - \mathrm{Corr})$ to $\mathbb{S}$-modules which is the identity on objects, and is given on morphisms by

$$[X, A \wedge Y] \to [X, \mathbb{S} \wedge Y] = [X, Y] \ .$$

In fact it is the composition of this functor with rationalization that I want to use:

Corollary: This homological functor $(A - \mathrm{Mod}) \to (\mathbb{Q} - \mathrm{Mod})$ lifts to the category of $\tilde{\mathbb{G}}(A \otimes \mathbb{Q})$-representations, yielding a spectral sequence

$$\mathrm{Ext}^{*,*}_{\tilde{\mathbb{G}}(A \otimes \mathbb{Q})-\mathrm{reps}}(X, Y) \Rightarrow \mathrm{Corr}^*_A(X, Y) \ .$$

Proof: $(X \wedge A) \wedge_A \mathbb{S} = X \ldots \ \Box$

2.4.2 A free Lie algebra has cohomological dimension one, so when A is $\mathrm{TC}(\mathbb{S}; p)$ and X and Y are spheres, this spectral sequence degenerates to

$$\mathrm{Ext}^1_{\tilde{\mathbb{G}}(\mathrm{TC} \otimes \mathbb{Q}_p)}(S^{2n}_{\mathbb{Q}_p}, S^0_{\mathbb{Q}_p}) \cong \mathrm{TC}_{2n-1}(\mathbb{S}, \mathbb{Q}_p) \ ,$$

with left-hand side isomorphic to

$$H^{1,0}_{\mathrm{Lie}}(\mathfrak{F}(\widetilde{\mathrm{TC}}^*[1]), S^{-2n}_{\mathbb{Q}_p}) \cong \mathrm{Hom}^0(\widetilde{\mathrm{TC}}^*[1], S^{-2n}_{\mathbb{Q}_p}) \ ,$$

which is just the one-dimensional vector space

$$\mathbb{Q}_p \langle e_{2n-1}[1] \beta^{-n} \rangle \ .$$

Similarly, at a regular odd prime p,

$$\mathrm{Wh}(*)/\Sigma\mathrm{coker}\, J \cong \Sigma\mathbb{H}P^\infty$$

[11, 16], yielding a spectral sequence

$$\mathrm{Ext}^*_{\tilde{\mathbb{G}}(A\otimes\mathbb{Q})}(S^{2n}_{\mathbb{Q}}, S^0_{\mathbb{Q}}) \Rightarrow A_{2n-*}(*) \otimes \mathbb{Q}$$

with $A_{4k+1}(*) \otimes \mathbb{Q} \cong \mathbb{Q}\langle e_{4k+1}[1]\beta^{-2k-1}\rangle$.

§3 A proposed category of A-theoretic motives

3.1 A retractive space Z over X is a diagram $X \xleftarrow{r} Z \xleftarrow{s} X$ which composes to the identity 1_X: it's a space over X with a section. Z is said to be finitely dominated if some finite complex is retractive over it.

Waldhausen showed that finitely dominated retractive spaces over X form a category with weak equivalences and cofibrations, and that the K-theory spectrum $A(X)$ of this category can be identified with $K(\mathbb{S}[\Omega X])$.

More generally, Z is relatively retractive over X, with respect to a map $p : X \to Y$, if the homotopy fiber of $p \circ r$ over any $y \in Y$ is finitely dominated as a retractive space over the homotopy fiber of p above y. The category $\mathcal{R}(p)$ of such spaces is again closed under cofibrations and weak equivalences, with an associated K-theory spectrum $A(X \to Y)$.

Bruce Williams [19 §4] (using a formalism developed in algebraic geometry by Fulton and MacPherson) shows that this functor has a rich bivariant structure: compositions

$$A(X \to Y) \wedge A(Y \to Z) \to A(X \to Z) \ ,$$

good behavior under products, &c. It behaves especially well on fibrations; in particular, the spectra

$$\forall A(X, Y) := A(X \times Y \to X)$$

(defined by relatively retractive spaces Z over $X \times Y \to X$) admit good products

$$\forall A(X, Y) \wedge \forall A(Y, Z) \to \forall A(X, Z) \ .$$

Let $A-\mathsf{Corr}$ be the triangulated envelope [6] of the symmetric monoidal additive category with finite CW complexes X, Y as objects, and $\forall A_0(X, Y) = \pi_0 \forall A(X, Y)$ as morphisms.

Composition

$$A(X \times Y \to Y) \to [X, A(Y)] \to [X, A \wedge Y]$$

of the standard assembly map with a slightly less familiar relative co-assembly map [10 §5] defines a monoidal stabilization functor

$$(A - \mathsf{Corr}) \to (A - \mathrm{Corr})$$

analogous to inverting the Tate motive, or to the introduction of desuspension in classical homotopy theory. However, A-theory of spaces is a highly nonlinear functor, and might possess other interesting stabilizations.

3.2 The motivic constructions of Suslin and Voevodsky begin with a category with morphism groups $\mathsf{SmCorr}(V, W)$ of (roughly) sums of irreducible subvarieties Z of $V \times W$ which are finite with respect to the projection $V \times W \to V$, and surjective on components of V.

When V and W are defined over a number field (eg $\mathbb{Q}$), I **believe** such subvarieties define relatively retractive spaces $Z(\mathbb{C})$ with respect to $V(\mathbb{C}) \times W(\mathbb{C}) \to V(\mathbb{C})$, and thus a homomorphism

$$\mathsf{SmCorr}(V, W) \to \forall A_0(V(\mathbb{C}), W(\mathbb{C})) ,$$

and hence a functor

$$V \mapsto V(\mathbb{C}) : (\mathsf{SmCorr}) \to (A - \mathsf{Corr}) .$$

My hope is that this will lead to an identification of the motivic group for the category of mixed Tate motives with $\tilde{\mathbb{G}}(A \otimes \mathbb{Q})$. It seems at least possible that Spec $(\mathbb{S} \wedge_{\mathrm{TC}} \mathbb{S}) \otimes \mathbb{Q}$ is the larger motivic group seen in physics [8] by Connes and Marcolli.

3.3 I don't want to end this sketch without mentioning one last possibility. Dundas and Østvær have proposed a bivariant K-theory based on categories $\mathcal{E}(E, F)$ of suitably exact functors between the categories of (cell) modules over (associative) ring-spectra E and F.

These module categories are to be understood as categories with weak equivalences and cofibrations; the exact functors are to preserve these structures, and be additive in a certain sense. $\mathcal{E}(E, F)$ is again a Waldhausen category, which suggests that the category (Alg_A) with associative ring-spectra E, F as its objects, and

$$\mathrm{Alg}_A(E, F) := K(\mathcal{E}(E, F))$$

as morphisms, is an interesting analog of categories of noncommutative correspondences proposed by various research groups [Kontsevich, Connes ...]. It seems reasonable to expect that this category will naturally be be enriched over A.

A space W over $X \times Y$ defines an X^D-Y^D bimodule W^D, and

$$W \mapsto \mathrm{Hom}_{X^D-\mathrm{Mod}}(W^D, -)$$

is a natural candidate for an exact functor, and hence a map

$$\mathcal{R}(X \times Y \to X) \to \mathcal{E}(X^D, Y^D) \ .$$

If so, this might define another interesting stabilization of $A - \mathsf{Corr}$, related more closely to the Waldhausen K-theory of Spanier-Whitehead duals than to spherical group rings.

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