

Logarithmic Geometry

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Emphasis

- ▶ What it's for
- ▶ How it works
- ▶ What it looks like

Motivating problem: Compactification

Consider

$$S^* \xrightarrow{j} S \xleftarrow{i} Z$$

j an open immersion, i its complementary closed immersion.

For example: S^* a moduli space of “smooth” objects, inside some space S of “stable” objects, Z the “degenerate” locus.

Log structure is “magic powder” which when added to S “remembers S^* .”

Motivating problem: Degeneration

Study **morphisms**

$$\begin{array}{ccccc} X^* & \longrightarrow & X & \xleftarrow{i} & Y \\ \downarrow f^* & & \downarrow f & & \downarrow g \\ S^* & \xrightarrow{j} & S & \xleftarrow{i} & Z \end{array}$$

Here f^* is smooth but f and g are only **log smooth**.

The log structure allows f and even g to somehow “remember” f^* .

Benefits

- ▶ Log smooth maps can be understood locally, (but are still much more complicated than classically smooth maps). They are not always flat!
- ▶ Degenerations can be studied locally on the singular locus Z .
- ▶ Log geometry has natural cohomology theories:
 - ▶ Betti
 - ▶ De Rham
 - ▶ Crystalline
 - ▶ Etale

Roots and ingredients

- ▶ Toroidal embeddings and toric geometry
- ▶ Regular singular points of ODE's, log poles and differentials
- ▶ Degenerations of Hodge structures

Remark: A key difference between local toric geometry and local log geometry:

- ▶ toric geometry based on study of **cones** and **monoids**.
- ▶ log geometry based on study of **morphisms** of cones and monoids.

Founders:

Deligne, Faltings, Fontaine–Illusie, Kazuya Kato, Chikara Nakayama, many others

Some applications

- ▶ Compactifying moduli spaces: K3's, abelian varieties, curves, covering spaces
- ▶ Moduli and degenerations of Hodge structures
- ▶ Crystalline and étale cohomology in the presence of bad reduction— C_{st} conjecture
- ▶ Work of Gabber and others on resolution of singularities (uniformization)
- ▶ Work of Gross and Siebert on mirror symmetry

Example: open subschemes

$$X^* \xrightarrow{j} X \xleftarrow{i} Z \quad (j \text{ open, } i \text{ closed}).$$

Instead of the sheaf of **ideals**:

$$I_Z := \{a \in \mathcal{O}_X : i^*(a) = 0\} \subseteq \mathcal{O}_X$$

consider the sheaf of **submonoids**:

$$\mathcal{M}_{X^*/X} := \{a \in \mathcal{O}_X : j^*(a) \in \mathcal{O}_{X^*}^*\} \subseteq \mathcal{O}_X.$$

Log structure: $\alpha_{X^*/X} : \mathcal{M}_{X^*/X} \rightarrow \mathcal{O}_X$ (the inclusion mapping)

Examples:

- ▶ $\mathcal{M}_{X/X} = \mathcal{O}_X^*$, the **trivial log structure**
- ▶ $\mathcal{M}_{\emptyset/X} = \mathcal{O}_X$, the **empty log structure**.

Notes

- ▶ This is generally useless unless $\text{codim}(Z, X) = 1$.
- ▶ $\mathcal{M}_{X^*/X}$ is a sheaf of **faces** of \mathcal{O}_X , i.e., a sheaf \mathcal{F} of submonoids such that $fg \in \mathcal{F}$ implies f and $g \in \mathcal{F}$.
- ▶ There is an exact sequence:

$$0 \rightarrow \mathcal{O}_X^* \rightarrow \mathcal{M}_{X^*/X} \rightarrow \Gamma_Z(\text{Div}_X^-) \rightarrow 0.$$

Definition of log structures

Let (X, \mathcal{O}_X) be a locally ringed space (e.g. a scheme).

A **prelog structure** on X is a morphism of sheaves of (commutative) monoids

$$\alpha_X: \mathcal{M}_X \rightarrow \mathcal{O}_X.$$

It is a **log structure** if

$$\alpha^{-1}(\mathcal{O}_X^*) \rightarrow \mathcal{O}_X^*$$

is an isomorphism. (In this case $\mathcal{M}_X^* \cong \mathcal{O}_X^*$.)

A **log space** is a triple $(X, \mathcal{O}_X, \alpha_X)$, and a **morphism of log spaces** is a triple (f, f^\sharp, f^\flat) .

$$f: X \rightarrow Y, f^\sharp: f^{-1}(\mathcal{O}_Y) \rightarrow \mathcal{O}_X, f^\flat: f^{-1}(\mathcal{M}_Y) \rightarrow \mathcal{M}_X$$

Notation

Note: If $X := (X, \alpha_X)$ is a log scheme let $\underline{X} := (X, \alpha_{X/X})$. There is a canonical map of log spaces:

$$X \rightarrow \underline{X}.$$

A log scheme: $(X, \mathcal{O}_X, \alpha_X: \mathcal{M}_X \rightarrow \mathcal{O}_X)$; **Possibilities:**

- ▶ (X, \mathcal{M}_X) or $X(\mathcal{M}_X)$
- ▶ (X, α_X) or $X(\alpha_X)$
- ▶ X a log scheme, \underline{X} for the underlying scheme, or, almost equivalently, for X with the “trivial” log structure.

Pictures?

some ideas later

Example: monoid schemes and torus embeddings

E.g. The **log line**: A^1 , with the log structure coming from $G_m \rightarrow A^1$.

Terminology: A commutative monoid Q is:

integral if $Q \subseteq Q^{gp}$

fine if Q is integral and finitely generated

saturated if Q is integral and $nx \in Q$ implies $x \in Q$, for $x \in Q^{gp}, n \in \mathbf{N}$

toric if Q is fine and saturated and Q^{gp} is torsion free

sharp if $Q^* = 0$.

Notation: $\overline{Q} := Q/Q^*$.

Generalization: toric varieties

Assume Q is toric. Let

$\underline{A}_Q^* := \text{Spec } R[Q^{gp}]$: a group scheme (torus)

$\underline{A}_Q := \text{Spec } R[Q]$: a monoid scheme

$\underline{A}_Q :=$ the log scheme given by the open immersion $j: \underline{A}_Q^* \rightarrow \underline{A}_Q$.

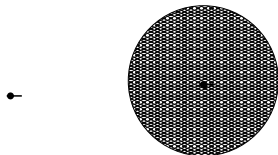
Pictures of Q :

$\text{Spec } Q$ is a finite topological space. Its points correspond to the orbits of the action of \underline{A}_Q^* on \underline{A}_Q , and to the faces of the cone C_Q spanned by Q . Embellish picture of a log scheme X by attaching $\text{Spec } \mathcal{M}_{X,x}$ to X at x .

Example: The log line ($Q = \mathbf{N}$, $C_Q = \mathbf{R}_{\geq}$)



$\text{Spec}(\mathbf{N})$

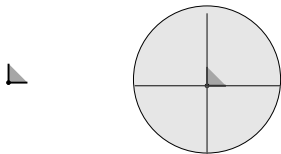


$\text{Spec}(\mathbf{N} \rightarrow \mathbf{C}[\mathbf{N}])$

Example: The log plane ($Q = \mathbf{N} \oplus \mathbf{N}$, $C_Q = \mathbf{R}_{\geq} \times \mathbf{R}_{\geq}$)



$\text{Spec}(\mathbf{N} \oplus \mathbf{N})$



$\text{Spec}(\mathbf{N} \oplus \mathbf{N} \rightarrow \mathbf{C}[\mathbf{N} \oplus \mathbf{N}])$

Log points

Standard log point

k a field, $t := \text{Spec } k$, Q a sharp monoid.

$$t_Q := k^* \oplus Q \rightarrow k \quad (u, q) \mapsto u$$

General log point:

If $\alpha_t: M_t \rightarrow k$ is a log structure on $t = \text{Spec } k$, one has:

$$1 \rightarrow k^* \rightarrow M_t \rightarrow \overline{M}_t \rightarrow 0$$

It splits (non-canonically) if $\text{Ext}^1(\overline{M}_t^{gp}, k^*) = 0$,
(for example, if \overline{M}_t^{gp} is torsion free, or if k is algebraically closed).

Example: log disks

V a discrete valuation ring, $K := \text{frac}(V)$, $m_V := \max(V)$,
 $k_V := V/m_V$, $\pi \in m_V$ uniformizer, $V' := V \setminus \{0\}$.

$T := \text{Spec } V$, $\tau := T^* := \text{Spec } K$, $t := \text{Spec } k$.

Log structures on T : $\Gamma(\alpha_T): \Gamma(T, M_T) \rightarrow \Gamma(T, \mathcal{O}_T)$:

trivial: $\alpha_{T/T} = V^* \rightarrow V$ (inclusion): T_{triv}

standard: $\alpha_{T^*/T} = V' \rightarrow V$ (inclusion): T_{std}

hollow: $\alpha_{hol} = V' \rightarrow V$ (inclusion on V^* , 0 on m_V): T_{hol}

split_m $\alpha_m = V^* \oplus \mathbf{N} \rightarrow V$ ($inc, 1 \mapsto \pi^m$): T_{spl_m}

Note: $T_{spl_1} \cong T_{std}$ and $T_{spl_m} \rightarrow T_{hol}$ as $m \rightarrow \infty$

Log disks restricting to log points

$$\begin{array}{ccc}
 t? & \longrightarrow & T? \\
 \downarrow & & \downarrow \\
 \underline{t} & \longrightarrow & \underline{T}
 \end{array}$$

$$\begin{array}{ccc}
 \tau? & \longrightarrow & T? \\
 \downarrow & & \downarrow \\
 \underline{\tau} & \longrightarrow & \underline{T}
 \end{array}$$

$$T_{triv} \times_{\underline{T}} \underline{t} = t_{triv} \quad , \quad T_{triv} \times_{\underline{T}} \underline{T} = \tau_{triv}$$

$$T_{std} \times_{\underline{T}} \underline{t} \cong t_{\mathbf{N}} \quad , \quad T_{std} \times_{\underline{T}} \underline{T} = \tau_{triv}$$

$$T_{hol} \times_{\underline{T}} \underline{t} \cong t_{\mathbf{N}} \quad , \quad T_{hol} \times_{\underline{T}} \underline{T} = \tau_{\mathbf{N}}$$

Charts

Let $\beta: Q \rightarrow \mathcal{M}$ be a morphism of sheaves of monoids. Form the **pushout** diagram:

$$\begin{array}{ccccc}
 \beta^{-1}(\mathcal{M}^*) & \longrightarrow & \mathcal{M}^* & & \\
 \downarrow & & \downarrow & \searrow & \\
 Q & \longrightarrow & Q_\beta & \xrightarrow{\tilde{\beta}} & \mathcal{M}
 \end{array}$$

Then $Q_\beta^* \cong \mathcal{M}^* \cong \tilde{\beta}^{-1}(\mathcal{M}^*)$.

β is a **chart** for \mathcal{M} if $\tilde{\beta}$ is an isomorphism.

$Q \rightarrow Q_\beta$ is always a chart for Q_β .

If $\beta: Q \rightarrow \mathcal{O}_X$ is a prelog structure, $Q_\beta \rightarrow \mathcal{O}_X$ is a log structure.

A sheaf of monoids or a log structure or scheme is said to be:

coherent if locally it admits a chart in which Q is a finitely generated (constant) monoid.

fine if it is coherent and integral

There is also a provision generalization: A sheaf of monoids \mathcal{F} is **relatively coherent** if locally there exist a coherent sheaf of monoids \mathcal{M} and a section f of \mathcal{M} such that \mathcal{F} is the sheaf of faces of \mathcal{M} generated by f .

- ▶ If Q is toric, the log structure \mathcal{M}_Q of A_Q is coherent, and $Q \rightarrow \mathcal{M}_Q$ is a chart.
- ▶ If $q \in Q$, and $U := D(q)$, $\mathcal{M}_{U/X}$ is relatively coherent.

Morphisms of monoids

Examples:

▶ $\mathbf{N} \rightarrow \mathbf{N} \oplus \mathbf{N}$

$$n \mapsto (n, n) \quad (\text{stable reduction})$$

▶ $\mathbf{N} \oplus \mathbf{N} \rightarrow \mathbf{N} \oplus \mathbf{N}$

$$(m, n) \mapsto (m, m + n) \quad (\text{blowup})$$

▶ $\mathbf{N} \rightarrow Q := \langle q_1, q_2, q_3, q_4 \rangle / (q_1 + q_2 = q_3 + q_4)$

$$n \mapsto nq_4 \quad (\text{not d-stable, appears in Dwork family})$$

Best handled by the relatively coherent log structure generated by q_4 .

Types of morphisms

A morphism $\theta: P \rightarrow Q$ of integral monoids is

strict if $\bar{\theta}: \bar{P} \rightarrow \bar{Q}$ is an isomorphism

local if $\theta^{-1}(Q^*) = P^*$

vertical if $Q/P := \text{Im}(Q \rightarrow \text{Cok}(\theta^{gp}))$ is a group.

exact if $P = (\theta^{gp})^{-1}(Q) \subseteq P^{gp}$

integral if every pushout $Q \oplus_P P'$ with P' integral is again integral.

saturated if P and Q are saturated and every pushout $Q \oplus_P P'$ with P' saturated is again saturated.

A morphism of log schemes $f: X \rightarrow Y$ has **P** if for every $x \in X$, the map $f^\flat: M_{Y, f(x)} \rightarrow M_{X, x}$ has **P**.

Note: Exactness + locality of $\text{Spec } \theta$ is stronger than exactness of θ , and is equivalent to locality + integrality of C_θ .

Fiber products

The category of coherent log schemes has fiber products.

$\mathcal{M}_{X \times_Z Y} \rightarrow \mathcal{O}_{X \times_Z Y}$ is the log structure associated to

$$p_X^{-1} \mathcal{M}_X \oplus_{p_Z^{-1} \mathcal{M}_Z} p_Y^{-1} \mathcal{M}_Y \rightarrow \mathcal{O}_{X \times_Z Y}.$$

Dangers

- ▶ $\mathcal{M}_{X \times_Z Y}$ may not be integral, so to get the fiber product in the category of fine log schemes we may have to pass to a closed subscheme.
- ▶ $\mathcal{M}_{X \times_Z Y}$ may not be saturated, so to get the fiber product in the category of fine saturated log schemes we may have to pass to a finite $W \rightarrow X \times_Z Y$.

Differentials

Let $f: X \rightarrow Y$ be a morphism of log schemes and let E be an \mathcal{O}_X -module.

$Der_{X/Y}(E) := \{(D, \delta) : D: \mathcal{O}_X \rightarrow E, \delta: \mathcal{M}_X \rightarrow E \text{ such that: } \}$

$$D(ab) = aD(b) + bD(a), D(a+b) = D(a) + D(b), a, b \in \mathcal{O}_X.$$

$$\delta(mn) = \delta(m) + \delta(n), m, n \in \mathcal{M}_X$$

$$D\alpha_X(m) = \alpha_X(m)\delta(m), m \in \mathcal{M}_X, \text{ so for } u \in \mathcal{O}_X^*, \delta(u) = d\log u$$

$$D(a) = 0, a \in f^{-1}(\mathcal{O}_Y)$$

$$D(m) = 0, m \in f^{-1}(\mathcal{M}_Y)$$

Universal derivation:

$$(d, \delta) : (\mathcal{O}_X, \mathcal{M}_X) \rightarrow \Omega_{X/Y}^1 \quad (\text{some write } \omega_{X/Y}^1)$$

Geometric construction:

Infinitesimal neighborhoods of diagonal $X \rightarrow X \times_Y X$ **made strict**:

$$X \rightarrow \mathcal{P}_{X/Y}^N, \quad \Omega_{X/Y}^1 = J/J^2.$$

$$d(a) = p_2^\sharp(a) - p_1^\sharp(a), \quad \delta(m) = u - 1, \quad \text{where } p_2^\flat(m) = up_1^\flat(m)$$

Example:

If $\alpha_X = \alpha_{X^*/X}$ where $Z := X \setminus X^*$ is a DNC relative to Y ,

$$\Omega_{X/Y}^1 = \Omega_{\underline{X}/\underline{Y}}^1(\log Z)$$

Geometric construction yields relation to deformation theory.

Smooth morphisms

The definition of smoothness follows Grothendieck's geometric idea. Let $f: X \rightarrow Y$ be a morphism of fine log schemes, locally of finite presentation. Consider diagrams:

$$\begin{array}{ccc}
 T & \overset{g}{\dashrightarrow} & X \\
 \downarrow i & & \downarrow f \\
 T' & \xrightarrow{h} & Y
 \end{array}$$

Here i is a **strict** nilpotent immersion. Then $f: X \rightarrow Y$ is

smooth if g always exists, locally on T ,

unramified if g is always unique,

étale if g always exists and is unique.

Examples: monoid schemes and tori

Let $\theta: P \rightarrow Q$ be a morphism of toric monoids. Over a base ring R , $A_\theta: A_Q \rightarrow A_P$ is

- ▶ smooth iff A_θ^* is smooth iff

$$R \otimes \text{Ker}(\theta^{gp}) = R \otimes \text{Cok}(\theta^{gp})_{tors} = 0$$

- ▶ unramified iff A_θ^* is unramified iff

$$R \otimes \text{Cok}(\theta^{gp}) = 0$$

- ▶ étale iff A_θ^* is étale iff

$$R \otimes \text{Cok}(\theta^{gp}) = R \otimes \text{Ker}(\theta^{gp}) = 0.$$

In general, smooth (resp. unramified, étale) maps look locally like these examples.

The space X_{log}

X/\mathbf{C} : (relatively) fine log scheme of finite type,

X_{an} : its associated log analytic space.

X_{log} : topological space, defined as follows:

Underlying set: the set of pairs (x, σ) , where $x \in X_{an}$ and

$$\begin{array}{ccc}
 \mathcal{O}_{X,x}^* & \xrightarrow{x^\sharp} & \mathbf{C}^* \\
 \downarrow & & \downarrow \text{arg} \\
 \mathcal{M}_{X,x} & \xrightarrow{\sigma} & \mathbf{S}^1
 \end{array}$$

commutes. Hence:

$$X_{log} \xrightarrow{\tau} X_{an} \longrightarrow X$$

Each $m \in \tau^{-1}M_X$ defines a function $\arg(m): X_{log} \rightarrow \mathbf{S}^1$.

X_{log} is given the weakest topology so that $\tau: X_{log} \rightarrow X_{an}$ and all $\arg(m)$ are continuous.

Get $\tau^{-1}\mathcal{M}_X^{gp} \xrightarrow{\arg} \underline{\mathbf{S}}^1$ extending \arg on $\tau^{-1}\mathcal{O}_X^*$.

We have $\exp: \mathbf{R}(1) \rightarrow \mathbf{S}^1$. Let $\mathcal{L}_X := \tau^{-1}M_X^{gp} \times_{\underline{\mathbf{S}}^1} \underline{\mathbf{R}}(1)$.

Get “exponential” sequence:

$$0 \rightarrow \mathbf{Z}(1) \rightarrow \mathcal{L}_X \rightarrow \tau^{-1}\mathcal{M}_X \rightarrow 0.$$

Define $\tau^{-1}\mathcal{O}_X \rightarrow \mathcal{L}_X : a \mapsto (\exp a, \text{Im}(a))$.

Construct universal sheaf of $\tau^{-1}\mathcal{O}_X$ -algebras \mathcal{O}_X^{log} containing \mathcal{L}_X

Compactification of open immersions

The map τ is an isomorphism over the set X^* where $\overline{\mathcal{M}} = 0$, so we get a diagram

$$\begin{array}{ccc}
 & & X_{log} \\
 & \nearrow^{j_{log}} & \downarrow \tau \\
 X_{an}^* & \xrightarrow{j} & X_{an}
 \end{array}$$

The map τ is proper, and for $x \in X$, $\tau^{-1}(x)$ is a torsor under $T_x := \text{Hom}(\overline{\mathcal{M}}_x^{gp}, \mathbf{S}^1)$ (a finite sum of compact tori).

We think of τ as a **relative compactification of j** .

Example: monoid schemes

$X = A_Q := \text{Spec}(Q \rightarrow \mathbf{C}[Q])$, with Q toric.

$$X_{log} = A_Q^{log} = R_Q \times T_Q,$$

where

$\underline{A}_Q(\mathbf{C}) = \{z: Q \rightarrow (\mathbf{C}, \cdot)\}$ (algebraic set)

$R_Q := \{r: Q \rightarrow (\mathbf{R}_{\geq}, \cdot)\}$ (semialgebraic set)

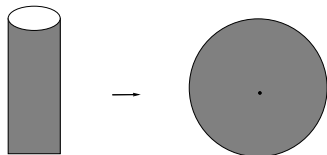
$T_Q := \{\zeta: Q \rightarrow (\mathbf{S}^1, \cdot)\}$ (compact torus)

$\tau: R_Q \times T_Q \rightarrow \underline{A}_Q(\mathbf{C})$ is multiplication: $z = r\zeta$.

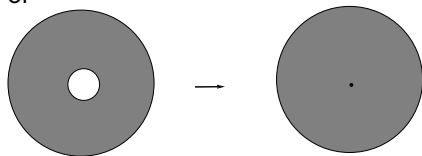
So A_Q^{log} means polar coordinates for \underline{A}_Q .

Example: log line, log point

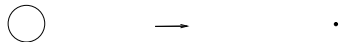
If $X = \mathbf{A}_N$, then $X_{log} = \mathbf{R}_{\geq} \times \mathbf{S}^1$.



or



(Real blowup)

If $X = P = x_{\mathbf{N}}$, $X_{log} = \mathbf{S}^1$.

Example: $\mathcal{O}_{P_{log}}^{log}$

$$\Gamma(P_{log}, \mathcal{O}_P^{log}) = \Gamma(\mathbf{S}_{log}^1, \mathcal{O}_P^{log}) = \mathbf{C}.$$

Pull back to universal cover $\exp : \mathbf{R}(1) \rightarrow \mathbf{S}^1$

$$\Gamma(\mathbf{R}(1), \exp^* \mathcal{O}_P^{log}) = \mathbf{C}[\theta],$$

generated by θ (identity map).

The log inertia group $I_P = \text{Aut}(\mathbf{R}(1)/\mathbf{S}^1) = \mathbf{Z}(1)$ acts, as the unique automorphism such that $\rho_\gamma(\theta) = \theta + \gamma$. In fact, if $N = d/d\theta$,

$$\rho_\gamma = e^{\gamma N}.$$

Compactification: The geometry of j_{log}

Theorem

If X/\mathbf{C} is (relatively) smooth, $j_{log}: X_{an}^* \rightarrow X_{log}$ is locally aspheric. In fact, $(X_{log}, X_{log} \setminus X_{an}^*)$ is a manifold with boundary.

Proof.

Reduce to the case $X = A_Q$. Reduce to (R_Q, R_Q^*) . Use the **moment map**:

$$(R_Q, R_Q^*) \cong (C_Q, C_Q^\circ) \quad : \quad r \mapsto \sum_{a \in A} r(a)a$$

where A is a finite set of generators of Q and C_Q is the real cone spanned by Q . □

Example: The log line



Degeneration: Submersivity of f_{log}

Theorem

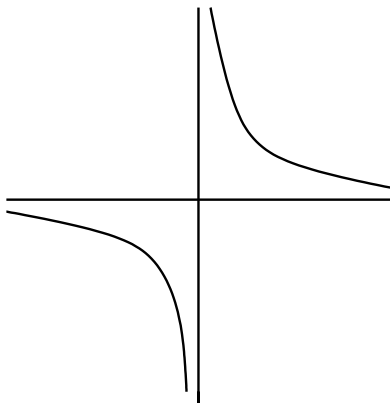
Let $f: X \rightarrow S$ be a (relatively) smooth exact morphism. Then $f_{log}: X_{log} \rightarrow S_{log}$ is a topological submersion, whose fibers are topological manifolds with boundary. The boundary corresponds to the set where f_{log} is not vertical.

Example

Semistable reduction $\mathbf{C} \times \mathbf{C} \rightarrow \mathbf{C} : (x_1, x_2) \mapsto x_1 x_2$

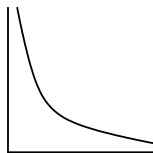
This is A_θ , where $\theta: \mathbf{N} \rightarrow \mathbf{N} \oplus \mathbf{N} : n \mapsto (n, n)$

Topology changes: (We just draw $\mathbf{R} \times \mathbf{R} \rightarrow \mathbf{R}$):

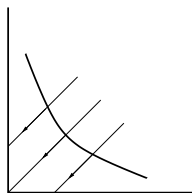


Log picture: $R_Q \times T_Q$

Just draw $R_Q \rightarrow R_N : \mathbf{R}_{\geq} \times \mathbf{R}_{\geq} \rightarrow \mathbf{R}_{\geq} : (x_1, x_2) \mapsto x_1 x_2$



Topology unchanged, and in fact is homeomorphic to projection mapping. Proof: (Key is *exactness* of f , *integrality* of C_θ .)



Consequence

Corollary

Let $f: X \rightarrow S$ be a (relatively) smooth, proper, and exact morphism of log schemes.

1. $f_{\log}: X_{\log} \rightarrow S_{\log}$ is a fiber bundle, and
2. $R^q f_{\log*}$ takes locally constant sheaves to locally constant sheaves.

Betti cohomology over \mathbf{C}

X/\mathbf{C} is a (relatively) coherent log scheme, X^* the open set where the log structure is trivial.

$$\begin{array}{ccc} H^*(X_{an}, \mathbf{Z}) & \xrightarrow{\tau^*} & H^*(X_{log}, \mathbf{Z}) \\ & \searrow & \downarrow j_{log}^* \\ & & H^*(X_{an}^*, \mathbf{Z}) \end{array}$$

Theorem

If X/\mathbf{C} is (relatively) smooth, j_{log}^* is an isomorphism.

Betti cohomology over log disks and points

$f: X \rightarrow S$ (relatively) smooth and exact, S a standard log disk.
 $P \rightarrow S$, log point. So $P_{log} = \mathbf{S}^1$. Let $Y := X \times_S P$.
Or: start with $g: Y \rightarrow P$, smooth and exact.

$$\begin{array}{ccc} Y_{log} & \xrightarrow{g_{log}} & \mathbf{S}^1 \\ \downarrow \tau & & \downarrow \tau \\ Y_{an} & \xrightarrow{g_{an}} & P_{an} \end{array}$$

Hence $Y_{log} \rightarrow Y_{an} \times \mathbf{S}^1$.

Monodromy

Pullback via $\exp: \mathbf{R}(1) \rightarrow \mathbf{S}^1$.

$$\begin{array}{ccccc}
 \tilde{Y}_{log} & \longrightarrow & Y_{an} \times \mathbf{R}(1) & \longrightarrow & \mathbf{R}(1) \\
 \downarrow \pi & \searrow \tilde{\tau} & \downarrow & & \downarrow \pi \\
 Y_{log} & \xrightarrow{\tau} & Y_{an} & \longrightarrow & P_{an}
 \end{array}$$

Theorem

$I_P := \text{Aut}(\mathbf{R}(1)/\mathbf{S}^1) = \mathbf{Z}(1)$ acts on \tilde{Y}_{\log} and hence on the (homotopy type of) the fibers, as well as on the complex of nearby cycles:

$$R\Psi := R\tilde{\tau}_*(\mathbf{Z}_{\tilde{Y}_{\log}}) \in D^+(Y_{an}, \mathbf{Z}[I_P]).$$

If Y/P is smooth and saturated, the action of I_P is *unipotent*.
In fact it is trivial on

$$\text{Gr}_{can}^{\cdot} \cong R^{\cdot}\tau_*\mathbf{Z} \cong \Lambda^{\cdot}\overline{\mathcal{M}}_Y^{gp}.$$

Connections and crystals

X/S (relatively) smooth map of log schemes.

Definition

A (log) connection on an \mathcal{O}_X -module E :

$$\nabla: E \rightarrow \Omega_{X/S}^1 \otimes E \quad \text{satisfying the Leibnitz rule}$$

Theorem (Riemann-Hilbert)

Let X/\mathbf{C} be (relatively) smooth. Then there is an equivalence of categories:

$$\begin{aligned} \text{MIC}_{\text{nil}}(X/\mathbf{C}) &\equiv L_{\text{un}}(X^{\text{log}}) \\ (E, \nabla) &\mapsto \text{Ker}(\tau^{-1}E \otimes \mathcal{O}_X^{\text{log}} \xrightarrow{\nabla} \tau^{-1}E \otimes \Omega_X^{\text{log}}) \end{aligned}$$

Example: $X := P$ (Standard log point)

$$\Omega_{P/\mathbf{C}}^1 \cong \mathbf{N} \otimes \mathbf{C} \cong \mathbf{C}, \text{ so}$$

$$MIC(P/\mathbf{C}) \equiv \{(E, N) : \text{vector space with endomorphism}\}$$

$P_{log} = \mathbf{S}^1$, so $L(P_{log})$ is cat of reps of the log inertia group
 $I_P = \mathbf{Z}(1)$. Thus:

$$L(P_{log}) \equiv \{(V, \rho) : \text{vector space with automorphism}\}$$

Conclusion:

$$\{(E, N) : N \text{ is nilpotent}\} \equiv \{(V, \rho) : \rho \text{ is unipotent}\}$$

Use $\mathcal{O}_P^{log} = \mathbf{C}[\theta]$:

$$(V, \rho) = \text{Ker}(\tau^* E \otimes \mathbf{C}[\theta] \rightarrow \tau^* E \otimes \mathbf{C}[\theta])$$

$$N \mapsto e^{2\pi i N}$$

De Rham cohomology over \mathbf{C}

X/\mathbf{C} (relatively) smooth log scheme.

$$H_{DR}(X) := H^*(X, \Omega_{X/\mathbf{C}}), \quad H_{DR}(X_{log}) := H^*(X_{log}, \Omega_{X/\mathbf{C}}^{log})$$

Theorem: There is a commutative diagram of isomorphisms:

$$\begin{array}{ccccc}
 H_{DR}(X) & \longrightarrow & H_{DR}(X^*) & & \\
 \downarrow & & \downarrow & \searrow & \\
 H_{DR}(X_{an}) & \longrightarrow & H_{DR}(X_{log}) & \longrightarrow & H_{DR}(X^*) \\
 & & \downarrow & & \downarrow \\
 & & H_B(X_{log}, \mathbf{C}) & \longrightarrow & H_B(X_{an}^*, \mathbf{C})
 \end{array}$$

De Rham cohomology over log disks and points

$f: X \rightarrow S$ (relatively) smooth and exact, $V := \mathbf{C}\{t\}$,
 $S := \text{Spec}(V' \rightarrow V)$, $P \rightarrow S$, log point. Let $Y := X \times_S P$.
Or: start with $g: Y \rightarrow P$, smooth and exact. $P_{\log} = \mathbf{S}^1$, so:

$$\begin{array}{ccc} Y_{\log} & \xrightarrow{g_{\log}} & \mathbf{S}^1 \\ \downarrow \tau & & \downarrow \tau \\ Y_{an} & \xrightarrow{g_{an}} & P_{an} \end{array}$$

Theorem

Assume X/S is saturated. Then $H_{DR}(X/S)$ is free over V , and $H_{DR}(Y/P) \cong \mathbf{C} \otimes_V H_{DR}(X/S)$. Gauss-Manin connection gives:

$$H_{DR}(Y/P) \rightarrow \Omega_{P/\mathbf{C}}^1 \otimes H_{DR}(Y/P).$$

$$N: H_{DR}(Y/P) \rightarrow H_{DR}(Y/P)$$

N is nilpotent, and corresponds to the (unipotent) monodromy action of I_P on $Rf_{log^*}(\mathbf{Z})$ (Betti cohomology).

Proof uses the log Poincaré lemma $\mathbf{C} \cong \Omega_{Y/S}^{log}$, and the map τ (to prove unipotency of monodromy).

The Steenbrink complex

The Steenbrink complex is an explicit representative of $R\Psi g$:

$$\Psi^\cdot := \mathcal{O}_P^{\log} \rightarrow \mathcal{O}_P^{\log} \otimes \Omega_{Y/S}^1 \otimes \cdots$$

Theorem

There is an isomorphism in the filtered derived category:

$$(R\Psi g, \text{Dec } T) \sim (\Psi^\cdot, \text{Dec } F).$$

Here T is the trivial filtration on $R\Psi g$ and F is the filtration given by the nilpotent endomorphism of N on $\mathcal{O}_P^{\log} = \mathbf{C}[\theta]$.

Corollary

There is a commutative diagram

$$\begin{array}{ccc}
 \mathrm{Gr}_{\mathrm{can}}^q R\Psi g & \longrightarrow & \Lambda^q \overline{\mathcal{M}}_Y^{\mathrm{gp}}[-q] \\
 \downarrow N & & \downarrow \cup \kappa \\
 \mathrm{Gr}_{\mathrm{can}}^{q-1} R\Psi g & \longrightarrow & \Lambda^{q-1} \overline{\mathcal{M}}_Y^{\mathrm{gp}}[1-q]
 \end{array}$$

where

$$\kappa: \overline{\mathcal{M}}_{Y/P} \rightarrow \mathbf{Z}[1]$$

is the (Kodaira-Spencer) map coming from the exact sequence

$$0 \rightarrow \mathbf{Z} \rightarrow \overline{\mathcal{M}}_Y^{\mathrm{gp}} \rightarrow \overline{\mathcal{M}}_{Y/P}^{\mathrm{gp}} \rightarrow 0$$

Conclusion

- ▶ Log geometry provides a uniform geometric perspective to treat compactification and degeneration problems in topology and in algebraic and arithmetic geometry.
- ▶ Log geometry incorporates many classical tools and techniques.
- ▶ Log geometry is not a revolution.
- ▶ Log geometry presents new problems and perspectives, both in fundamentals and in applications.