Logarithmic Geometry

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Outline II

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Emphasis

- What it's for
- How it works
- What it looks like

Motivating problem: Compactification

Consider

$$S^* \xrightarrow{j} S \xleftarrow{i} Z$$

j an open immersion, i its complementary closed immersion.
For example: S^* a moduli space of "smooth" objects, inside some space S of "stable" objects, Z the "degenerate" locus.

Log structure is "magic powder" which when added to S "remembers S^*."

Motivating problem: Degeneration

Study morphisms

$$\begin{array}{ccccc} X^* & \longrightarrow & X & \xleftarrow{\;i\;} & Y \\ \downarrow{\scriptstyle f^*} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} \\ S^* & \xrightarrow{\;j\;} & S & \xleftarrow{\;i\;} & Z \end{array}$$

Here f^* is smooth but f and g are only log smooth.
The log structure allows f and even g to somehow “remember” f^*.

Benefits

- Log smooth maps can be understood locally, (but are still much more complicated than classically smooth maps). They are not always flat!
- Degenerations can be studied locally on the singular locus Z.
- Log geometry has natural cohomology theories:
  - Betti
  - De Rham
  - Crystalline
  - Etale

Roots and ingredients

- Toroidal embeddings and toric geometry
- Regular singular points of ODE's, log poles and differentials
- Degenerations of Hodge structures

Remark: A key difference between local toric geometry and local log geometry:

- toric geometry based on study of cones and monoids.
- log geometry based on study of morphisms of cones and monoids.

Founders:
Deligne, Faltings, Fontaine–Illusie, Kazuya Kato, Chikara Nakayama, many others

Some applications

- Compactifying moduli spaces: K3's, abelian varieties, curves, covering spaces
- Moduli and degenerations of Hodge structures
- Crystalline and étale cohomology in the presence of bad reduction—C_{st} conjecture
- Work of Gabber and others on resolution of singularities (uniformization)
- Work of Gross and Siebert on mirror symmetry

Example: open subschemes

$$X^* \xrightarrow{j} X \xleftarrow{i} Z \qquad (j \text{ open, } i \text{ closed}).$$

Instead of the sheaf of ideals:

$$I_Z := \{a \in \mathcal{O}_X : i^*(a) = 0\} \subseteq \mathcal{O}_X$$

consider the sheaf of submonoids:

$$\mathcal{M}_{X^*/X} := \{a \in \mathcal{O}_X : j^*(a) \in \mathcal{O}^*_{X^*}\} \subseteq \mathcal{O}_X.$$

Log structure: $\alpha_{X^*/X} \colon \mathcal{M}_{X^*/X} \to \mathcal{O}_X$ (the inclusion mapping)
Examples:

- $\mathcal{M}_{X/X} = \mathcal{O}^*_X$, the trivial log structure
- $\mathcal{M}_{\emptyset/X} = \mathcal{O}_X$, the empty log structure .

Notes

- This is generally useless unless codim $(Z, X) = 1$.
- $\mathcal{M}_{X^*/X}$ is a sheaf of faces of $\mathcal{O}_X$, i.e., a sheaf $\mathcal{F}$ of submonoids such that $fg \in \mathcal{F}$ implies f and $g \in \mathcal{F}$.
- There is an exact sequence:

$$0 \to \mathcal{O}_X^* \to \mathcal{M}_{X^*/X} \to \underline{\Gamma}_Z(Div_X^-) \to 0.$$

Definition of log structures

Let $(X, \mathcal{O}_X)$ be a locally ringed space (e.g. a scheme).
A prelog structure on X is a morphism of sheaves of (commutative) monoids

$$\alpha_X \colon \mathcal{M}_X \to \mathcal{O}_X.$$

It is a log structure if

$$\alpha^{-1}(\mathcal{O}_X^*) \to \mathcal{O}_X^*$$

is an isomorphism. (In this case $\mathcal{M}_X^* \cong \mathcal{O}_X^*$.)

A log space is a triple $(X, \mathcal{O}_X, \alpha_X)$, and a morphism of log spaces is a triple $(f, f^\sharp, f^\flat)$.

$$f \colon X \to Y, f^\sharp \colon f^{-1}(\mathcal{O}_Y) \to \mathcal{O}_X, f^\flat \colon f^{-1}(\mathcal{M}_Y) \to \mathcal{M}_X$$

Notation

Note: If $X := (X, \alpha_X)$ is a log scheme let $\underline{X} := (X, \alpha_{X/X})$. There is a canonical map of log spaces:

$$X \to \underline{X}.$$

A log scheme: $(X, \mathcal{O}_X, \alpha_X \colon \mathcal{M}_X \to \mathcal{O}_X)$; Possibilities:

- $(X, \mathcal{M}_X)$ or $X(\mathcal{M}_X)$
- (X, α_X) or $X(\alpha_X)$
- X a log scheme, $\underline{X}$ for the underlying scheme, or, almost equivalently, for X with the "trivial" log structure.

Pictures?

some ideas later

Example: monoid schemes and torus embeddings

E.g. The log line: A^1, with the log structure coming from $\mathrm{G_m} \to A^1$.

Terminology: A commutative monoid Q is:

- **integral** if $Q \subseteq Q^{gp}$
- **fine** if Q is integral and finitely generated
- **saturated** if Q is integral and $nx \in Q$ implies $x \in Q$, for $x \in Q^{gp}, n \in \mathbf{N}$
- **toric** if Q is fine and saturated and Q^{gp} is torsion free
- **sharp** if $Q^* = 0$.

Notation: $\overline{Q} := Q/Q^*$.

Generalization: toric varieties

Assume Q is toric. Let

$\underline{\mathsf{A}}_{\mathsf{Q}}^{*} := \operatorname{Spec} R[Q^{gp}]$: a group scheme (torus)

$\underline{\mathsf{A}}_{\mathsf{Q}} := \operatorname{Spec} R[Q]$: a monoid scheme

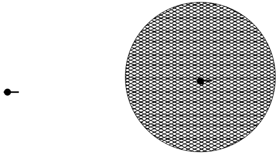
$\mathsf{A}_{\mathsf{Q}} :=$ the log scheme given by the open immersion $j\colon \underline{\mathsf{A}}_{\mathsf{Q}}^{*} \to \underline{\mathsf{A}}_{\mathsf{Q}}$.

Pictures of Q:

$\operatorname{Spec} Q$ is a finite topological space. Its points correspond to the orbits of the action of $\underline{\mathsf{A}}_{\mathsf{Q}}^{*}$ on $\underline{\mathsf{A}}_{\mathsf{Q}}$, and to the faces of the cone C_Q spanned by Q. Embellish picture of a log scheme X by attaching $\operatorname{Spec} \mathcal{M}_{X,x}$ to X at x.

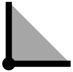
Example: The log line ($Q = \mathbf{N}$, $C_Q = \mathbf{R}_{\geq}$)

$\mathrm{Spec}(\mathbf{N})$

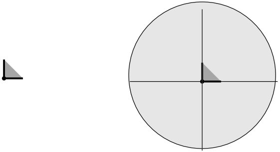


$\mathrm{Spec}(\mathbf{N} \to \mathbf{C}[\mathbf{N}])$

Example: The log plane ($Q = \mathbf{N} \oplus \mathbf{N}$, $C_Q = \mathbf{R}_{\geq} \times \mathbf{R}_{\geq}$)



$\mathrm{Spec}(\mathbf{N} \oplus \mathbf{N})$



$\mathrm{Spec}(\mathbf{N} \oplus \mathbf{N} \to \mathbf{C}[\mathbf{N} \oplus \mathbf{N}])$

Log points

Standard log point

k a field, $t := \operatorname{Spec} k$, Q a sharp monoid.

$$t_Q := k^* \oplus Q \to k \quad (u, q) \mapsto u$$

General log point:

If $\alpha_t \colon M_t \to k$ is a log structure on $t = \operatorname{Spec} k$, one has:

$$1 \to k^* \to M_t \to \overline{M}_t \to 0$$

It splits (non-canonically) if $Ext^1(\overline{M}_t^{gp}, k^*) = 0$,
(for example, if $\overline{M}_t^{gp}$ is torsion free, or if k is algebraically closed).

Example: log disks

V a discrete valuation ring, $K := frac(V)$, $m_V := max(V)$, $k_V := V/m_V$, $\pi \in m_V$ uniformizer, $V' := V \setminus \{0\}$.

$T := \mathrm{Spec}\, V$, $\tau := T^* := \mathrm{Spec}\, K$, $t := \mathrm{Spec}\, k$.

Log structures on T: $\Gamma(\alpha_T)\colon \Gamma(T, M_T) \to \Gamma(T, \mathcal{O}_T)$:

trivial: $\alpha_{T/T} = V^* \to V$ (inclusion): T_{triv}

standard: $\alpha_{T^*/T} = V' \to V$ (inclusion): T_{std}

hollow: $\alpha_{hol} = V' \to V$ (inclusion on V^*, 0 on m_V): T_{hol}

split$_m$ $\alpha_m = V^* \oplus \mathbf{N} \to V \quad (inc, 1 \mapsto \pi^m)$: T_{spl_m}

Note: $T_{spl_1} \cong T_{std}$ and $T_{spl_m} \to T_{hol}$ as $m \to \infty$

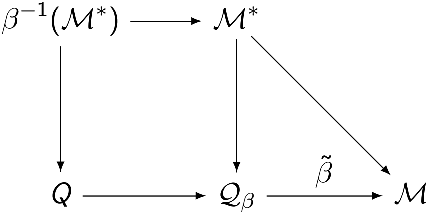
Log disks restricting to log points

$$\begin{array}{ccc} t_? & \longrightarrow & T_? \\ \downarrow & & \downarrow \\ \underline{t} & \longrightarrow & \underline{T} \end{array} \qquad\qquad \begin{array}{ccc} \tau_? & \longrightarrow & T_? \\ \downarrow & & \downarrow \\ \underline{\tau} & \longrightarrow & \underline{T} \end{array}$$

$$\begin{array}{rcl} T_{triv} \times_{\underline{T}} \underline{t} = t_{triv} & , & T_{triv} \times_{\underline{T}} \underline{\tau} = \tau_{triv} \\ T_{std} \times_{\underline{T}} \underline{t} \cong t_{\mathbf{N}} & , & T_{std} \times_{\underline{T}} \underline{\tau} = \tau_{triv} \\ T_{hol} \times_{\underline{T}} \underline{t} \cong t_{\mathbf{N}} & , & T_{hol} \times_{\underline{T}} \underline{\tau} = \tau_{\mathbf{N}} \end{array}$$

Charts

Let $\beta\colon Q \to \mathcal{M}$ be a morphism of sheaves of monoids. Form the pushout diagram:



Then $\mathcal{Q}_\beta^* \cong \mathcal{M}^* \cong \tilde{\beta}^{-1}(\mathcal{M}^*)$.

β is a chart for $\mathcal{M}$ if $\tilde{\beta}$ is an isomorphism.

$Q \to \mathcal{Q}_\beta$ is always a chart for $\mathcal{Q}_\beta$.

If $\beta\colon Q \to \mathcal{O}_X$ is a prelog structure, $\mathcal{Q}_\beta \to \mathcal{O}_X$ is a log structure.

A sheaf of monoids or a log structure or scheme is said to be:

coherent: if locally it admits a chart in which Q is a finitely generated (constant) monoid.

fine: if it is coherent and integral

There is also a provision generalization: A sheaf of monoids $\mathcal{F}$ is relatively coherent if locally there exist a coherent sheaf of monoids $\mathcal{M}$ and a section f of $\mathcal{M}$ such that $\mathcal{F}$ is the sheaf of faces of $\mathcal{M}$ generated by f.

- If Q is toric, the log structure $\mathcal{M}_Q$ of A_Q is coherent, and $Q \to \mathcal{M}_Q$ is a chart.
- If $q \in Q$, and $U := D(q)$, $\mathcal{M}_{U/X}$ is relatively coherent.

Morphisms of monoids

Examples:

- $\mathbf{N} \to \mathbf{N} \oplus \mathbf{N}$
  $n \mapsto (n, n)$ (stable reduction)
- $\mathbf{N} \oplus \mathbf{N} \to \mathbf{N} \oplus \mathbf{N}$
  $(m, n) \mapsto (m, m + n)$ (blowup)
- $\mathbf{N} \to Q := \langle q_1, q_2, q_3, q_4 \rangle / (q_1 + q_2 = q_3 + q_4)$
  $n \mapsto nq_4$ (not d-stable, appears in Dwork family)
  Best handled by the relatively coherent log structure generated by q_4.

Types of morphisms

A morphism $\theta \colon P \to Q$ of integral monoids is

- **strict** if $\overline{\theta} \colon \overline{P} \to \overline{Q}$ is an isomorphism
- **local** if $\theta^{-1}(Q^*) = P^*$
- **vertical** if $Q/P := Im(Q \to Cok(\theta^{gp}))$ is a group.
- **exact** if $P = (\theta^{gp})^{-1}(Q) \subseteq P^{gp}$
- **integral** if every pushout $Q \oplus_P P'$ with P' integral is again integral.
- **saturated** if P and Q are saturated and every pushout $Q \oplus_P P'$ with P' saturated is again saturated.

A morphism of log schemes $f \colon X \to Y$ has **P** if for every $x \in X$, the map $f^\flat \colon M_{Y,f(x)} \to M_{X,x}$ has **P**.

Note: Exactness + locality of $\mathrm{Spec}\,\theta$ is stronger than exactness of θ, and is equivalent to locality + integrality of C_θ.

Fiber products

The category of coherent log schemes has fiber products. $\mathcal{M}_{X\times_Z Y} \to \mathcal{O}_{X\times_Z Y}$ is the log structure associated to

$$p_X^{-1}\mathcal{M}_X \oplus_{p_Z^{-1}\mathcal{M}_Z} p_Y^{-1}\mathcal{M}_Y \to \mathcal{O}_{X\times_Z Y}.$$

Dangers

- $\mathcal{M}_{X\times_Z Y}$ may not be integral, so to get the fiber product in the category of fine log schemes we may have to pass to a closed subscheme.
- $\mathcal{M}_{X\times_Z Y}$ may not be saturated, so to get the fiber product in the category of fine saturated log schemes we may have to pass to a finite $W \to X \times_Z Y$.

Differentials

Let $f\colon X \to Y$ be a morphism of log schemes and let E be an $\mathcal{O}_X$-module.
$Der_{X/Y}(E) := \{(D, \delta) : D\colon \mathcal{O}_X \to E, \delta\colon \mathcal{M}_X \to E \text{ such that: } \}$

$D(ab) = aD(b) + bD(a), D(a+b) = D(a) + D(b), a, b \in \mathcal{O}_X.$

$\delta(mn) = \delta(m) + \delta(n), m, n \in \mathcal{M}_X$

$D\alpha_X(m) = \alpha_X(m)\delta(m), m \in \mathcal{M}_X$, so for $u \in \mathcal{O}_X^*$, $\delta(u) = d\log u$

$D(a) = 0, a \in f^{-1}(\mathcal{O}_Y)$

$D(m) = 0, m \in f^{-1}(\mathcal{M}_Y)$

Universal derivation:

$$(d, \delta) : (\mathcal{O}_X, \mathcal{M}_X) \to \Omega^1_{X/Y} \quad \text{(some write } \omega^1_{X/Y}\text{)}$$

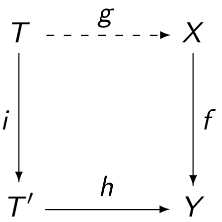
Geometric construction:
Infinitesimal neighborhoods of diagonal $X \to X \times_Y X$ made strict:
$X \to \mathcal{P}^N_{X/Y}$, $\Omega^1_{X/Y} = J/J^2$.
$d(a) = p_2^\sharp(a) - p_1^\sharp(a)$, $\delta(m) = u - 1$, where $p_2^\flat(m) = up_1^\flat(m)$
Example:
If $\alpha_X = \alpha_{X^*/X}$ where $Z := X \setminus X^*$ is a DNC relative to Y,

$$\Omega^1_{X/Y} = \Omega^1_{\underline{X}/\underline{Y}}(\log Z)$$

Geometric construction yields relation to deformation theory.

Smooth morphisms

The definition of smoothness follows Grothendieck's geometric idea. Let $f\colon X \to Y$ be a morphism of fine log schemes, locally of finite presentation. Consider diagrams:



Here i is a strict nilpotent immersion. Then $f\colon X \to Y$ is

smooth if g always exists, locally on T,

unramified if g is always unique,

étale if g always exists and is unique.

Examples: monoid schemes and tori

Let $\theta\colon P \to Q$ be a morphism of toric monoids. Over a base ring R, $A_\theta\colon A_Q \to A_P$ is

- smooth iff A^*_θ is smooth iff

$$R \otimes \mathrm{Ker}(\theta^{gp}) = R \otimes \mathrm{Cok}(\theta^{gp})_{tors} = 0$$

- unramified iff A^*_θ is unramified iff

$$R \otimes \mathrm{Cok}(\theta^{gp}) = 0$$

- étale iff A^*_θ is étale iff

$$R \otimes \mathrm{Cok}(\theta^{gp}) = R \otimes Ker(\theta^{gp}) = 0.$$

In general, smooth (resp. unramified, étale) maps look locally like these examples.

The space X_{log}

$X/\mathbf{C}$: (relatively) fine log scheme of finite type,

X_{an} : its associated log analytic space.

X_{log}: topological space, defined as follows:

Underlying set: the set of pairs (x, σ), where $x \in X_{an}$ and

$$\begin{array}{ccc} \mathcal{O}^*_{X,x} & \xrightarrow{x^\sharp} & \mathbf{C}^* \\ \downarrow & & \downarrow \text{arg} \\ \mathcal{M}_{X,x} & \xrightarrow{\sigma} & \mathbf{S}^1 \end{array}$$

commutes. Hence:

$$X_{log} \xrightarrow{\tau} X_{an} \longrightarrow X$$

Each $m \in \tau^{-1} M_X$ defines a function $\arg(m) \colon X_{log} \to \mathbf{S}^1$.
X_{log} is given the weakest topology so that $\tau \colon X_{log} \to X_{an}$ and all $\arg(m)$ are continuous.

Get $\tau^{-1}\mathcal{M}_X^{gp} \xrightarrow{\arg} \underline{\mathbf{S}}^1$ extending arg on $\tau^{-1}\mathcal{O}_X^*$.

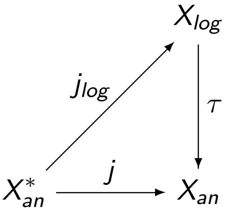
We have $\exp \colon \mathbf{R}(1) \to \mathbf{S}^1$. Let $\mathcal{L}_X := \tau^{-1} M_X^{gp} \times_{\underline{\mathbf{S}}^1} \underline{\mathbf{R}}(1)$.
Get "exponential" sequence:

$$0 \to \mathbf{Z}(1) \to \mathcal{L}_X \to \tau^{-1}\mathcal{M}_X \to 0.$$

Define $\tau^{-1}\mathcal{O}_X \to \mathcal{L}_X : a \mapsto (\exp a, Im(a))$.
Construct universal sheaf of $\tau^{-1}\mathcal{O}_X$-algebras $\mathcal{O}_X^{log}$ containing $\mathcal{L}_X$

Compactification of open immersions

The map τ is an isomorphism over the set X^* where $\overline{\mathcal{M}} = 0$, so we get a diagram



The map τ is proper, and for $x \in X$, $\tau^{-1}(x)$ is a torsor under $T_x := \mathrm{Hom}(\overline{\mathcal{M}}^{gp}_x, \mathbf{S}^1)$ (a finite sum of compact tori).
We think of τ as a relative compactification of j.

Example: monoid schemes

$X = A_Q := \mathrm{Spec}(Q \to \mathbf{C}[Q])$, with Q toric.

$$X_{log} = A_Q^{log} = R_Q \times T_Q,$$

where

$\underline{A}_Q(\mathbf{C}) = \{z\colon Q \to (\mathbf{C}, \cdot)\}$ (algebraic set)

$R_Q := \{r\colon Q \to (\mathbf{R}_{\geq}, \cdot)\}$ (semialgebraic set)

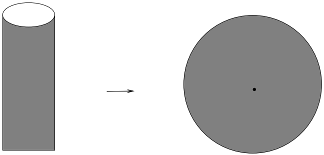
$T_Q := \{\zeta\colon Q \to (\mathbf{S}^1, \cdot)\}$ (compact torus)

$\tau\colon R_Q \times T_Q \to \underline{A}_Q(\mathbf{C})$ is multiplication: $z = r\zeta$.

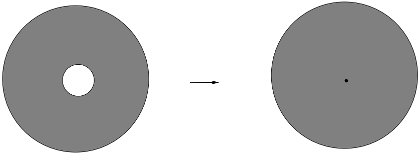
So A_Q^{log} means polar coordinates for $\underline{A}_Q$.

Example: log line, log point

If $X = A_{\mathbf{N}}$, then $X_{log} = \mathbf{R}_{\geq} \times \mathbf{S}^1$.



or



(Real blowup)

If $X = P = x_{\mathbf{N}}$, $X_{log} = \mathbf{S}^1$.

Example: $\mathcal{O}_{P^{log}}^{log}$

$$\Gamma(P_{log}, \mathcal{O}_P^{log}) = \Gamma(\mathbf{S}^1_{log}, \mathcal{O}_P^{log}) = \mathbf{C}.$$

Pull back to universal cover $\exp : \mathbf{R}(1) \to \mathbf{S}^1$

$$\Gamma(\mathbf{R}(1), \exp^* \mathcal{O}_P^{log}) = \mathbf{C}[\theta],$$

generated by θ (identity map).
The log inertia group $I_P = Aut(\mathbf{R}(1)/\mathbf{S}^1) = \mathbf{Z}(1)$ acts, as the unique automorphism such that $\rho_\gamma(\theta) = \theta + \gamma$. In fact, if $N = d/d\theta$,

$$\rho_\gamma = e^{\gamma N}.$$

Compactification: The geometry of j_{log}

Theorem

If $X/\mathbf{C}$ is (relatively) smooth, $j_{log}: X^*_{an} \to X_{log}$ is locally aspheric. In fact, $(X_{log}, X_{log} \setminus X^*_{an})$ is a manifold with boundary.

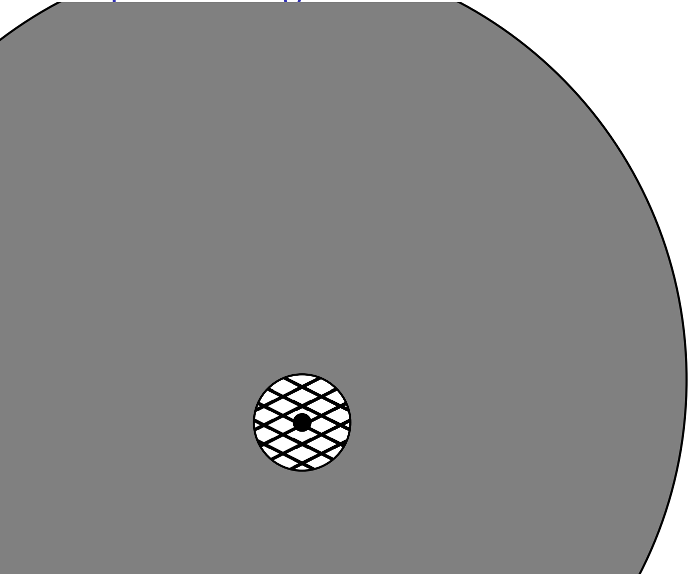
Proof.

Reduce to the case $X = \mathrm{A}_\mathrm{Q}$. Reduce to (R_Q, R^*_Q). Use the moment map:

$$(R_Q, R^*_Q) \cong (C_Q, C^o_Q) \quad : \quad r \mapsto \sum_{a \in A} r(a)a$$

where A is a finite set of generators of Q and C_Q is the real cone spanned by Q. □

Example: The log line



Degeneration: Submersivity of f_{log}

Theorem

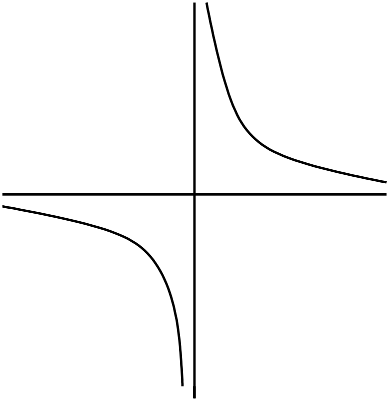
Let $f : X \to S$ be a (relatively) smooth exact morphism. Then $f_{log} : X_{log} \to S_{log}$ is a topological submersion, whose fibers are topological manifolds with boundary. The boundary corresponds to the set where f_{log} is not vertical.

Example

Semistable reduction $\mathbf{C} \times \mathbf{C} \to \mathbf{C} : (x_1, x_2) \mapsto x_1 x_2$

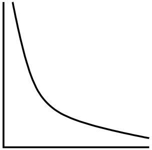
This is A_θ, where $\theta \colon \mathbf{N} \to \mathbf{N} \oplus \mathbf{N} : n \mapsto (n, n)$

Topology changes: (We just draw $\mathbf{R} \times \mathbf{R} \to \mathbf{R}$):

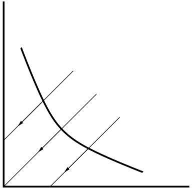


Log picture: $R_Q \times T_Q$

Just draw $R_Q \to R_{\mathbf{N}} : \mathbf{R}_{\geq} \times \mathbf{R}_{\geq} \to \mathbf{R}_{\geq} : (x_1, x_2) \mapsto x_1 x_2$



Topology unchanged, and in fact is homeomorphic to projection mapping. Proof: (Key is *exactness* of f, *integrality* of C_θ.)



Consequence

Corollary

Let $f : X \to S$ be a (relatively) smooth, proper, and exact morphism of log schemes.

1. $f_{log} : X_{log} \to S_{log}$ *is a fiber bundle, and*
2. $R^q f_{log*}$ *takes locally constant sheaves to locally constant sheaves.*

Betti cohomology over $\mathbf{C}$

$X/\mathbf{C}$ is a (relatively) coherent log scheme, X^* the open set where the log structure is trivial.

$$\begin{array}{ccc} H^*(X_{an}, \mathbf{Z}) & \xrightarrow{\tau^*} & H^*(X_{log}, \mathbf{Z}) \\ & \searrow & \downarrow j^*_{log} \\ & & H^*(X^*_{an}, \mathbf{Z}) \end{array}$$

Theorem

If $X/\mathbf{C}$ is (relatively) smooth, j^*_{log} is an isomorphism.

Betti cohomology over log disks and points

$f\colon X \to S$ (relatively) smooth and exact, S a standard log disk.
$P \to S$, log point. So $P_{log} = \mathbf{S}^1$. Let $Y := X \times_S P$.
Or: start with $g\colon Y \to P$, smooth and exact.

$$\begin{array}{ccc} Y_{log} & \xrightarrow{g_{log}} & \mathbf{S}^1 \\ {\scriptstyle \tau}\downarrow & & \downarrow{\scriptstyle \tau} \\ Y_{an} & \xrightarrow{g_{an}} & P_{an} \end{array}$$

Hence $Y_{log} \to Y_{an} \times \mathbf{S}^1$.

Monodromy

Pullback via $\exp\colon \mathbf{R}(1) \to \mathbf{S}^1$.

$$\begin{array}{ccccc}
\tilde{Y}_{log} & \longrightarrow & Y_{an} \times \mathbf{R}(1) & \longrightarrow & \mathbf{R}(1) \\
\downarrow \pi & \searrow^{\tilde{\tau}} & \downarrow & & \downarrow \pi \\
Y_{log} & \xrightarrow{\tau} & Y_{an} & \longrightarrow & P_{an}
\end{array}$$

Theorem

$I_P := Aut(\mathbf{R}(1)/\mathbf{S}^1) = \mathbf{Z}(1)$ acts on $\tilde{Y}_{log}$ and hence on the (homotopy type of) the fibers, as well as on the complex of nearby cycles:

$$R\Psi := R\tilde{\tau}_*(\mathbf{Z}_{\tilde{Y}_{log}}) \in D^+(Y_{an}, \mathbf{Z}[I_P]).$$

If Y/P is smooth and saturated, the action of I_P is unipotent. In fact it is trivial on

$$\mathrm{Gr}^{\cdot}_{can} \cong R^{\cdot}\tau_*\mathbf{Z} \cong \Lambda^{\cdot}\overline{\mathcal{M}}^{gp}_Y.$$

Connections and crystals

X/S (relatively) smooth map of log schemes.

Definition

A (log) connection on an $\mathcal{O}_X$-module E:

$$\nabla\colon E \to \Omega^1_{X/S} \otimes E \quad \text{satisfying the Leibnitz rule}$$

Theorem (Riemann-Hilbert)

Let $X/\mathbf{C}$ be (relatively) smooth. Then there is an equivalence of categories:

$$MIC_{nil}(X/\mathbf{C}) \equiv L_{un}(X^{log})$$

$$(E, \nabla) \mapsto \mathrm{Ker}(\tau^{-1}E \otimes \mathcal{O}_X^{log} \xrightarrow{\nabla} \tau^{-1}E \otimes \Omega_X^{1\,log})$$

Example: $X := P$ (Standard log point)

$\Omega^1_{P/\mathbf{C}} \cong \mathbf{N} \otimes \mathbf{C} \cong \mathbf{C}$, so

$$MIC(P/\mathbf{C}) \equiv \{(E, N) : \text{vector space with endomorphism}\}$$

$P_{log} = \mathbf{S}^1$, so $L(P_{log})$ is cat of reps of the log inertia group $I_P = \mathbf{Z}(1)$. Thus:

$$L(P_{log}) \equiv \{(V, \rho) : \text{vector space with automorphism}\}$$

Conclusion:

$$\{(E, N) : N \text{ is nilpotent}\} \equiv \{(V, \rho) : \rho \text{ is unipotent}\}$$

Use $\mathcal{O}^{log}_P = \mathbf{C}[\theta]$:

$$(V, \rho) = \mathrm{Ker}\,(\tau^* E \otimes \mathbf{C}[\theta] \to \tau^* E \otimes \mathbf{C}[\theta])$$

$$N \mapsto e^{2\pi i N}$$

De Rham cohomology over $\mathbf{C}$

$X/\mathbf{C}$ (relatively) smooth log scheme.

$$H_{DR}(X) := H^*(X, \Omega^{\cdot}_{X/\mathbf{C}}), \quad H_{DR}(X_{log}) := H^*(X_{log}, \Omega^{log\cdot}_{X/\mathbf{C}})$$

Theorem: There is a commutative diagram of isomorphisms:

$$\begin{array}{ccccc}
H_{DR}(X) & \longrightarrow & H_{DR}(X^*) & & \\
\downarrow & & \downarrow & \searrow & \\
H_{DR}(X_{an}) & \longrightarrow & H_{DR}(X_{log}) & \longrightarrow & H_{DR}(X^*) \\
 & & \downarrow & & \downarrow \\
 & & H_B(X_{log}, \mathbf{C}) & \longrightarrow & H_B(X^*_{an}, \mathbf{C})
\end{array}$$

De Rham cohomology over log disks and points

$f\colon X \to S$ (relatively) smooth and exact, $V := \mathbf{C}\{t\}$,
$S := \mathrm{Spec}(V' \to V)$, $P \to S$, log point. Let $Y := X \times_S P$.
Or: start with $g\colon Y \to P$, smooth and exact. $P_{log} = \mathbf{S}^1$, so:

$$\begin{array}{ccc} Y_{log} & \xrightarrow{g_{log}} & \mathbf{S}^1 \\ \downarrow{\scriptstyle \tau} & & \downarrow{\scriptstyle \tau} \\ Y_{an} & \xrightarrow{g_{an}} & P_{an} \end{array}$$

Theorem

Assume X/S is saturated. Then $H_{DR}(X/S)$ is free over V, and $H_{DR}(Y/P) \cong \mathbf{C} \otimes_V H_{DR}(X/S)$. Gauss-Manin connection gives:

$$H_{DR}(Y/P) \to \Omega^1_{P/\mathbf{C}} \otimes H_{DR}(Y/P).$$

$$N\colon H_{DR}(Y/P) \to H_{DR}(Y/P)$$

N is nilpotent, and corresponds to the (unipotent) monodromy action of I_P on $Rf_{log*}(\mathbf{Z})$ (Betti cohomology).

Proof uses the log Poincaré lemma $\mathbf{C} \cong \Omega^{log\cdot}_{Y/S}$, and the map τ (to prove unipotence of monodromy).

The Steenbrink complex

The Steenbrink complex is an explicit representative of $R\Psi g$:

$$\Psi^{\cdot} := \mathcal{O}_P^{log} \to \mathcal{O}_P^{log} \otimes \Omega^1_{Y/S} \otimes \cdots$$

Theorem

There is an isomorphism in the filtered derived category:

$$(R\Psi g, Dec\ T) \sim (\Psi^{\cdot}, Dec\ F).$$

Here T is the trivial filtration on $R\Psi g$ and F is the filtration given by the nilpotent endomorphism of N on $\mathcal{O}_P^{log} = \mathbf{C}[\theta]$.

Corollary

There is a commutative diagram

$$\begin{array}{ccc} \mathrm{Gr}^{q}_{can} R\Psi g & \longrightarrow & \Lambda^q \overline{\mathcal{M}}^{gp}_Y[-q] \\ \downarrow N & & \downarrow \cup\kappa \\ \mathrm{Gr}^{q-1}_{can} R\Psi g & \longrightarrow & \Lambda^{q-1} \overline{\mathcal{M}}^{gp}_Y[1-q] \end{array}$$

where

$$\kappa\colon \overline{\mathcal{M}}_{Y/P} \to \mathbf{Z}[1]$$

is the (Kodaira-Spencer) map coming from the exact sequence

$$0 \to \mathbf{Z} \to \overline{\mathcal{M}}^{gp}_Y \to \overline{\mathcal{M}}^{gp}_{Y/P} \to 0$$

Conclusion

- Log geometry provides a uniform geometric perspective to treat compactification and degeneration problems in topology and in algebraic and arithmetic geometry.
- Log geometry incorporates many classical tools and techniques.
- Log geometry is not a revolution.
- Log geometry presents new problems and perspectives, both in fundamentals and in applications.