LOG THH AND TC

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In this talk I wil discuss:

(1) Aims for how to understand algebraic K-theory of commutative $\mathbb{S}$-algebras, and the hope to create higher $K(n)$-local fields using log $\mathbb{S}$-algebras.
(2) A construction of log topological Hochschild homology $\mathrm{THH}(A, M)$ for arbitrary log $\mathbb{S}$-algebras (A, M), modeling the log de Rham–complex $\Omega^*_{(A,M)}$.
(3) A construction of log topological restriction homology $\mathrm{TR}(A, M)$ and log topological cyclic homology $\mathrm{TC}(A, M)$ for log $\mathbb{S}$-algebras (A, M) with M p-exact, modeling the log de Rham–Witt complex $W\Omega^*_{(A,M)}$ and its $F = id$ part, respectively
(4) Localization sequences for these log theories, for suitable M.

Algebraic K-theory

Let A be an $\mathbb{S}$-algebra ($= A_\infty$-ring).

When $A = HR$ is the Eilenberg–Mac Lane spectrum of a number ring R, $K(A) = K(R)$ contains number-theoretic information about R.

When $A = KU$ represents topological K-theory, $K(KU)$ is related to elliptic cohomology. More precisely, $V(1)_*K(KU) = K/(p, v_1)_*(KU)$ is essentialy a finitely generated free module over $\mathbb{F}_p[v_2] = P(v_2)$. (Ausoni–Rognes, Baas-Dundas-Richter-Rognes)

When $A = \mathbb{S}[\Omega X]$ is the spherical group ring of the loop group of a compact manifold X, $K(A) = A(X)$ is related to the automorphism group of X, either in the topological, piecewise-linear or differentiable categories. (Waldhausen-Jahren-Rognes)

To understand $K(R)$ for many discrete commutative rings R, we can use localization sequences to reduce to the case $R = F$ is a field. Then we can use Galois descent to reduce to the case $R = \bar{F}$ is algebraically closed, at least after completion at a prime p and up to some moderate error. Finally, $K(\bar{F}) \simeq ku$ after p-competion. (Suslin)

$$R \xrightarrow{j} f\!f(R) = F \xrightarrow{G_F} \bar{F}\,.$$

We would like to understand algebraic K-theory of commutative $\mathbb{S}$-algebras ($= E_\infty$-rings) in similar terms. This leads to the question of what plays the role of the residue rings and fraction fields of a commutative $\mathbb{S}$-algebra.

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

In the basic case of the sphere spectrum itself, $A = \mathbb{S}$, the map from the p-localization to the rationalization factors through an infinite tower of distinct chromatic localizations:

$$\mathbb{S}_{(p)} \to \cdots \to L_n\mathbb{S} \to L_{n-1}\mathbb{S} \to \cdots \to L_0\mathbb{S} = H\mathbb{Q}\,.$$

A formal neighborhood of the n-th layer here is modeled by the localization (= a kind of completion) $L_n\mathbb{S} \to \hat{L}_n\mathbb{S}$ with respect to the n-th Morava K-theory spectrum, with

$$\pi_* K(n) = \mathbb{F}_p[v_n^{\pm 1}]$$

where $|v_n| = 2(p^n - 1)$, and there is a $K(n)$-local pro-Galois extension $\hat{L}_n\mathbb{S} \to E_n$ (Morava, Devinatz–Hopkins) where E_n is the n-th Lubin–Tate spectrum, with

$$\pi_* E_n = W(\mathbb{F}_{p^n})[[u_1, \dots, u_{n-1}]][u^{\pm 1}]\,.$$

Here $\pi_0 E_n = W(\mathbb{F}_{p^n})[[u_1, \dots, u_{n-1}]]$ is the complete local ring supporting the universal deformation of Honda's height n formal group law over $\mathbb{F}_{p^n}$. However, the $\mathbb{S}$-algebra $K(n)$ is not commutative, and neither $\hat{L}_n\mathbb{S}$ nor E_n seem to properly behave as fields.

Is there a construction of a fraction field $\mathit{ff}(E_n)$ of E_n? Does it admit a algebraic closure Ω_n in a category of $K(n)$-local fields? The extensions

$$\mathit{ff}(\hat{L}_n\mathbb{S}) \to \mathit{ff}(E_n) \to \Omega_n$$

should then determine an extension of Galois groups

$$G_{\mathit{ff}(E_n)} \to G_{\mathit{ff}\hat{L}_n\mathbb{S}} \to \mathbb{G}_n$$

where $\mathbb{G}_n$ is the n-th extended Morava stabilizer group, essentially given by the automorphisms of the height n formal group law. Conjecturally $\mathit{ff}(E_n)$ is an $(n+1)$-dimensional higher local field, of p-cohomological dimension $(n+2)$, with arithmetic duality of Tate–Poitou/Deninger–Winberg type, given by a perfect pairing landing in $H^{n+2}_{Gal}(\mathit{ff}(E_n); \mathbb{F}_{p^{n+1}}(n+1))$.

$$\begin{array}{ccccccccc}
 & & & & \Omega_n & & & & H\bar{\mathbb{Q}}_p \\
 & & & & \uparrow{\scriptstyle G_{\mathit{ff}(E_n)}} & & & & \uparrow{\scriptstyle G_{\mathbb{Q}_p}} \\
 & & E_n & \xrightarrow{j} & \mathit{ff}(E_n) & & & & H\mathbb{Q}_p \\
 & & {\scriptstyle \mathbb{G}_n}\uparrow & & \uparrow{\scriptstyle \mathbb{G}_n} & & & & \uparrow{\scriptstyle =} \\
 & & \hat{L}_n\mathbb{S} & \xrightarrow{j} & \mathit{ff}(\hat{L}_n\mathbb{S}) & & & & H\mathbb{Q}_p \\
 & & \uparrow & & & & & & \uparrow \\
\mathbb{S}_{(p)} & \longrightarrow \cdots \longrightarrow & L_n\mathbb{S} & \longrightarrow & L_{n-1}\mathbb{S} & \longrightarrow & \cdots & \longrightarrow & H\mathbb{Q}
\end{array}$$

To simplify, we look at the first non-algebraic case $n = 1$, when $E_1 = KU_p$ is p-adic K-theory, with

$$\pi_* KU_p = \mathbb{Z}_p[u^{\pm 1}]\,.$$

The obvious localization $KU_p[p^{-1}] = KU\mathbb{Q}_p$ is an algebra over $\mathbb{S}_p[p^{-1}] = H\mathbb{Q}_p$. Hence its algebraic K-theory $K(KU\mathbb{Q}_p)$ is an algebra over $K(\mathbb{Q}_p)$, which is v_2-torsion. Hence this localization destroys the relation between $K(KU)$ and elliptic cohomology.

We therefore think that this $\mathbb{S}$-algebraic localization is too drastic. Instead, we propose to work with the milder localization given by a topological log structure on KU_p suitably generated by p. The log $\mathbb{S}$-algebra

$$ff(KU_p) = (KU_p, \langle p\rangle)$$

is then our candidate for the fraction field of KU_p. What is the appropriate topological monoid $\langle p\rangle$? We get a factorization

$$KU_p \to (KU_p, \langle p\rangle) \to KU\mathbb{Q}_p$$

of the localization map.

The calculations with Ausoni suggest that $K(\Omega_1)$ is a connective form of E_2, so that $\hat{L}_2K(\Omega_1) \simeq E_2$, the descent spectral sequence

$$E^2_{s,t} = H^s_{mot}(ff(KU_p); \mathbb{F}_{p^2}(t/2)) \Longrightarrow K/(p, v_1)_{t-s}(ff(KU_p))$$

collapses for $p \geq 5$, the change-of-topology map

$$H^s_{mot}(ff(KU_p); \mathbb{F}_{p^2}(t/2)) \to H^s_{Gal}(ff(KU_p); \mathbb{F}_{p^2}(t/2))$$

is an isomorphism for $0 \leq s \leq t/2$, that $H^s_{Gal} = 0$ for $s > 3$ and that the cup product to

$$H^3_{Gal}(ff(KU_p); \mathbb{F}_{p^2}(2)) \cong \mathbb{Z}/p$$

is a perfect pairing.

Topological cyclic homology

For an $\mathbb{S}$-algebra A, there is a cyclotomic trace map

$$trc\colon K(A) \to TC(A)$$

to topological cyclic homology. (We suppress p from the usual notation $TC(A;p)$.) This is a very strong p-adic invariant when A is connective and $\pi_0(A)$ is finitely generated as a module over $W(k)$ for a perfect field k of characteristic p. In these cases, the induced homomorphism $K_*(A) \to TC_*(A)$ is a p-adic isomorphism for all $* \geq 0$, while $\mathrm{TC}_{-1}(A) \cong W(\pi_0 A)_F$. (Hesselholt–Madsen, using McCarthy, Dundas).

This applies, for example, when $A = \mathbb{Z}_p$, ku_p or ku/p. However, it does not apply when $A = \mathbb{Q}_p$, KU_p or KU/p. Note that these are obtained from the previous examples by inverting p, u and u, respectively.

To extend the utility of the cyclotomic trace map, we therefore wish to extend the definition of topological cyclic homology to log $\mathbb{S}$-algebras (A, M), so as to construct log topological cyclic homology $TC(A, M)$, for suitable commutative ($\mathcal{I}$-space) monoids M. We aim to get log cyclotomic trace maps

$$trc\colon K(A, M) \to TC(A, M)$$

from the log K-theory of (A, M), and expect that

$$K(A, M) \simeq K(A[M^{-1}])$$

for suitable M. For example, when $A = \mathbb{Z}_p$ and $M = \langle p \rangle \cong \mathbb{N}_0$, we have a p-adic equivalence

$$K(\mathbb{Q}_p) \to TC(\mathbb{Z}_p, \langle p \rangle)$$

in positive degrees.

Given this extension to log $\mathbb{S}$-algebras, we also wish to interpret $K(\mathit{ff}(KU_p))$ as $K(KU_p, \langle p \rangle)$, and to compute it using the p-adic equivalence

$$K(KU_p, \langle p \rangle) \to TC(KU_p, \langle p \rangle)$$

in positive degrees.

Topological Hochschild homology

Topological cyclic homology is constructed from topological Hochschild homology, $\mathrm{THH}(A) = T(A) = HH^{\mathbb{S}}(A)$, which is the same as Hochschild homology with base ring the sphere spectrum $\mathbb{S}$. For A smooth over $\mathbb{S}$ it is a model for the de Rham complex $\Omega^*_{A/\mathbb{S}}$ of A, while for $A = H\mathbb{F}_p$ Bökstedt (and Breen) computed

$$\pi_* \mathrm{THH}(\mathbb{F}_p) \simeq P(\mu_0)$$

with $|\mu_0| = 2$. Of course $\pi_* HH^{\mathbb{F}_p}(\mathbb{F}_p) = \mathbb{F}_p$, so the extra factor $P(\mu_0)$ arises from the change of base along $\mathbb{S} \to H\mathbb{F}_p$.

There is a Connes cyclic structure on $\mathrm{THH}(A)$, making it an S^1-equivariant spectrum. In fact Bökstedt's specific construction makes it is a p-cyclotomic spectrum, in the sense of Hesselholt–Madsen, so that there is an equivariant equivalence

$$r \colon \Phi^{C_p} \mathrm{THH}(A) \xrightarrow{\simeq} \mathrm{THH}(A)$$

from the geometric fixed point spectrum on the left. When combined with the natural equivariant map

$$\mathrm{THH}(A)^{C_p} \to \Phi^{C_p} \mathrm{THH}(A)$$

from the categorical fixed points, we obtain the equivariant restriction map

$$R \colon \mathrm{THH}(A)^{C_p} \to \mathrm{THH}(A)$$

which induces a tower of restriction maps

$$\dots \xrightarrow{R} \mathrm{THH}(A)^{C_{p^n}} \xrightarrow{R} \mathrm{THH}(A)^{C_{p^{n-1}}} \xrightarrow{R} \dots \xrightarrow{R} \mathrm{THH}(A)$$

for all $n \geq 1$, upon passing to $C_{p^{n-1}}$-fixed points. The homotopy groups

$$\pi_* \mathrm{TR}^n(A) = \pi_* \mathrm{THH}(A)^{C_{p^{n-1}}}$$

then form a diagram of de Rham–Witt type, like $W_n\Omega^*_A$. In particular,

$$\pi_0 \operatorname{TR}^n(A) \cong W_n(\pi_0 A)$$

when A is connective, and

$$\pi_* \operatorname{TR}^1(A) = \pi_* \operatorname{THH}(A)$$

is our topological model for the de Rham complex. We define

$$\operatorname{TR}(A) = \operatorname*{holim}_{n,R} \operatorname{TR}^n(A) = \operatorname*{holim}_{n,R} \operatorname{THH}(A)^{C_{p^n}}$$

as the topological model of the de Rham–Witt complex, computing crystalline cohomology.

There is also an equivariant forgetful map

$$F\colon \operatorname{THH}(A)^{C_p} \to \operatorname{THH}(A)$$

that induces maps

$$F\colon \operatorname{THH}(A)^{C_{p^n}} \to \operatorname{THH}(A)^{C_{p^{n-1}}}$$

commuting with the R-maps. The homotopy equalizer of F and id on $\operatorname{TR}(A)$ defines the topological cyclic homology, and the trace map $K(A) \to \operatorname{THH}(A)$ lifts all the way through that homotopy equalizer:

$$\begin{array}{ccccc} K(A) & \xrightarrow{tr} & \operatorname{THH}(A) & & \\ {\scriptstyle trc}\downarrow & & \uparrow & & \\ TC(A) & \xrightarrow{\pi} & \operatorname{TR}(A) & \underset{id}{\overset{F}{\rightrightarrows}} & \operatorname{TR}(A)\,. \end{array}$$

Hence $TC(A)$ is a topological model for the $F = id$ part of the de Rham–Witt complex, which in particular contains the image from algebraic K-theory. For example, for each unit x in A, the symbol $\{x\}$ in $K_1(A)$ maps to $x^{-1}dx$ in $\Omega^1_A \cong \operatorname{THH}_1(A)$.

Thus $TC(A)$ seems to be a topological model for the $r = 0$ part of the syntomic complexes. It would be interesting to identify the J-adic filtration on $TR(A)$, so as to also get topological models for the complexes $s^r_\bullet$ computing syntomic cohomology (Fontaine–Messing).

The cyclic bar construction

Let M be a commutative $\mathcal{I}$-space monoid. For reasonable M there are weak equivalences

$$(M \boxtimes M)_{h\mathcal{I}} \simeq M_{h\mathcal{I}} \times M_{h\mathcal{I}}$$

so for simplicity we may think of $M \boxtimes M$ as the product $M \times M$, and similarly with multiple factors.

Definition (Waldhausen(?)). The **cyclic bar construction** on M, denoted $B^{cy}(M)$, is a cyclic object

$$[q] \mapsto M \boxtimes M \boxtimes \cdots \boxtimes M$$

with $(1+q)$ copies of M in degree q. The cyclic structure is similar to that for the Hochschild homology, with face maps

$$d_i(m_0, \dots, m_q) = (m_0, \dots, m_i m_{i+1}, \dots, m_q)$$

for $0 \le i < q$ and

$$d_q(m_0, \dots, m_q) = (m_q m_0, m_1, \dots, m_{q-1}),$$

degeneracy maps

$$s_j(m_0, \dots, m_q) = (m_0, \dots, m_j, 1, m_{j+1}, \dots, m_q)$$

for $0 \le j \le q$, and cyclic operators

$$t_q(m_0, \dots, m_q) = (m_q, m_0, \dots, m_{q-1}).$$

In particular there is an S^1-action on $B^{cy}(M)$, which acts on a vertex (m) (a 0-simplex) by moving it once around the loop $t_1 s_0(m) = (1, m)$ (a 1-simplex with identical endpoints).

Let $\mathbb{S}[M]$ be the spherical monoid ring of M, which is a commutative $\mathbb{S}$-algebra. There are natural isomorphisms of commutative $\mathbb{S}$-algebras

$$\mathbb{S}[M \boxtimes \cdots \boxtimes M] \cong \mathbb{S}[M] \wedge \cdots \wedge \mathbb{S}[M],$$

which induce an S^1-equivariant isomorphism

$$\mathbb{S}[B^{cy}(M)] \cong \mathrm{THH}(\mathbb{S}[M]).$$

Now let (A, M) be a prelog $\mathbb{S}$-algebra. The prelog structure map $\alpha \colon M \to \Omega^{\mathcal{I}} A$ is right adjoint to a commutative $\mathbb{S}$-algebra map $\bar{\alpha} \colon \mathbb{S}[M] \to A$, which induces a map

$$\mathbb{S}[B^{cy}(M)] \cong \mathrm{THH}(\mathbb{S}[M]) \xrightarrow{\bar{\alpha}} \mathrm{THH}(A)$$

of commutative $\mathbb{S}$-algebras. This makes $(\mathrm{THH}(A), B^{cy}(M))$ a prelog $\mathbb{S}$-algebra.

The replete bar construction

There is an augmentation map $\epsilon \colon B^{cy}(M) \to M$ with

$$\epsilon(m_0, \dots, m_q) = m_0 \cdot \ldots \cdot m_q,$$

which is well-defined since M is commutative, and $(\mathrm{THH}(A), B^{cy}(M))$ is therefore naturally a prelog $\mathbb{S}$-algebra augmented over (A, M). However, the commutative diagram

$$\begin{array}{ccc} B^{cy}(M) & \xrightarrow{\gamma} & B^{cy}(M^{gp}) \\ \downarrow{\scriptstyle \epsilon} & & \downarrow{\scriptstyle \epsilon} \\ M & \xrightarrow{\gamma} & M^{gp} \end{array}$$

is usually not a homotopy pullback diagram, so ϵ is usually not exact. In other words, $B^{cy}(M)$ is not replete over the base M.

A standard premise in classical logarithmic geometry is that one only works with commutative monoids that are fine, or fine and saturated. For the topological theory, we get a satisfactory theory by fixing a base commutative $\mathcal{I}$-space monoid, M in this case, and to only consider commutative $\mathcal{I}$-space monoids $\epsilon\colon N \to M$ that are augmented over that base by a map that is replete, i.e., virtually surjective and exact. Whenever we encounter an N over M that is not replete, we replace it with its repletion N^{rep}, defined as the right hand homotopy pullback in the diagram

$$\begin{array}{ccccc} N & \longrightarrow & N^{rep} & \longrightarrow & N^{gp} \\ {\scriptstyle\epsilon}\downarrow & & \downarrow & & \downarrow{\scriptstyle\epsilon} \\ M & \xrightarrow{=} & M & \xrightarrow{\gamma} & M^{gp} \end{array}$$

of commutative $\mathcal{I}$-space monoids.

Definition. The **replete bar construction** $B^{rep}(M) = B^{cy}(M)^{rep}$ is the repletion over M of the cyclic bar construction $B^{cy}(M)$, given by the homotopy pullback of ϵ along γ:

$$\begin{array}{ccccc} B^{cy}(M) & \longrightarrow & B^{rep}(M) & \longrightarrow & B^{cy}(M^{gp}) \\ {\scriptstyle\epsilon}\downarrow & & \downarrow & & \downarrow{\scriptstyle\epsilon} \\ M & \xrightarrow{=} & M & \xrightarrow{\gamma} & M^{gp} \end{array}$$

The canonical map $B^{cy}(M) \to B^{rep}(M)$ is called the **repletion map**. It is a map of cyclic commutative $\mathcal{I}$-space monoids.

Lemma. *The projection $\pi\colon B^{cy}(M) \to BM$ to the usual bar construction*

$$\pi(m_0, m_1, \dots, m_q) = [m_1|\dots|m_q]\,,$$

induces equivalences

$$(\epsilon, \pi)\colon B^{cy}(M^{gp}) \xrightarrow{\simeq} M^{gp} \times BM^{gp}$$

and

$$(\epsilon, \pi)\colon B^{rep}(M) \xrightarrow{\simeq} M \times BM^{gp}$$

but

$$(\epsilon, \pi)\colon B^{cy}(M) \to M \times BM$$

is usually not an equivalence.

Log topological Hochschild homology

Recall that (A, M) is a given prelog $\mathbb{S}$-algebra, with "suspension"

$$(\mathrm{THH}(A), B^{cy}(M)) = S^1 \otimes (A, M)$$

in the category of prelog $\mathbb{S}$-algebras over (and under) (A, M). After expanding the cyclic bar construction $B^{cy}(M)$ to its repletion, $B^{rep}(M)$, it may no longer map to

$\Omega^{\mathcal{I}}\,\mathrm{THH}(A)$, so to obtain a suspension in the category of replete prelog $\mathbb{S}$-algebras over (and under) (A, M), we must replace $\mathrm{THH}(A)$ by the pushout $\mathrm{THH}(A, M)$ in the diagram

$$\begin{array}{ccc} \mathbb{S}[B^{cy}(M)] & \longrightarrow & \mathbb{S}[B^{rep}(M)] \\ {\scriptstyle \bar{\alpha}}\downarrow & & \downarrow{\scriptstyle \bar{\alpha}} \\ \mathrm{THH}(A) & \longrightarrow & \mathrm{THH}(A, M) \end{array}$$

of cyclic commutative $\mathbb{S}$-algebras.

Definition. *The* **log topological Hochschild homology** *of* (A, M) *is the cyclic commutative* $\mathbb{S}$*-algebra*

$$\mathrm{THH}(A, M) = \mathrm{THH}(A) \wedge_{\mathbb{S}[B^{cy}(M)]} \mathbb{S}[B^{rep}(M)]\,.$$

This is the topological model for the log de Rham complex of (A, M).

For each $m \in M$ the loop $(1, m)$ at (m) in $B^{cy}(M)$ maps to the differential dm in $\Omega^1_A \to \mathrm{THH}_1(A)$, while there is also a loop (m^{-1}, m) at (1) in $B^{rep}(M)$ that maps to the log differential $d\log m$ in $\Omega^1_{(A,M)} \to \mathrm{THH}_1(A, M)$. The formula $(m) \cdot (m^{-1}, m) = (1, m)$ implies the key relation $m \cdot d\log m = dm$.

We can rewrite the pushout square above as:

$$\begin{array}{ccc} \mathrm{THH}(\mathbb{S}[M]) & \longrightarrow & \mathrm{THH}(\mathbb{S}[M], M) \\ {\scriptstyle \bar{\alpha}}\downarrow & & \downarrow{\scriptstyle \bar{\alpha}} \\ \mathrm{THH}(A) & \longrightarrow & \mathrm{THH}(A, M) \end{array}$$

By a similar construction, suspending infinitely often in the category of replete prelog $\mathbb{S}$-algebras over and under (A, M), we can define the log toplogical André–Quillen homology of (A, M), denoted $\mathrm{TAQ}(A, M)$. It corepresents log derivations

$$\mathrm{Mod}_A(\mathrm{TAQ}(A, M), J) \simeq \mathrm{Der}((A, M), J)\,,$$

and plays the role of the log Kähler differentials $\Omega^1_{(A,M)}$, or more precisely, the log cotangent complex of (A, M). There is a Quillen spectral sequence from the graded symmetric A-algebra on $\Sigma\,\mathrm{TAQ}(A, M)$ converging to $\mathrm{THH}(A, M)$ for connective A, which collapses when (A, M) is log smooth over the base (in this case $(\mathbb{S}, 1)$). This is analogous to the expression of the algebraic log de Rham–complex as the exterior algebra on the log Kähler differentials.

A strictly commutative case

When $M = \mathbb{N}_0$, $M^{gp} = \mathbb{Z}$, $BM^{gp} \simeq S^1$ and $B^{cy}(M^{gp}) \simeq \mathbb{Z} \times S^1$. More precisely

$$B^{cy}(M^{gp}) \simeq \coprod_{j\in\mathbb{Z}} S^1(j)$$

where the S^1-action on $S^1(j)$ has degree j. Taking pullback along $\gamma\colon \mathbb{N}_0 \to \mathbb{Z}$ we get

$$B^{rep}(M) \simeq \coprod_{j\geq 0} S^1(j)\,.$$

Inside of this we have

$$B^{cy}(M) \simeq * \sqcup \coprod_{j>0} S^1(j)$$

with the repletion map taking $* = (0)$ to a point in $S^1(0)$. This implies:

Lemma. *For $M = \mathbb{N}_0$, there is a cofiber sequence of $B^{cy}(M)$-spaces*

$$S^0(0) = 1_+ \to B^{cy}(M) \to B^{rep}(M)$$

and a cofiber sequence of $\mathbb{S}[B^{cy}(M)]$-modules

$$\mathbb{S} \xrightarrow{i_*} \mathbb{S}[B^{cy}(M)] \xrightarrow{j^*} \mathbb{S}[B^{rep}(M)]\,.$$

Proposition. *Let (A, M) be a prelog $\mathbb{S}$-algebra with $M \cong \mathbb{N}_0$. There is a cofiber sequence*

$$\mathrm{THH}(A//M) \xrightarrow{i_*} \mathrm{THH}(A) \xrightarrow{j^*} \mathrm{THH}(A, M)$$

where $A//M = A \wedge_{\mathbb{S}[M]} \mathbb{S}$.

Proof. We base change the cofiber sequence above along $\bar{\alpha}\colon \mathbb{S}[B^{cy}(M)] \to A$. The left hand term is

$$\begin{aligned}\mathrm{THH}(A) \wedge_{\mathbb{S}[B^{cy}(M)]} \mathbb{S} &\cong \mathrm{THH}(A) \wedge_{\mathrm{THH}(\mathbb{S}[M])} \mathrm{THH}(\mathbb{S}) \\ &\cong \mathrm{THH}(A \wedge_{\mathbb{S}[M]} \mathbb{S}) = \mathrm{THH}(A//M)\,.\end{aligned}$$

The right hand term is $\mathrm{THH}(A, M)$, by definition. □

Example. Let A be a discrete ring and $x \in A$ an element that does not divide zero. Then $A//\langle x\rangle = A/(x)$ and there is a cofiber sequence

$$\mathrm{THH}(A/(x)) \xrightarrow{i_*} \mathrm{THH}(A) \xrightarrow{j^*} \mathrm{THH}(A, \langle x\rangle)\,.$$

In particular, when A is a discrete valuation ring with uniformizer π, residue field $k = A/(\pi)$ and fraction field $K = A[\pi^{-1}]$, there is a cofiber sequence

$$\mathrm{THH}(k) \xrightarrow{i_*} \mathrm{THH}(A) \xrightarrow{j^*} \mathrm{THH}(A, \langle \pi\rangle)$$

that agrees with the ad hoc cofiber sequence of Hesselholt and Madsen

$$\mathrm{THH}(k) \xrightarrow{i_*} \mathrm{THH}(A) \xrightarrow{j^*} \mathrm{THH}(A|K)$$

which is only defined in this special setting. Here $\pi_* \mathrm{THH}(A|K) \cong \Omega^*_{(A,M)} \otimes P(\kappa)$ with $|\kappa| = 2$.

Note that for (A, M) to be a prelog $\mathbb{S}$-algebra when $M = \mathbb{N}_0$, the image x of $1 \in \mathbb{N}_0$ must be "strictly self-commuting" in A, in the sense that the free prelog structure map

$$\mathbb{S}[\coprod_{j\ge 0} B\Sigma_j] \to A$$

generated by x extends over

$$\coprod_{j\ge 0} B\Sigma_j \to \mathbb{N}_0\,.$$

This implies that all higher Dyer–Lashof operations on $[x] \in H_0(A)$ vanish, which does not hold for p in $\Omega^{\mathcal{I}} ku_p$.

Computation of $THH(\mathbb{Z}, \langle p\rangle)$

We explain this in the case $A = \mathbb{Z}_p$ and $\pi = p$. For the p-adic part of the computation we may as well work with $A = \mathbb{Z}$.

Bökstedt computed that

$$\pi_* THH(\mathbb{Z}/p) \cong P(\mu_0)$$

where $|\mu_0| = 2$, so

$$\pi_*(THH(\mathbb{Z}/p); \mathbb{Z}/p) \cong E(\epsilon_0) \otimes P(\mu_0)$$

where $|\epsilon_0| = 1$. Here $P(-)$ and $E(-)$ denote the polynomial and exterior algebras over $\mathbb{Z}/p$ on the given generators. Bökstedt also computed that

$$\pi_*(THH(\mathbb{Z}); \mathbb{Z}/p) \cong E(\lambda_1) \otimes P(\mu_1)$$

where $|\lambda_1| = 2p - 1$ and $|\mu_1| = 2p$.

It is known that

$$\begin{aligned} H_*(B^{cy}(M)) &\cong P(x) \otimes E(dx) \\ H_*(B^{rep}(M)) &\cong P(x) \otimes E(d\log x) \end{aligned}$$

where $|x| = 0$, $|dx| = 1$ and $|d\log x| = 1$. Here $H_*(-)$ denotes homology with $\mathbb{Z}/p$-coefficients. The repletion map takes dx to $x \cdot d\log x$.

We can use this to determine the structure of the Künneth (= bar) spectral sequence

$$\begin{aligned} E^2_{**} &= \mathrm{Tor}^{H_*(B^{cy}(M))}_{**}(\pi_*(THH(\mathbb{Z}); \mathbb{Z}/p), \mathbb{Z}/p) \\ &\Longrightarrow \pi_*(THH(\mathbb{Z}/p); \mathbb{Z}/p)\,. \end{aligned}$$

Here the E^2-term is

$$\begin{aligned} E^2_{**} &\cong \mathrm{Tor}^{P(x)\otimes E(dx)}_{**}(E(\lambda_1) \otimes P(\mu_1), \mathbb{Z}/p) \\ &\cong E(\lambda_1) \otimes P(\mu_1) \otimes E([x]) \otimes \Gamma([dx]) \end{aligned}$$

where $[x]$ has bidegree $(1, 0)$ and $[dx]$ has bidegree $(1, 1)$. We write $\Gamma(-)$ for the divided power algebra on the given generator.

To get the given abutment, we must have a differential

$$d^p(\gamma_p([dx])) = \lambda_1$$

up to a unit in $\mathbb{Z}/p$. Here $\gamma_p([dx]))$ denotes the p-th divided power on $[dx]$, in bidegree (p, p). This leaves the following term

$$E^{p+1}_{**} \cong P(\mu_1) \otimes E([x]) \otimes P_p([dx])$$

where $P_p(-)$ denotes the truncated polynomial algebra of height p. There is no room for further differentials, so $E^{p+1}_{**} = E^\infty_{**}$. It follows that ϵ_0 is represented by $[x]$, μ_0 is represented by $[dx]$, and μ_0^p is represented by μ_1. Hence there is a multiplicative extension $[dx]^p = \mu_1$ in the abutment.

Lemma. *In the Künneth spectral sequence for the $\mathbb{Z}/p$-homotopy of*

$$THH(\mathbb{Z}/p) \simeq THH(\mathbb{Z}) \wedge_{\mathbb{S}[B^{cy}(M)} \mathbb{S}$$

there is a differential

$$d^p(\gamma_p([dx])) = \lambda_1$$

and a multiplicative extension

$$[dx]^p = \mu_1 \,.$$

We now use naturality along $\mathbb{S}[B^{rep}(M)] \to \mathbb{S}$ to compute the $\mathbb{Z}/p$-homotopy of

$$THH(\mathbb{Z}, \langle p \rangle) \simeq THH(\mathbb{Z}) \wedge_{\mathbb{S}[B^{cy}(M)]} \mathbb{S}[B^{rep}(M)] \,.$$

There is a Künneth spectral sequence

$$\begin{aligned}
E^2_{**} &= \mathrm{Tor}_{**}^{H_*(B^{cy}(M))}(\pi_*(THH(\mathbb{Z}); \mathbb{Z}/p), H_*(B^{rep}(M))) \\
&\cong \mathrm{Tor}_{**}^{P(x)\otimes E(dx)}(E(\lambda_1) \otimes P(\mu_1), P(x) \otimes E(d\log x)) \\
&\cong E(\lambda_1, d\log x) \otimes P(\mu_1) \otimes \Gamma([dx]) \\
&\Longrightarrow \pi_*(THH(\mathbb{Z}, \langle p \rangle); \mathbb{Z}/p) \,.
\end{aligned}$$

By naturality with respect to the map

$$THH(\mathbb{Z}, \langle p \rangle) \to THH(\mathbb{Z}/p)$$

induced by the augmentation $\mathbb{S}[B^{rep}(M)] \to \mathbb{S}$, there must be a differential

$$d^p(\gamma_p([dx])) = \lambda_1$$

leaving the term

$$E^{p+1}_{**} = E(d\log x) \otimes P(\mu_1) \otimes P_p([dx])$$

which must equal the E^∞-term for degree reasons. Also by naturality there must be a multiplicative extension

$$[dx]^p = \mu_1 \,.$$

Proposition. *There is an isomorphism*

$$\pi_*(THH(\mathbb{Z}, \langle p \rangle); \mathbb{Z}/p) \cong E(d\log p) \otimes P(\kappa) \,,$$

with $d\log p$ represented by $d\log x$ and κ represented by $[dx]$ in the abutment of the Künneth spectral sequence.

Since $\Omega^*_{(\mathbb{Z}, \langle p \rangle)} \cong E(d\log p)$, this agrees with the Hesselholt–Madsen calculation.

Non-strictly commutative cases

More generally, if $M_+ \to P_+$ is a map of commutative $\mathcal{I}$-space monoids with zero, such that there is a cofiber sequence

$$B^{cy}(P)_+ \to B^{cy}(M) \to B^{rep}(M)$$

of $B^{cy}(M)$-spaces, then there is a cofiber sequence

$$\mathrm{THH}(A \wedge_{\mathbb{S}[M]} \mathbb{S}[P]) \to \mathrm{THH}(A) \to \mathrm{THH}(A, M)\,.$$

It seems to be an interesting problem to find the M for which such a monoid P exists.

I believe this happens for $M = Q_{\geq 0}S^0 \subset QS^0$ and $P = Q_0S^0$, where $M_+ \to P_+$ takes the components $Q_jS^0 \subset M$ with $j > 0$ to the base point.

One might ask if there is a prelog structure $\alpha\colon Q_{\geq 0}S^0 \to \Omega^{\mathcal{I}} ku_p$ that takes the generator to p, i.e., such that the free prelog structure extends over the partial group completion map

$$\coprod_{j\geq 0} B\Sigma_j \to Q_{\geq 0}S^0\,.$$

This appears to be unlikely, because the cofiber sequence $\mathbb{S}[M] \to \mathbb{S}[M] \to \mathbb{S}[P]$ would give $ku_p \wedge_{\mathbb{S}[M]} \mathbb{S}[P] \simeq ku/p$ a commutative $\mathbb{S}$-algebra structure, which is impossible.

To obtain a prelog structure on ku_p generated by the Bott element $u \in \pi_2(ku_p)$, we start with a map $\Sigma^\infty S^2 \to ku_p$ representing u, and extend it freely to a commutative $\mathbb{S}$-algebra map

$$\bigvee_{j\geq 0} \Sigma^\infty(E\Sigma_{j+} \wedge_{\Sigma_j} S^{2j}) \to ku_p\,.$$

We have stabilization maps

$$S^2 \wedge (E\Sigma_{j+} \wedge_{\Sigma_j} S^{2j}) \to E\Sigma_{(j+1)+} \wedge_{\Sigma_{j+1}} S^{2(j+1)}$$

acting on the left hand side, and inverting these yields $\bigvee_{j\in\mathbb{Z}} M_j$, where

$$M_j = \operatorname*{colim}_k \Sigma^{-2k} E\Sigma_{(j+k)+} \wedge_{\Sigma_{j+k}} S^{2(j+k)}$$

This may not be realized by a commutative $\mathcal{I}$-space monoid, but perhaps by a commutative $\mathcal{J}$-space monoid. Letting

$$M = \bigvee_{j\geq 0} M_j$$

and $P = M_0$, we get a cofiber sequence

$$\Sigma^2 M \to M \to P\,.$$

If the free prelog structure above extends over

$$\bigvee_{j\geq 0} E\Sigma_{j+} \wedge_{\Sigma_j} S^{2j} \to M$$

we get that

$$ku_p \wedge_{\mathbb{S}[M]} \mathbb{S}[P] \simeq H\mathbb{Z}_p$$

and there will be a cofiber sequence

$$\mathrm{THH}(\mathbb{Z}_p) \xrightarrow{i_*} \mathrm{THH}(ku_p) \xrightarrow{j^*} \mathrm{THH}(ku_p, M)$$

with this prelog structure.

Cyclotomic structure

The natural equivariant equivalence

$$r\colon \Phi^{C_p}\,\mathrm{THH}(A) \xrightarrow{\simeq} \mathrm{THH}(A)$$

specializes to an equivariant equivalence

$$r\colon \Phi^{C_p}\,\mathrm{THH}(\mathbb{S}[M]) \xrightarrow{\simeq} \mathrm{THH}(\mathbb{S}[M])\,.$$

The suspension spectrum functor translates fixed points of spaces to geometric fixed points of spectra, so the left hand side can be rewritten

$$\Phi^{C_p}\,\mathrm{THH}(\mathbb{S}[M]) \cong \Phi^{C_p}\mathbb{S}[B^{cy}(M)] \cong \mathbb{S}[B^{cy}(M)^{C_p}]$$

and the cyclic bar construction is self-similar in the sense that a certain p-th power map

$$B^{cy}(M) \xrightarrow{\cong} B^{cy}(M)^{C_p}$$

is an isomorphism, so that we get an agreement with the right hand side

$$\mathrm{THH}(\mathbb{S}[M]) \cong \mathbb{S}[B^{cy}(M)]\,.$$

(To see the details here, one must use p-fold edgewise subdivision to get a model $sd_p B^{cy}(M)$ for $B^{cy}(M)$ with a simplicial C_p-action, and then contemplate its C_p-fixed points in each simplicial degree.)

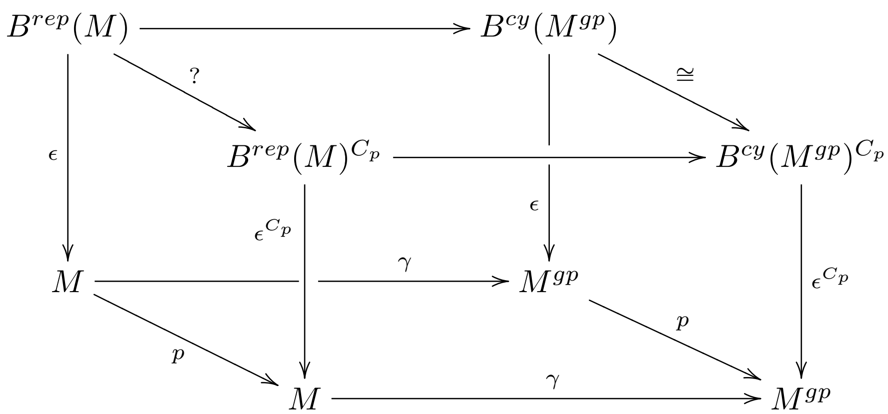
Geometric fixed points also commutes with smash products of spectra, so in order to get an equivariant equivalence

$$r\colon \Phi^{C_p}\,\mathrm{THH}(A, M) \xrightarrow{\simeq} \mathrm{THH}(A, M)$$

it will suffice to show that the p-th power map

$$B^{rep}(M) \xrightarrow{?} B^{rep}(M)^{C_p}$$

is an equivalence. By the following commutative diagram,



with homotopy pullback squares in the front and back, it follows that we will need to assume that M is p-exact in the following sense.

Definition. We say that a commutative $\mathcal{I}$-space monoid M is p**-exact** if the multiplication-by-p map $p\colon M \to M$ is exact, meaning that the square

$$\begin{array}{ccc} M & \xrightarrow{\gamma} & M^{gp} \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p} \\ M & \xrightarrow{\gamma} & M^{gp} \end{array}$$

is a homotopy pullback square.

For discrete M, integral and saturated implies p-exact for all p.

Lemma. *For M p-exact, the p-th power map*

$$B^{rep}(M) \xrightarrow{\simeq} B^{rep}(M)^{C_p}$$

is an equivalence.

Proposition. *Let (A, M) be a prelog $\mathbb{S}$-algebra with M p-exact. Then the vertical equivariant equivalences*

$$\begin{array}{ccccc} \Phi^{C_p} \operatorname{THH}(A) & \longleftarrow & \Phi^{C_p}\mathbb{S}[B^{cy}(M)] & \longrightarrow & \Phi^{C_p}\mathbb{S}[B^{rep}(M)] \\ {\scriptstyle r}\downarrow & & \downarrow{\scriptstyle r} & & \downarrow{\scriptstyle r} \\ \operatorname{THH}(A) & \longleftarrow & \mathbb{S}[B^{cy}(M)] & \longrightarrow & \mathbb{S}[B^{rep}(M)] \end{array}$$

induce an equivariant equivalence

$$r\colon \Phi^{C_p} \operatorname{THH}(A, M) \to \operatorname{THH}(A, M)$$

of pushouts, making $\operatorname{THH}(A, M)$ *a p-cyclotomic spectrum.*

Proof.

$$\begin{aligned} \Phi^{C_p} \operatorname{THH}(A, M) &= \Phi^{C_p}(\operatorname{THH}(A) \wedge_{\mathbb{S}[B^{cy}(M)]} \mathbb{S}[B^{rep}(M)]) \\ &\cong \Phi^{C_p} \operatorname{THH}(A) \wedge_{\Phi^{C_p}\mathbb{S}[B^{cy}(M)]} \Phi^{C_p}\mathbb{S}[B^{rep}(M)] \\ &\cong \operatorname{THH}(A) \wedge_{\mathbb{S}[B^{cy}(M)]} \mathbb{S}[B^{rep}(M)] \\ &= \operatorname{THH}(A, M)\,. \end{aligned}$$

□

Examples. The strictly commutative monoid $M = \mathbb{N}_0$ is p-exact for all p, as is the commutative $\mathcal{I}$-space monoid $M = Q_{\geq 0}S^0$, and probably the commutative $\mathcal{J}$-space monoid $M = \bigvee_{j \geq 0} M_j$. The free commutative $\mathcal{I}$-space monoid $\coprod_{j \geq 0} B\Sigma_j$ is not p-exact, and likewise for $\bigvee_{j \geq 0} E\Sigma_{j+} \wedge_{\Sigma_j} S^{2j}$.

Log topological cyclic homology

The cyclotomic structure map for $\mathrm{THH}(A,M)$ gives rise to a tower of restriction maps

$$\dots \xrightarrow{R} \mathrm{THH}(A,M)^{C_{p^n}} \xrightarrow{R} \mathrm{THH}(A,M)^{C_{p^{n-1}}} \xrightarrow{R} \dots \xrightarrow{R} \mathrm{THH}(A,M)$$

for all $n \geq 1$, as in the classical case. We let $\mathrm{TR}^n(A,M) = \mathrm{THH}(A,M)^{C_{p^{n-1}}}$. The homotopy groups

$$\pi_* \mathrm{TR}^n(A,M) = \pi_* \mathrm{THH}(A,M)^{C_{p^{n-1}}}$$

then form a diagram of log de Rham–Witt type, with

$$\pi_0 \mathrm{TR}^n(A,M) \cong W_n(\pi_0 A)$$

for connective A, and

$$\pi_* \mathrm{TR}^1(A,M) = \pi_* \mathrm{THH}(A,M)$$

our topological model for the log de Rham complex. We define **log topological restriction homology**

$$\mathrm{TR}(A,M) = \operatorname*{holim}_{n,R} \mathrm{TR}^n(A,M) = \operatorname*{holim}_{n,R} \mathrm{THH}(A)^{C_{p^n}}$$

as the topological model for the log de Rham–Witt complex, computing log crystalline cohomology.

The forgetful maps

$$F\colon \mathrm{THH}(A,M)^{C_{p^n}} \to \mathrm{THH}(A,M)^{C_{p^{n-1}}}$$

commute with the R-maps, hence induce a self-map of $\mathrm{TR}(A,M)$, and we define **log topological cyclic homology** as the homotopy equalizer

$$TC(A,M) \xrightarrow{\pi} \mathrm{TR}(A,M) \underset{id}{\overset{F}{\rightrightarrows}} \mathrm{TR}(A,M)\,.$$

The log cyclotomic trace

When (A,M) is such that there is a a localization cofiber sequence

$$\mathrm{THH}(A/\!/M) \xrightarrow{i_*} \mathrm{THH}(A) \xrightarrow{j^*} \mathrm{THH}(A,M)$$

of cyclotomic spectra, we also get cofiber sequences

$$\mathrm{TR}(A/\!/M) \xrightarrow{i_*} \mathrm{TR}(A) \xrightarrow{j^*} \mathrm{TR}(A,M)$$

and

$$\mathrm{TC}(A/\!/M) \xrightarrow{i_*} \mathrm{TC}(A) \xrightarrow{j^*} \mathrm{TC}(A,M)\,.$$

Suppose that there is also a localization sequence

$$K(A/\!/M) \xrightarrow{i_*} K(A) \xrightarrow{j^*} K(A[M^{-1}])\,.$$

and that the cyclotomic trace maps for A and $A//M$ commute with the i_*-maps. Then there will exist a log cyclotomic trace map, as on the right in the following diagram:

$$\begin{array}{ccccc} K(A//M) & \xrightarrow{i_*} & K(A) & \xrightarrow{j^*} & K(A[M^{-1}]) \\ \downarrow{\scriptstyle trc} & & \downarrow{\scriptstyle trc} & & \downarrow{\scriptstyle trc} \\ TC(A//M) & \xrightarrow{i_*} & TC(A) & \xrightarrow{j^*} & TC(A,M) \end{array}$$

When A and $A//M$ are connective with π_0 finite over $W(k)$, then

$$K_*(A[M^{-1}]) \to TC_*(A,M)$$

will be a p-adic isomorphism for $* > 0$.

For an alternative approach, one may consider the category of (A,M)-log modules (Sagave–Rognes), with algebraic K-theory the log K-theory $K(A,M)$. The topological Hochschild homology of that category may be equivalent, by a Morita equivalence, to $\mathrm{THH}(A,M)$, so that the Dundas–McCarthy construction of the cyclotomic trace map defines a map

$$K(A,M) \to TC(A,M)\,.$$

If there is a cofiber sequence

$$K(A//M) \xrightarrow{i_*} K(A) \xrightarrow{j^*} K(A,M)$$

and similarly for TC, then we can again conclude that

$$K_*(A,M) \to TC_*(A,M)$$

will be a p-adic isomorphism for $* > 0$.